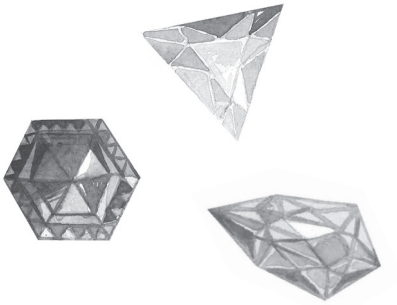


Introduction

> Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. There are ambitious expectations for all, with accommodation for those who need it. Knowledgeable teachers have adequate resources to support their work and are continually growing as professionals. The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. Technology is an essential component of the environment. Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it. (*Principles and Standards for School Mathematics,* National Council of Teachers of Mathematics [NCTM] 2000, p. 3)

Forty years ago, NCTM initiated widespread dialogue about the teaching and learning of mathematics with the publication of *An Agenda for Action* (NCTM 1980) and its leadoff recommendation that problem solving should be the focus of school mathematics. Over the ensuing years this dialogue has been reiterated, refined, and expanded by NCTM through such vehicles as the *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), *Assessment Standards for School Mathematics* (1995), *Principles and Standards for School Mathematics* (2000), and other

documents both from NCTM and in collaboration with other educational partners. Among the relevant documents that we consider here, the following are major sources:

Principles and Standards for School Mathematics (NCTM 2000) [PSSM]

Common Core State Standards for Mathematics (NGA Center and CCSSO 2010) [CCSSM]

Principles to Actions: Ensuring Mathematical Success for All (NCTM 2014) [PtA]

Closely aligned and complementary to the above reports are two additional NCTM documents, *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (2006) and *Focus in High School Mathematics: Reasoning and Sense Making* (2009), along with *Adding It Up: Helping Children Learn Mathematics* (2001), a report from the National Research Council (NRC).

Each of these documents considers the process of teaching and learning mathematics, along with curricular content, and each offers a vision of high-quality mathematics programs and the educational experiences, practices, and outcomes that are deemed essential components of such programs. While the language varies a bit, common themes resonate. Consider, for example, the PSSM Process Standards (see figure Intro.1), the CCSSM Standards for Mathematical Practice (see figure Intro.2), and the PtA Mathematics Teaching Practices (see figure Intro.3). Comparing these three sets of standards or practices, we observe strong convergence on the following elements:

- Emphasis on reasoning and sense making in order to develop conceptual understanding and facilitate problem solving
- Flexibility in investigating and representing mathematical problems and developing mathematical arguments and proofs
- Connecting mathematical concepts, representations, and processes, and using those connections to extend understanding and solutions
- Communicating mathematical ideas and reasoning, and evaluating the communication of others
- Developing procedural fluency anchored in conceptual understanding and flexible thinking

FIG. INTRO.1

NCTM Process Standards (*Principles and Standards for School Mathematics,* NCTM 2000)

NCTM Process Standards (PSSM)

Problem Solving: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and in other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving

Reasoning and Proof: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

Communication: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas precisely

Connections: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Recognize and use connections among mathematical ideas
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- Recognize and apply mathematics in contexts outside of mathematics

Representation: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena

FIG. INTRO.2

CCSSM Standards for Mathematical Practice (NGA Center and CCSSO, 2010)

CCSSM Mathematical Practices
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

FIG. INTRO.3

Mathematics Teaching Practices (*Principles to Actions,* NCTM 2014)

Principles to Actions Mathematics Teaching Practices
1 Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2 Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3 Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4 Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Principles to Actions Mathematics Teaching Practices ***(continued)***

5 Pose purposeful questions.

Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

6 Build procedural fluency from conceptual understanding.

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

7 Support productive struggle in learning mathematics.

Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

8 Elicit and use evidence of student thinking.

Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

We can compare the enterprise of mathematics education to a woven fabric in which two sets of threads, the *warp* and the *weft* (or *woof*), are interlaced to produce a finished cloth. In weaving, the warp threads are stretched lengthwise on the loom and form the longitudinal elements of the fabric, while the weft threads are woven transversely, over and under the warp, to yield the finished cloth. In mathematics education, the warp can be likened to the curriculum strands, such as the elements elaborated in the PSSM content standards of number and operations, algebra, geometry, measurement, and data analysis and probability that span the PK–12 curriculum and beyond, while the weft is embodied in the practices of reasoning and sense making essential for mathematical proficiency at all levels. Those process threads have also been summarized succinctly as five interwoven and interdependent strands of mathematical proficiency (*Adding It Up,* NRC 2001, p. 115):

conceptual understanding—comprehension of mathematical concepts, operations, and relations

procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

strategic competence—ability to formulate, represent, and solve mathematical problems

adaptive reasoning—capacity for logical thought, reflection, explanation, and justification

productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

Both components of the woven fabric are essential to the strength, usefulness, integrity, as well as the beauty, of the finished cloth, just as the components of both content and process

are essential for mathematical proficiency and understanding. PSSM stressed that the content and process standards are inextricably linked, and NCTM complemented the publication of PSSM by commissioning the series of 35 Navigations books and accompanying CDs. These Navigations volumes serve as guides to help educators set a course for successful implementation of the PSSM vision by providing examples of high-quality tasks that promote reasoning and sense making across the fabric of mathematics education in PK–12.

