CHAPTER 3

Making Mathematics Reasonable in School

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Benny, a sixth grader whose teacher regards him as one of her best pupils, is interviewed by a classroom visitor (Erlwanger, 1973). The interviewer probes his knowledge of decimals:

Interviewer: What would you get if you add .3 + .4?

Benny: That would be… oh, seven [07]… point oh seven [.07].

Interviewer: How do you decide where to put the point?

Benny: Because there's two points: at the front of the 4 and the front of the 3. So you have to have two numbers after the decimal, because… you know… two decimals. Now like if I had .44, .44 (i.e., .44 + .44), I have to have four numbers after the decimal (i.e., .0088). (p. 4)

He does the same thing when he multiplies decimals.

Interviewer: What about .7 × .5?

Benny: That would be .35.

Interviewer: And how do you decide on the point?

Benny: Because there's two points, one in front of each number; so you have to add both of the numbers left… 1 and 1 is 2; so there has to be two numbers left for the decimal. (p. 5)

Using these methods, Benny produces such answers as 4 + 1.6 = 2.0 and 7.48 – 7 = 7.41 and yet is unaware that his answers are wrong. Because Benny knows that numbers can be represented in different ways, if his answer is different from the teacher's answer or from the answer in the answer key, he simply assumes that his answer is just another way to write the correct one. "It's like a wild goose chase," he explains emotionally . Mathematics is anything but reasonable for Benny. His experience and the conclusions he draws about mathematics are also not uncommon. Contrast his experience in school with that of the children in the following excerpt from Deborah Ball's third-grade class (transcript of class, 19 January 1990). In this example, all names are pseudonyms, standardized across published analyses of these data and selected to be culturally similar to the children's real names. Near the end of a class, the children are concluding a discussion with their teacher:

Riba: (to the class) So Sean is saying that some even numbers, in a pattern, can be even and odd, and some can't. Four can't, because it's two groups. Six can. Eight can't. Ten can. *(Pointing at the number line above the chalkboard, she uses a pointer to mark off consecutive even numbers.)* Can't. Can. Can't. Can….

Ofala: Well, I just think that just because twenty-two is eleven groups, that doesn't mean it's an odd number. My conjecture, I think it's always true, is that if all twos are circled in a number, then it's an even number.

Sean: What conjecture?

Ball: Ofala, tell him what you're talking about when you talk about your conjecture. He's not sure what you're referring to.

Ofala: That conjecture I already…

Sean: That's not a conjecture. That's a *definition*.

These children and their classmates are struggling with concepts of evenness and oddness as a consequence of one child's claim that six could be even, and it could also be odd because you have an odd number of groups of two. Not unlike Benny, they are confronting puzzling mathematics. They had thought that they understood even and odd numbers. How could the number six be both even and odd? But rather than simply accept the student's notion, appeal to the teacher, or dismiss what might be seen as nonsense, the student's classmates are actively reasoning about elaborations of the claim and about the student's reasons for it. They have developed resources for inspecting and judging mathematical claims and arguments and for revising and developing mathematical ideas. In the process, they are solidifying their understanding of the definitions of even and odd. For them, unlike Benny, mathematics is reasonable, that is, something about which one can reason.

Mathematical Reasoning and Proof: Essential as Both End and Means

NCTM's *Principles and Standards for School Mathematics* (2000) makes a strong statement about the centrality of mathematical reasoning by including a major standard on reasoning and proof for all grades (p. 56):

Reasoning and Proof

Instructional programs pre-kindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Some may regard a standard on reasoning and proof as a nice, but esoteric, embellishment to the main curricular goals in mathematics. Some might even consider the standard expendable. Quite the contrary. Mathematical reasoning is no less than a basic skill. Why do we make this claim?

First, the notion of *mathematical understanding* is meaningless without a serious emphasis on reasoning. What, after all, would mathematical "understanding" mean if it were not founded on mathematical reasoning? Take, for example, understanding multiplication of decimals. Benny, like many adults, counted decimal places to determine the number of places in an answer, but he had no idea what it meant. Not understanding the reasons underlying the procedure meant that he made senseless mistakes. Why does multiplying .7 × .5 produce an answer with two decimal places—.35—whereas adding the same numbers, .7 + .5, yields an answer with just one decimal place—1.2? Unjustified knowledge is unreasoned and, hence, easily becomes unreasonable.

