PRACTICE 1

Make Sense of Problems and Persevere in Solving Them

Practice 1: Make sense of problems and persevere in solving them

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches. (CCSSI 2010, p. 6)

Unpacking the Practice

The first mathematical principle in the *Common Core State Standards for Mathematics* (CCSSM; CCSSI 2010) centers on *problem solving*—making sense of problems and persevering in solving them. In this section, we show how this practice is aligned with each of the National

Council of Teachers of Mathematics Process Standards (NCTM 2000). In addition, because problem solving is so fundamental to learning mathematics with understanding, we briefly mention how certain aspects of this mathematical practice are explored further in later chapters focused on the other mathematical practices.

Problem Solving Standard

Although problem solving is just one of the Process Standards in NCTM's (2000) *Principles and Standards for School Mathematics*, it is central to the kind of mathematics learning NCTM advocates. Because problem solving is part of all content areas, problem-solving activities should not be an isolated part of a lesson, unit, or curriculum but should instead be integrated into students' experiences, involve important mathematics, and connect to multiple process and content strands.

"Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings" (NCTM 2000, p. 52). This means that many of the tasks we ask students to engage in must be problems to solve, not simply exercises to do. These tasks should allow students to enter problems via multiple entry points and to invent and use strategies that make sense to them. It is through this meaning making that students can develop and deepen their mathematical understanding.

If we want students to develop what *Adding It Up* (NRC 2001) calls a *productive disposition* as a mathematical problem solver, we must support them in taking up a different role than typically seen in traditional classrooms. *Adding It Up* defines a productive disposition as "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that the steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (p. 131). Students should be active participants in the sense-making process, responsible for making sense of the problems before them and allowed to invent solution strategies by building on the knowledge they already possess. We know from the work of Carpenter and colleagues (1999) that even young children can solve many different kinds of problems—without having to be explicitly taught—by using their own informal mathematics knowledge; in fact, "children may actually understand the concepts that we are trying to teach but be unable to make sense of specific procedures that we are asking them to use" (Carpenter et al. 1999, p. xiv). When teachers allow students to use and build on their own knowledge, students can strengthen and extend what they know as well as develop new mathematical understanding when problem solving.

The teacher plays a crucial role supporting students in problem solving. Although many teachers have textbooks to guide their instruction, it is their responsibility to select, adapt, design, and implement appropriate mathematical tasks for the students in front of them. If mathematical tasks are posed to students in ways that allow them to build on what they already know, students are more able to develop their own methods for solving problems as well as create new knowledge in the process (NCTM 2000). Teachers should create classrooms where students are encouraged to "explore, take risks, share failures and successes, and question one another…[so that] they will be more likely to pose problems and to persist with challenging problems" (NCTM 2000, p. 52), just as this mathematical practice calls for.

Reasoning and Proof Standard

The Reasoning and Proof Standard makes brief mention of the importance of making conjectures to assist in solving a problem (NCTM 2000, p. 55). Students should have opportunities to make and communicate conjectures, to explore these conjectures, and to analyze and justify these conjectures. Making and investigating conjectures supports students in developing knowledge that can be used to understand a problem, see how a problem is connected to a larger body of mathematics, and learn new mathematics. However, because this idea is more fully developed in practice 3: construct viable arguments and critique the reasoning of others, please see that chapter for more detail.

Representations Standard

Although practice 1 states that "*younger* students might rely on using concrete objects or pictures to help conceptualize and solve a problem" (CCSSI 2010, p. 6; italics added), students of all ages can benefit from using manipulatives or drawing pictures to help understand and solve problems. By using concrete objects or pictures, students can understand the underlying concepts in a problem and may be able to make connections to more abstract ways of representing the problem.

NCTM's *Principles and Standards* describes several ways students should learn to work with representations in pre-K–12 classrooms: "create and use representations…; select, apply, and translate among representations…; use representations to model…phenomena" (NCTM 2000, p. 67).

