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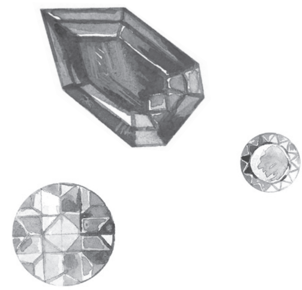

Discovering Primes As the Ancient Mathematicians Did

OVERVIEW

In this investigation, fourth graders explore concepts of factors, multiples, and prime and composite numbers. Using the hundreds chart and the Sieve of Eratosthenes, they begin to relate these concepts to real life.

These concepts are part of the field called *number theory*, a branch of mathematics that deals with the properties and relationships of numbers, especially whole numbers. Number theory is understandable to fourth graders, and they can use it to work on very old problems, some of which have not yet been solved!

Derived from "Discovering Primes As the Ancient Mathematicians Did" in *Navigating through Problem Solving and Reasoning in Grade 4*, pp. 8–13.

The tasks in this investigation help students become more fluent with multiplication and division, using factors as a sort of "un-multiplication." Such activities also prepare students for later work with fractions, where determining factors and multiples are key components of many computations. Since these operations are approached in a problem-solving mode, as part of trying to find all the primes from 1–100, the learning will be built on a foundation of conceptual understanding. The tasks of the lesson lead students to reasoning and problem solving, as they engage in mathematical discourse.

GOALS

- Define, differentiate among, and apply factors, multiples, prime numbers, and composite numbers in mathematical situations for the numbers 1–100.
- Use the Sieve of Eratosthenes to identify prime numbers in the range of 1–100.

MATERIALS NEEDED

Copies of the Using the Sieve of Eratosthenes activity sheet (pp. 112–113, and available at this book's more4U page), one for each student

Copies of the Hundreds Chart sheet (p. 114 and at more4U), one for each student

Six colors of markers, crayons, or pencils

Engage

Present the following scenario to your students:

> **Oceanside Elementary School is preparing for its annual spring carnival. Mr. Johnson has issued a challenge to his fourth-grade students: If his class can determine the total number of prime numbers on a hundreds chart, he will volunteer to participate in the "Dunk the Teacher" booth at the carnival. Mr. Johnson suggests that students use the procedure for the Sieve of Eratosthenes as they try to meet his challenge.**

Read aloud to the class the story *The Librarian Who Measured the Earth* by Kathryn Lasky (1994). Mixing history and science, Lasky tells how Eratosthenes calculated the circumference of the earth through a process involving the sun and shadows.

After you have finished reading the book, explain that one of the many other mathematics problems explored by Eratosthenes was finding out how many numbers from 1 to 100 are prime numbers. Tell the students that we can still use his method, which is called the Sieve of Eratosthenes. Ask the students if they know what a sieve is, and invite them to offer their ideas. Demonstrate with a strainer or colander, explaining that it is used to for separating sand from stones or ore, or in the kitchen to remove liquid from produce or pasta, and for many other uses. This investigation will use Eratosthenes' sieve to separate prime numbers from composite numbers. After we finish the activity, the numbers left in our "sieve" will all be prime numbers.

Ask your students, "Do you think the fourth graders at Oceanside Elementary School will be able to rise to Mr. Johnson's challenge and find all the prime numbers on a hundreds chart?"

In applying the concepts of prime numbers and factors, students need to develop a clear understanding of the meaning of these mathematical ideas. If the terms *prime, composite, factor, and multiple* are new to your students, you could either let the meanings emerge from the activity, pointing out what they are learning along the way, or stop to define those terms here.

- A *prime number* is a whole number with exactly two distinct factors, itself and 1. (For example, 3 is a prime number because its only factors are 3 and 1.)
- A *composite number* is a whole number that has more than two factors. (For instance, 4 is a composite number because it has three factors: 1, 2, and 4.)

- A *factor* is a whole-number component of a multiplication computation. (The number 15 has the factors 1, 3, 5, and 15, for example.)
- A *multiple* is a whole-number product of a given number. (The multiples of 3 are 3, 6, 9, 12, and so on.)

Explore

For students to develop their thinking skills, they must believe that they are working in a safe classroom environment where risk-taking is acceptable and errors are learning opportunities. When teachers support and extend students' mathematical understanding, students can safely reach out and try new mathematical ideas.