A second reason for claiming that mathematical reasoning is a basic skill is that such reasoning is fundamental to using mathematics. Knowing particular mathematical ideas and procedures as mere fact or routine is insufficient for using those ideas flexibly in diverse cases. First graders who have learned to use the equals sign to signal the result of an operation on two numbers are baffled when presented with 8 = __ + 5. They think that the 8 does not "tell" them to "do" anything, and so they often say there is no number to write in the blank. Their thinking results from using the equals sign in number sentences without reasoning about the concept of equality (Carpenter & Franke, 2001; Falkner, Levi, & Carpenter, 1999). Or consider people who know that the probability of two independent events can be calculated by multiplying the probability of the first event by the probability of the second. For flips of two fair coins, they thus may correctly calculate the probability of two heads, or of two tails, as 1/4 (= 1/2 × 1/2) and yet say that the probability of one head and one tail is 1/3, seeing this case as one of three possibilities (two heads, two tails, or mixed), failing to take account of the fact that the mixed case has two ways of occurring.

Third, mathematical reasoning is fundamental to reconstructing faded knowledge when a demand for it arises. A person who once knew how to divide fractions but has forgotten the algorithm can rebuild a reasonable procedure if he can use the meaning of division and of fractions to reason about dividing one fraction by another. Analyzing the basic meaning of division allows him to see dividing fractions as not essentially different from dividing any whole number by another: Dividing 4/5 by 2/3 is conceptually like dividing 6 by 3, even though the numbers involved in the former example have more complex descriptions. Or, consider a person who learned—but has since forgotten—the formulas to calculate probabilities of independent and of mutually exclusive events. When asked what the probability is of tossing two coins and getting two the same (i.e., two heads or two tails), she may be unsure what operations to perform. Should she multiply the probabilities of each outcome? Add them? What are the outcomes? If she can reason, however, about the logic of a probabilistic situation to analyze whether the outcomes are independent or mutually exclusive, she will likely be able to figure out that she must first multiply the probability of flipping one head times the probability of flipping a second and must do the same for two tails; the successive tosses are independent. Each of these probabilities is 1/2 ×

1/2, or 1/4. But to calculate the probability of getting two coins the same, she will realize that she then must add the chance of tossing two heads to that of tossing two tails; the two-heads and two-tails outcomes are mutually exclusive. Being able to reason mathematically allows her to recapture a way to work successfully with the problem.

Our point is that mathematical reasoning is as fundamental to knowing and using mathematics as comprehension of text is to reading. Readers who can only decode words can hardly be said to know how to read. Reading competently depends on being able to understand the structures of texts and nuances of language; to interpret authors' ideas; and to visualize, evaluate, and infer meanings. Likewise, merely being able to operate mathematically does not assure being able to do and use mathematics in useful ways. Procedural operations are fundamental to reasonable mathematical activity but are by themselves little more than the analog of reciting text based on the phonetic and structural analysis of words. Making mathematics reasonable means making it reasoned and, therefore, known in useful and usable ways.

This chapter examines what is entailed by mathematical reasoning, and what this looks like as it develops in students in classrooms. Drawing from work on elementary school teaching and learning, and on the practices of mathematics, we discuss a framework for what we call the reasoning of justification—or proof. Next, we turn to what it might take to make mathematics reasonable—that is, what can support the reasoning about mathematics in school.

What Do We Mean by Mathematical Reasoning?

Making mathematics reasonable is more than individual sense making. Making sense refers to making mathematical ideas sensible, or perceptible, and allows for understanding based only on personal conviction. Reasoning, as we use it, comprises a set of practices and norms that are collective, not merely individual or idiosyncratic, and rooted in the discipline. Making mathematics reasonable entails making it subject to, and the result of, such reasoning. That an idea makes sense to me is not the same as reasoning toward understandings that are shared by others with whom I discuss and critically examine that idea toward a shared conviction.