Teachers should provide opportunities for students to learn about and use conventional representations as well as opportunities for students to invent and develop their own ways of representing mathematical ideas and thinking. In lower grade classrooms, this might simply be encouraging students to draw pictures as a way to solve a problem. In upper grade classrooms, students might "transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need," as practice 1 states. Students at all grade levels should be supported in developing their own representations and challenged to explain the connections between their representations and the problem itself, as well as the connections among representations. Finally, students should have opportunities to model various phenomena using appropriate representations, something that is explored in more detail in the chapter on practice 4: model with mathematics.

Connections Standard

NCTM, in its *Principles and Standards,* states that students should "recognize and use connections among mathematical ideas" (2000, p. 64) that can support them in "consider[ing] analogous problems, and try[ing] special cases and simpler forms of the original problem in order to gain insight into its solution" (practice 1; CCSSI 2010, p. 6). If students can do this, they are better able to develop a richer and deeper understanding of the problem at hand and to understand the discipline of mathematics as an integrated body of knowledge rather than discrete, unrelated topics.

NCTM's Connections Standard says that students should have opportunities to recognize and use mathematics in contexts outside of mathematics (2000, p. 64). Although this idea is not an explicit part of CCSSM's practice 1, it is closely connected. Teachers can broaden their students' understanding of mathematics by connecting to other school subjects like science and social studies. Teachers can also engage students in understanding mathematics more deeply and personally by connecting to, including, and building on students' *community mathematics knowledge* (Civil 2007; Gutstein 2006, 2007). Community mathematics knowledge is the mathematical knowledge that students can gain and use in their out-of-school experiences. By grounding problem solving in students' experiences and focusing on the assets that students bring to the classroom, teachers can highlight the important mathematical practices in which students and their families engage, as well as the relevance of mathematics to their lives. This doesn't mean that all tasks need to be set in real-world contexts. Abstract tasks that develop, support, and challenge students' understanding of and connections among mathematical content and processes are also important.

Communication Standard

Practice 1 ends with a statement about the need for students to understand others' solution strategies as well as understand the mathematical connections between different strategies, which aligns with important aspects of NCTM's Communication Standard. In fact in meeting the Standard, not only will students learn to understand and evaluate others' strategies, but when they engage in mathematical arguments where they must justify their own solutions, they "will gain a better mathematical understanding as they work to convince their peers about different points of view" (NCTM 2000, p. 60, citing Hatano and Inagaki 1991).

Mathematical discussions should center on ideas and solution strategies so as to place attention on mathematical understanding rather than simply solutions (Hiebert et al. 1997). It is important to note that these ideas and strategies should, for the most part, come from the students themselves. Teacher and students should have joint responsibility for sharing and clarifying important mathematical ideas. This does not mean that teachers should be passive observers waiting for their students to "discover" the mathematics, but rather that teachers must play an important role in discussions in which students have opportunities to explain ideas, to show solution strategies, and to determine the correctness of an answer themselves. When students develop their own methods, their opportunity to learn mathematics with understanding increases (Carpenter and Lehrer 1999; Hiebert et al. 1997).

Classroom Examples

In this section, we present three classroom vignettes. Although there are many different ways that teachers can encourage students to make sense of problems and persevere in solving them, we provide one elementary grades vignette and two secondary grades vignettes to illustrate several important aspects of the practice.

Elementary Grades Vignette: How Many Teams?

To engage young students in problem solving, teachers can and should use problems that come from the students' worlds (NCTM 2000). Although some elementary school teachers may be reluctant to spend the majority of their time on story problems because they fear that they are too difficult or that students must be able to recall number facts first, students in classrooms with a strong focus on problem solving perform just as well as students in classrooms that have a more traditional focus on only number facts (e.g., see Peterson et al. 1989).

In the following vignette, a teacher poses a division problem arising from an event taking place in his class. Although the teacher has not formally introduced division to his second graders, he is confident that they will be able to solve the problem by using strategies that make sense to them.