Group your students in pairs and distribute a copy of the Using the Sieve of Eratosthenes activity sheet and the Hundreds Chart sheet to each student. Ask each pair to predict the total number of primes they expect to find on the chart and to write it on a corner of the hundreds chart. Then ask each pair to use the following procedure to find the answer to Mr. Johnson's question.

Have the students cross off the number 1 on the hundreds chart because it is not prime. This action may well elicit questions from one or more students unless you have explained the definitions above, which make it clear why the number 1 cannot be prime. The issue of whether 1 is prime, composite, or neither is a very interesting question mathematically. You may want to set it aside for further discussion later. See the article "Ryan's Primes" (Juraschek and Evans 1997; available at this book's more4U page) for insight on this question.

Have the students circle the number 2 in green. How many factors does the number 2 have? [*2.*] Now they can cross out in green all the multiples of 2 on the hundreds chart. Ask, "How many factors does each multiple of 2 have? [*At least 3: 1, 2, and the number. Some, such as 24, have many more.*] Ask the students to try to find any patterns they can for those numbers that are crossed out in green, and to come up with a way to identify the numbers that are multiples of 2. [*They are even, or they end in 0, 2, 4, 6, or 8.*]

Now have the students circle the number 3 in red. Ask, "How many factors does 3 have? [*2: 1 and 3.*] Is 3 a prime number? [*Yes.*] Use red to cross out all the multiples of 3. Encourage students to reflect on the number of factors for each of the multiples of three that they cross off. [*3 or more factors.*] Are these multiples prime or composite? [*Composite.*] Some students may pause when they come to 6, 12, and so on, since they are already crossed off in green. They should still cross them off with red, indicating that both 2 and 3 are factors. Ask students to look for any patterns they can find in the numbers crossed off in red. [*This is more challenging than identifying multiples of 2. If you wish to give a hint, you could suggest adding the digits of two-digit multiples of 3, since the sum of the digits of multiples of 3 will be divisible by 3. Or, if you wish, you could leave this for a future lesson, since the goal of this lesson is to find primes, and not to focus on multiples.*]

As student pairs work on finding primes, circulate around the classroom, noting progress and any questions students are asking. This is part of the assessment for the lesson.

This would be a good time to revisit the students' estimates of how many primes there are between 1 and 100, which they noted on the corner of their hundreds chart. Some may wish to revise their estimate upward or downward.

Ask students to find the next number they should use in their search for primes. The number 4 is already crossed off, since is it a multiple of 2, so the next prime is 5. Use blue to circle 5, and cross off all multiples of 5. Each of these numbers has at least three factors—itself, 5, and 1. If students have already crossed off any of these numbers in green or red, have them cross it off again in blue. When they are finished, again ask them to notice any patterns in the crossed-off blue numbers. [*Most students will notice that multiples of 5 end in 5 or 0.*]

Have students repeat the process that they have been following for circling prime numbers and crossing off numbers that are multiples, following the directions on their worksheets. They should use a different color each time and continue working until the chart has no numbers left to cross off or circle.

Since identifying multiples of 7 and 11 is more sophisticated than identifying those for 2, 3, and 5, it is not advised to look for patterns for those numbers at this time.

To summarize the search for primes after all students have finished, you could ask questions such as the following. All of these questions could be approached with a Think/Pair/Share strategy, giving students a minute or two to think, then discuss with partner, and finally share with class:

1. How many prime numbers did you find in the hundreds chart? [*25.*]

There are 25 prime numbers between 1 and 100. They are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

2. If you wanted to search for the primes in the numbers from 101 to 200, then from 201 to 300, and so forth, how could you do it? [*Extend the Sieve of Eratosthenes to larger number charts, remembering to cross out all the multiples of all the primes you have already found.*]
3. What is the first multiple of 7 that you found that you hadn‘t already crossed off as a multiple of a smaller prime? [*49.*]
4. If you had just circled 11 as a prime and were crossing off multiples of 11, what number would you need to have in your chart to be able to cross off a multiple of 11 for the first time? [*121.*]
5. How many primes did you find in the first 50 numbers? [*15.*] This represents what fraction of all the primes on the hundreds chart? [*3/5.*]

6. What is the difference between a prime number and a composite number? [*See the definitions earlier in the lesson. Students' definitions may vary, but should include key ideas of two distinct factors for primes, more than two for composites.*]
7. Is the number 1 a prime number? Is it a composite number? Why? [*No, 1 is neither prime nor composite, since it does not have two distinct factors.*]
8. What do you think the next five primes numbers after 100 would be? [*They are 101, 103, 107, 109, 113. Do not expect students to get all of them immediately.*]

To return to the original context and question, tell your students that the fourth graders at Oceanside Elementary School met the deadline and located all the prime numbers on the hundreds chart. Mr. Johnson did indeed participate in the spring carnival and was frequently dunked at the "Dunk the Teacher" booth.