Yet despite our belief in the essential connectedness of mathematical understanding and competencies, current discussions (and assessments) of mathematics education, both among educators and the general public alike, run the risk of focusing disproportionately on the warp of the educational fabric while overlooking the vital role of the weft. With that in mind, our goal with this series of books has been to select activities from throughout the Navigations series (in each of the four grade bands and across subject areas) that embody the Mathematics Teaching Practices described in PtA and that illustrate how each selected activity can provide opportunities for students to develop the mathematical understanding and proficiency that we value as goals for all students.

We will be paying particular attention to those teaching practices throughout this book. Each activity will establish clear goals that drive the flow of the lesson. The tasks are chosen to promote reasoning and problem solving. Multiple representations of concepts are not only used but are also connected in classroom discussion. Throughout each activity, teachers are guided and encouraged to facilitate mathematical discourse, posing purposeful questions to guide discussions. For the young learners engaging with these activities, the emphasis is on developing conceptual understanding, with procedural fluency following as each student is ready.

Perhaps the most difficult practice for many teachers is supporting productive struggle. All teachers have a strong desire to help students. Often, at the first sign of struggle, the teacher steps in and offers assistance to lift students over the trouble spot, despite knowing at some deep level that there can be rich learning for students who figure out a way to climb over the trouble spots on their own or with just a tiny boost. The challenge for teachers is to offer just enough encouragement and a small hint to students so that they can continue finding solutions without taking away the thinking and satisfaction of finally solving the problem on their own.

Finally, effective teachers monitor students' thinking throughout the lesson, adjust as needed, and reflect soon afterward to note suggestions for improving the lesson for the future. Reflecting on the activities sampled in this book, and trying them with your own students, can provide opportunities for gaining deeper insight into your students' thinking and understanding. Doing this together with colleagues and examining the teaching practices that you employed and their effects with your own students can contribute even more to your professional growth.

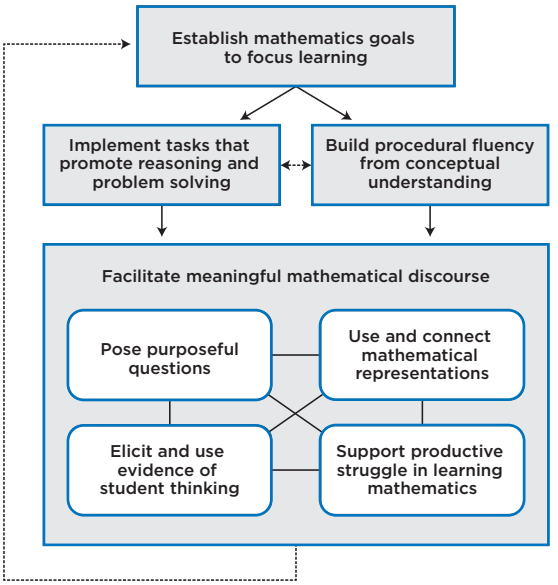
Developing Reasoning and Sense Making in Grades 3–5

Students should enter grade 3 with a foundation of mathematical reasoning, sense making, and problem solving that was laid in their early school experiences. During PK–grade 2, they were expected to develop an understanding of whole numbers as well as a level of facility with the operations of addition and subtraction that relies on an additive model in which their computing strategies usually involve counting on or counting back. Now, in grades 3–5, students will expand their thinking as the emphasis extends to multiplicative reasoning that involves making comparisons between quantities, applying geometric models to multiplication problems, reasoning algebraically and generalizing patterns, and recognizing equivalent representations of numbers and shapes. The sets of numbers themselves will also expand as students encounter fractions and decimals. PSSM summarizes the goals for grades 3–5 as recognizing that mathematics learning is both about making sense of mathematical ideas and about acquiring skills and insights to solve problems (NCTM 2000, p. 144).

To achieve those dual goals, students engage with physical objects and visual representations to model problem situations that lead to representing the same problems in a variety of ways, such as the following: (a) when they are expected to explain their thinking and give reasons to support their conclusions, (b) when they are encouraged to ask questions and develop strategies for working out answers to those questions, and (c) when they are given ample exposure to mathematical relationships and everyday applications. These are exactly the outcomes described in the *Principles to Actions* Teaching Practices (see figure Intro.3), practices that, as illustrated in figure Intro.4, are interconnected and mutually supportive.

FIG. INTRO.4

Mathematics Teaching Framework (from Berry 2019)



Students in grades 3–5 develop their reasoning and sense making by building on and extending the concepts, skills, and strategies initiated in PK–2. They do this by such means as extending number understandings to fractions and decimals, extending operations on whole numbers to multiplication and division, extending measurement to area and volume, and putting more emphasis on properties in geometric figures. They engage with physical objects and visual representations to model problems that will lead to representing the same problem symbolically. They are expected to explain their thinking and give reasons to support their conclusions. As in PK–2, students are encouraged to ask questions and develop strategies for working out answers to those questions, and they are given ample exposure to mathematical relationships in the everyday world around them.

The NCTM Navigations Series offered a rich collection of activities to support teachers and students, addressing content and practice, goals and outcomes, through implementation of the strategies reiterated in the framework shown in figure Intro.4. The Navigations activities were developed to align with the PSSM standards, which may differ slightly from CCSSM or local standards. Because of this, teachers are encouraged to try using tasks in different grade levels besides those suggested and to otherwise adapt the activities as appropriate for their own students. The Navigations Gems assembled in the current volume provide examples of teachers and students in grades 3–5 actively exploring, thinking, connecting, discussing, justifying, and advancing in their mathematical understanding and confidence.