Mr. Guzman poses the following problem to his students: "As you know, we are joining Ms. Leavell's class to play math games this afternoon. If there are 36 kids and we need to make teams of 4, how many teams will there be?" The teacher allows several minutes for the students to explore the problem individually using various strategies, then asks students to discuss their strategies in small groups. Finally, students are asked to share their solution methods with the whole class.

Mr. Guzman: Who would like to explain how they solved the problem to the class?

Suzanna: I can. I drew a picture. All I had to do was draw 36 dots, and then I drew circles around groups of 4.

Mr. Guzman: Why did you use 36 dots?

Suzanna: I used 36 dots because there were 36 kids. I didn't want to draw actual people…it would take too long, so I just used dots.

Mr. Guzman: Did anyone else use a drawing to help them solve the problem?

[*A few students raise their hands, and Mr. Guzman asks the students to briefly discuss how the drawings are similar to or different from Suzanna's and from one another's.*]

Mr. Guzman: Did anyone use another strategy?

Katy: Yeah, I used blocks.

Mr. Guzman: How did you use blocks to help you solve this problem?

Katy: Well, first I counted out 36 blocks to stand for each kid. Then I just started making them up into groups of 4 to make the teams.

Mr. Guzman: And then what?

Katy: Then I just counted up how many groups I made.

[*Again Mr. Guzman asks if any other students used manipulatives and very briefly discusses how the manipulatives were used in similar or different ways and even connects them back to the drawings previously discussed.*]

Mr. Guzman: Are there any other strategies?

Carlos: I just started adding 4s.

Mr. Guzman: OK, how did you know to start adding 4s?

Carlos: Because I knew there would be 4 people on each team.

Mr. Guzman: OK, so when you added 4 and 4, that would be 8 people or 2 teams. Then you added another 4?

Carlos: Yep.

Mr. Guzman: OK, so then what? How did you know how many 4s to add?

Carlos: Well, I just kept adding, like 4, 8, 12, until I got to 36 because that would mean that I included all 36 kids. To figure out the answer, I then just had to count how many 4s I added up.

Mr. Guzman: And what did each 4 stand for?

Carlos: One team.

Mr. Guzman: So what was your answer? How many teams will there be?

Carlos: 9.

Mr. Guzman asks Suzanna, Katy, and the rest of the class if they also got 9 teams as the answer and then begins a discussion about the mathematical connections among the solution strategies (e.g., when Suzanna circled groups of 4 dots this was similar to Katy dividing up the blocks in groups of 4).

Notice that when Mr. Guzman asked the students about how they solved the problem, the main focus was on students' problem-solving strategies, not the answer. While the answer is important, Mr. Guzman wanted his students to focus on their own solution strategies and then to consider mathematical connections among these strategies in this mathematical discussion.

Although the problem in this example is a partitive division problem and may at first seem difficult for primary students, it affords multiple entry points, which allow students to use different kinds of strategies (e.g., drawing a picture, using manipulatives, using addition) and shows that young children can successfully solve this problem using various strategies (Carpenter et al. 1999). This vignette illustrates how students who have never been formally taught division can still solve the problem in ways that make sense to them. By having tasks that include multiple solution paths, students are challenged not only to make sense of the problem but also to understand it more deeply by reflecting on the mathematical connections among the different strategies

Middle School Vignette: The Border Problem

This example features the Border Problem and shows how a teacher might introduce middle school students to algebraic thinking and representation (see Burns and Humphreys [1990] and Boaler and Humphreys [2005] for more detailed information and analysis of teacher moves and student strategies). It is similar to the elementary-grades example in that the teacher selects a problem-solving task that has multiple entry points for students, encourages them to make sense of different solution strategies, and provides ways for students to see the connections among strategies.

Ms. Waters: For today's problem, I'm going to show you a picture on the overhead. It's a 10 x 10 grid, and some of the squares are colored in. What I want you to figure out is how many of the squares are shaded in on the 10 x 10 square and to do it without counting each of the squares individually. I'd like you to try to come up with more than one strategy so that you can check your answer, since I don't want you to count them.