Extend

Among the further explorations you can do are the following questions and investigations.

1. Students can explore **Goldbach's Conjecture,** which states: Any even number greater than 2 can be written as a sum of two primes. For example, 8 = 5 + 3. Primes can be repeated, as in 6 = 3 + 3. The word *conjecture* means that so far no one has proved that this is true for all numbers, nor has anyone proved that it is not true. Ask students to test Goldbach's Conjecture for even numbers up to 50.

Christian Goldbach was a German mathematician who lived from 1690 to 1764.

2. **Twin primes** are pairs of primes separated by one even number. There are eight such pairs less than 100. Can you find them? [*They are 3-5, 5-7, 11-13, 17-19, 29-31, 41-43, 59-61, and 71-73.*]

3-5-7 is the only case of "triplet primes." Can you explain why?

3. What do you think is the largest prime number? Students can suggest large numbers. Mathematicians believe that there is not a largest prime, and some search for the largest known prime. Periodically, they publish the newest such prime. You could look for the newest, or at least a recent entry, which are usually represented as a **Mersenne prime number,** a prime of the form $2^p - 1$, where p is prime. Your students may not have encountered exponents yet, but they can still understand the repeated multiplication. So, with p = 2, 2 × 2 – 1 = 3. 3 is prime. Mersenne primes are special. Most primes are not Mersenne primes, and Mersenne primes will turn out to be important in later mathematical studies. Students now can try to find all Mersenne primes less than 100. [*Only 3, 7, and 31 are Mersenne primes less than 100***.**]

Marin Mersenne was a French monk who lived from 1588 to 1648.

As of December 2018, mathematicians had found 51 Mersenne primes. The largest one is 24,862,048 digits long. If you printed that number with 10 digits per inch, your printout would be 39.24 miles long.

Not all primes, p, will produce a Mersenne prime using the form $2^p - 1$. (For example, p = 11 yields 2047 = 23 × 89.) But in order for a Mersenne number to be prime, the exponent, p, must be prime.

4. How many even primes are there? [*2 is the only even prime.*] Why is this true? [*All other even numbers are multiples of 2.*]
5. Can you find any strings of consecutive composite number with 4 or more in a row? How many? What is the longest series of composites in the numbers less that 100? [*There are seven strings of 5, and one string of 7.*]
6. The **fundamental theorem of arithmetic** states that every composite number can be expressed in only one way as the product of primes. [*Note that rearranging the order of factors does not count; that is, 2 × 3 × 2 is considered the same as 2 × 2 × 3.*] See if you can find the prime factorization of the first 15 composite numbers.

For further activities to strengthen concepts of prime and composite numbers, see the following:

- "Mersenne's Fun House" in *Math Trek: Adventures in the Math Zone* (Peterson and Henderson 2000, pp. 29–38)
- "Number Name Discoveries" in *Navigating through Number and Operations in Grades 3–5,* pp. 22–26, available at more4U. This lesson focuses on primes, composites, and square numbers from a geometric and visual approach, forming numbers with arrays and tiles.
- A delightful story from children's literature is *A Remainder of One* by Eleanor J. Pinczes (1995). A Tile Rectangles activity sheet for this lesson is available on p. 115 and at more4U.

SUMMARY

Throughout this lesson we have used research-informed practice, establishing clear goals for learning and incorporating tasks that encourage reasoning and problem solving. Students work in pairs, giving ample opportunities for mathematical discussion. The teacher poses purposeful questions to assist students in making sense of important ideas and concepts. A safe learning environment supports students as they struggle and are free to make mistakes. Those mistakes lead to deeper learning. Throughout the lesson, the teacher assesses student thinking, using evidence of student thinking to both measure progress and adjust instruction. Further assessment can be done by giving students a set of numbers and asking them to identify the primes. Each of the students can then explain how they determined the prime numbers. This can be a short quiz, an exit slip, or any other suitable form for daily evaluation. [*Example: 107, 113, 117, 152, 255, 256, 363. Only the first 2 are prime.*]