[*Ms. Waters then shows figure 1.1 and gives them a short time to think.*]

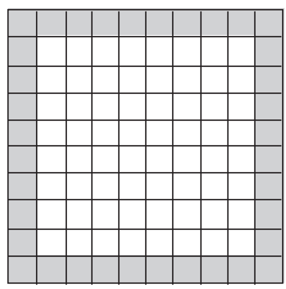

Fig. 1.1. The Border Problem

Ms. Waters: I bet you might have a few ideas of how to figure this out, but first I want to hear what you think the answer might be. How many are shaded in?

Larissa: 40!

Finn: 36!

Johnetta: 38?

Ms. Waters: Hmmm, so right now we have 40 and 36 and 38 as possible answers on the table. Did anyone else get another answer?

[*No one offers another answer.*]

Ms. Waters: OK, so do you think that this is the kind of problem that could have more than one answer?

Larissa: No, because we're all counting the same squares. There should be only one right answer.

Ms. Waters: OK, so take some more time to continue working. Be sure to come up with a strategy that you think will help you arrive at the right answer. Remember, I also want you to challenge yourself to come up with multiple solution strategies. You can also work with people at your table.

[*After several minutes, Ms. Waters asks students to share their strategies with the class, and these get recorded on the board for the students to see.*]

Ms. Waters: Finn, can you share your strategy?

Finn: I think there are 36 colored squares.

Ms. Waters: OK, why? How did you get that answer?

Finn: I got this because I added 10 for the top row, 10 for the bottom row, and 8 for each of the columns on the right and left sides.

Ms. Waters: Tell me more. How did you decide that there were 8 on each side?

Finn: Well, at first I thought there were 40 squares shaded in…

Larissa: Me too!

Finn: But then I realized that I was counting some of the squares more than once—the squares on the corners.

Larissa: I also thought there were 40 squares, but I realized that wasn't right because that would mean there were 60 squares that weren't shaded. And I knew that wasn't right.

Ms. Waters: How did you know that wasn't right?

Larissa: When we had time to work with people at our table, Joslyn's strategy made me realize this.

Ms. Waters: Joslyn, can you tell us what you did?

Joslyn: Well, I wasn't sure this was going to work at first, but I decided that instead of counting the shaded squares, I could figure out how many were not shaded and then subtract that number from 100.

Ms. Waters: You subtracted it from 100?

Joslyn: Yes, because you told us that it was a 10 x 10 grid, which means there were 100 squares total. I could visualize the 8 x 8 square in the middle, which would be 64. 100 minus 64 is 36.

Ms. Waters: You said that you visualized 64. Were you sure it was 64?

Joslyn: Yes, because one row across the whole square would be 10, but there is a square shaded on both ends which makes the inside square only 8 across.

Ms. Waters: Did anyone else try this strategy?

[*Only one other student raises his hand.*]

Ms. Waters: Ok, so take a moment, and talk with someone close by about why this might be a good strategy.

After a short time, Ms. Waters asks students to share their strategies. Not only are these strategies discussed and recorded, the teacher and students discuss the mathematical connections among the strategies. Ms. Waters will follow up by using different-sized square grids to support students in beginning to generalize their solutions for n by n grids.

In this vignette, the students are engaged in the problem not because it is from a context taken from their everyday lives, but because it is a problem that has multiple entry points students can use to solve it. Although at first the students arrive at different answers, it is clear that the students are able to persevere in figuring out the correct answer. The students also begin to discuss the connections between the diagram and their strategies, eventually connecting to generalized equations that would allow them to solve related problems.

High School Vignette: Completing the Square

This final vignette features a high school teacher inspired by Vinogradova's (2007) description of a lesson introducing students to "completing the square." When completing the square is introduced to students, it is most often explained from a procedural perspective because many students will not use this method in actual practice when solving quadratic equations unless they are asked to. The completing the square method (CTS) becomes important as students learn to solve quadratic equations and use translation methods to graph functions (and conics). Additionally, CTS is used in integral calculus.

This activity, however, supports students in understanding the underlying concepts through making connections between the algebraic procedure of completing the square and its geometric meaning. The following vignette involves students in an algebra II class; they have just finished sections on graphing quadratic equations from tables of values. At this point, the students know how x-intercepts connect to solutions of quadratic equations and have some knowledge of squares and experience with algebra tiles.

To begin the activity, students are asked to use algebra tiles to square binomials, for example $(x + 3)^{2.}$ The common error students make is to square only the x and the 3, yielding an incorrect answer of $x^2 + 9$. Since this activity has been done before, students working in groups are able to build the correct representation of $x^2 + 6x + 9$ with their tiles. Note: the algebra tiles consist of x-by-x squares, 1-by-x rods, and 1-by-1 small squares (unit squares). The teacher poses a question to the class.

Ms. Liston: Would you be able to work "backward"? By that I mean, if you started with the trinomial $x^2 + 6x + 9$, would you be able to write the binomial you would square to get that trinomial?

Sam: Sure, $(x + 3)^2$

Ms. Liston: Right. Would you be willing to try another one?

[*Sam nods.*]

Ms. Liston: How about $x^2 + 8x + 16$? I'd like all the groups to work on this while Sam is thinking about his answer. Sam, feel free to talk with your group.

Sam: I got $(x + 4)^2$.

Zoe: I got $(x + 8)^2$.

Ms. Liston: Zoe, tell me about that. What was your thinking?

Zoe: Well, you need to get 16, and 2 times 8 is 16.

Sam: No, you get the 16 from 4 times 4 because the last number is a square number.

Zoe: Oh, yeah, you're right.

Ms. Liston: Great, what do others think? [*The class agrees with Sam and Zoe that* $(x + 4)^2$ *is correct*.] OK, let's move on to a new idea that will connect to what we've been doing. [*The lesson begins with familiarizing students with an arithmetic-geometry perspective*.]

Ms. Liston: Use your unit tiles to build a 3-by-5 rectangle [*see fig. 1.2*]. How many square units make up the area of this rectangle?

Zoe: 15.

Ms. Liston: How did you get 15?

Zoe: I just did 3 times 5.

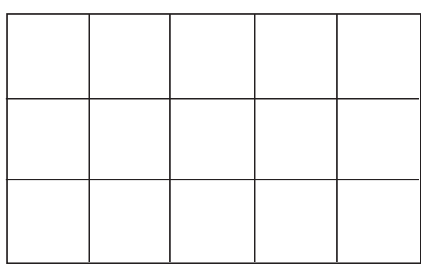

Fig. 1.2. A 3-by-5 rectangle (re-created from Vinogradova [2007, p. 404])

Ms. Liston: OK, very nice. Here's what I would like you to try to do: I would like to see if you can make a square out of this rectangle of 15 square units.

Sam: You can't.

Ms. Liston: Tell me, why do you say that?

Sam: Because 15 is not a square number.

Ms. Liston: But 15 is close to 16.

Sam: Yeah, it's one less.

Ms. Liston: Yes. Can everyone show me by rearranging the tiles how our rectangle of 15 unit squares could "almost" look like a square and record your ideas on paper?

[*The students draw and redraw and working together they come up with "squares" that have a missing piece. See fig. 1.3.*]

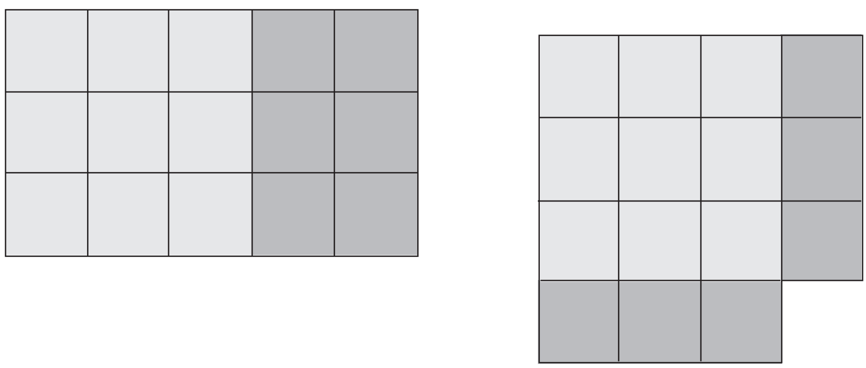

Fig. 1.3. Recomposing a rectangle into a "square" (re-created from Vinogradova [2007, p. 404])

Ms. Liston: Good job. One way to express our square with a missing piece is to write $15 = (3 + 1)^2 - 1$. Try making a square out of 22 square units. Remember, you might have some missing pieces. [*She lets the students work for a few minutes.*] Mateo, what did you get, and can you show us your drawing?

Mateo: I got $(4 + 1)^2 - 3$.

[*Students then practice the same process of completing the square using algebraic tiles.*]

Ms. Liston: So far we have been working with a rectangle with set dimensions and then making that into something that is "almost" a square. But what if we don't know the exact dimensions of the rectangle?

Mateo: We'll have to use variables.

Ms. Liston: How would you like to do that?

Mateo: We could use x and y.

Ms. Liston: OK. Will you come up to the board and draw your rectangle for the class?

[*Mateo comes to the board and draws a rectangle with y being longer than x.*]

Ms. Liston: How can we decide how long y is in comparison to x?

Mateo: It's longer.

Ms. Liston: How much longer?

Mateo: Let's make it two longer.

Ms. Liston: Can one of your group members tell us how to use algebra to write that relationship?

Jessica: In place of y you could put $x + 2$.

[*Students nod, so Ms. Liston continues.*]

Ms. Liston: Okay, would everyone use the tile blocks and build Mateo's rectangle so that it is x on one dimension and $x + 2$ on the other? [*See fig. 1.4.*]

[*Ms. Liston walks around and checks on the groups.*]

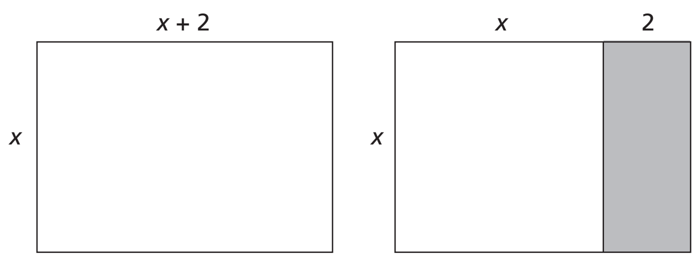

Fig. 1.4. Building an x by $x + 2$ rectangle
(re-created from Vinogradova [2007, p. 404])

Ms. Liston: Your tile rectangles look good. What's the area of your rectangle?

Zoe: Mine is $x(x + 2)$.

Sam: I got $x^2 + 2$.

Zoe: Well, you could do that, but you have to have $x^2 + 2x$.

Sam: That's what I meant to say.

Ms. Liston: Why are both answers, $x(x + 2)$ and $x^2 + 2x$ correct?

Sam: Because you distribute the x.

Ms. Liston: Great. What can we do to try to make a square out of our rectangle?

Sam: Well, you could move one of the x-rods like this. [*Shows moving x-rod as in fig. 1.5.*]

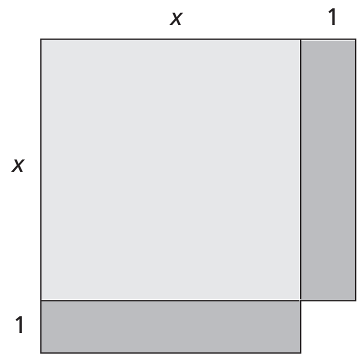

Fig. 1.5. Making a "square" out of the rectangle (re-created from Vinogradova [2007, p. 404])

Ms. Liston: Very nice. This is very close to a square now. How much is missing?

Sam: Only one unit-square.

Ms. Liston: I would like everyone to think about how to write the area of your rectangle, remembering that it is almost a square; it just has one unit-square missing.

Sam: The big square is $(x + 1)^2$. Then you have to subtract the 1, so $(x + 1)^2 - 1$.

Ms. Liston: That looks different from $x^2 + 2x$. Can you convince me that they are the same?

[*Students distribute* $(x + 1)^2$*, subtract 1, and find that the expressions are equivalent.*]

What is interesting about this introduction to completing the square is that with these examples, students begin to realize (by manipulation of the algebra tiles) that rectangles can be viewed as squares with missing pieces. In addition, the number of 1-by-x rods must always be divided into two parts. As the idea of CTS continues, students can be asked, what number of square units will complete the square? After the connection is made by students that "what's missing from the square" is what is needed to complete the square, the generalization in figure 1.6 can be made.

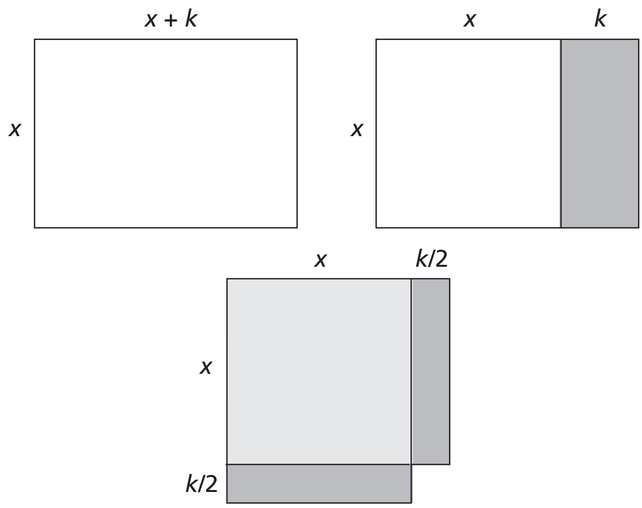

Fig. 1.6. Recomposing a rectangle into a "square"
(re-created from Vinogradova [2007, p. 405])

These three vignettes show how students can be engaged in problems by making sense of them and persevere in solving them by using strategies that make sense to them and others. Students are able to develop and communicate their strategies and to modify their strategies if needed. In doing so, they construct a new understanding of the mathematics concepts involved, as well as strengthening their knowledge of related concepts.

Resources

This resource describes the teacher's role in supporting mathematical thinking and problem solving in the classroom by showing how teachers elicit student thinking, as well as promoting reflection and sense making.

- Rigelman, N. R. "Fostering Mathematical Thinking and Problem Solving: The Teacher's Role." *Teaching Children Mathematics* 13, no. 6 (2007): 308–14.

This resource presents a classroom vignette where a teacher poses a problem to students and highlights important aspects of her pedagogy that support English language learners (ELLs) in the problem-solving process.

- Wiest, L. R. "Problem Solving Support for English Language Learners." *Teaching Children Mathematics* 14 (April 2008): 479–84.

This resource describes pedagogical practices that can be used to support ELLs in understanding and succeeding in mathematics.

- Brown, C. L., J. A. Cady, and P. M. Taylor. "Problem Solving and the English Language Learner." *Mathematics Teaching in the Middle School* 14, no. 9 (2009): 532–39.

This resource describes a problem-centered curriculum for high school students that uses problem sets to engage students in problem solving. Students gain mathematical understanding of the concepts in the context of the problems.

- Campbell, W. E., J. C. Kemp, and J. H. Zia. "Bugs, Planes, and Ferris Wheels: A Problem-Centered Curriculum." *The Mathematics Teacher* 99, no. 6 (2006): 406–13.

This resource presents teaching through problem solving pedagogy that aims to engage students in problem solving as a tool to facilitate learning important mathematics subject matter and mathematical practices.

- Fi, C. D., and K. M. Degner. "Teaching through Problem Solving." *The Mathematics Teacher* 105, no. 6 (2012): 455–59.

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