The desire to know and to understand has led people to develop disciplined means of reasoning, of exploring and verifying, of hypothesizing and justifying, in many arenas of human activity. Historians reason about evidence from the past, physicians reason about patients' symptoms, chefs reason about composing ingredients under particular conditions, and pilots reason about instrument readings. In none of these examples do individuals make sense in whatever ways they choose. Instead, in each of these arenas, people have developed methods of reliable thinking that afford inspection, analysis, judgment, and conclusions. These methods of reasoning are the particular means of constructing and evaluating knowledge in a domain.

Much has been written in recent years about constructivist theories of learning and their implications for instruction. Indeed, *constructivism* has arguably been one of the most influential—and most multiply interpreted—ideas in mathematics education. Our research analyzes classroom mathematics learning and teaching in light of ideas about constructing knowledge that are rooted in mathematics as a discipline. Lampert (1990, 1992) has been exploring similar resonances between the practices of knowing mathematics in school and those of knowing mathematics in the discipline. When students are at work in a mathematics class, for example, we see them as constructing mathematical knowledge. Looking at the development of students' knowledge in this way highlights the fundamentally mathematical nature of their—and hence, their teachers'—work. The ways in which students seek to justify claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge have powerful resonances with mathematicians' work. As students explore problems, make and inspect claims, and seek to prove their validity, even as young children, they engage in substantial forms of mathematical reasoning and make use of mathematical resources. Smith (1999) provides a vivid portrait of this through his close analyses of four nine-year-olds' individual mathematical reasoning. His account focuses on their use of language and representations as they draw together and use mathematical resources to solve problems. This mathematical perspective makes visible some fundamental aspects of mathematics teaching and learning that are hidden when instruction is viewed from a purely cognitive or sociocultural perspective. In particular, this analysis allows for and explores a subject-specific view of learning.

This work finds company in recent advances in other fields (e.g., Wilson, 2001; Wineburg, 1996). Shari Levine Rose (1999), in a study of fourth graders' learning of history in her own classroom, distinguishes between what she was able to see in her students' work when she viewed it from the perspective of generic theories of learning and when she later began to view their work using a lens of historical reasoning. Initially, she explains, she was "influenced by constructivist theories of learning,… [believing] that children drew upon knowledge, values, and beliefs in actively *making sense* of new information." She argues, however, that the generic perspective did not help her see how the children were constructing meaning of historical events. But with the historical lens, she was struck by the "historical nature" of children's sense making. They repeatedly sought understanding through constructing stories, much as historians fashion narratives, embedding meaning and interpretation in context. Rose writes about how the historical perspectives that she brought to bear in hearing and interpreting her students made visible how the children came to know the past and constructed meaning of historical events in ways that were much more rooted in the nature of historical reasoning.

Viewed from the perspective of the practicing mathematician, reasoning is one of the principal instruments for developing mathematical understanding and for constructing new mathematical knowledge. Mathematical reasoning can serve as an instrument of inquiry in discovering and exploring new ideas, a process that we call the *reasoning of inquiry*. Mathematical reasoning also functions centrally in justifying or proving mathematical claims, a process that we call the *reasoning of justification*, the focus of this chapter.

Historically, this sort of mathematical reasoning has primarily been found in the high school geometry curriculum in the context of constructing two-column proofs, sometimes treated more as ritual than as an instrument of sense making. What might be entailed by a broader conception and practice of mathematical reasoning in school, as called for by the NCTM's standard on reasoning and proof? In this chapter, we offer and illustrate a conceptual framework for learning and teaching mathematical reasoning.

Teaching Commitments to Mathematics, Students, and Community

Our study of mathematical reasoning is framed by a conception of teaching founded on three specific commitments—to the integrity of the discipline, to taking individual students' thinking seriously, and to the collective as an intellectual community. These commitments orient, but do not determine, practice. First is a commitment to draw from mathematics as a discipline in intellectually sound and honest ways (Ball, 1993; Ball & Bass, 2000a, 2000b; Bruner 1960; Lampert, 1990, 1992, 2001). So in the case of this analysis, we ask, What mathematical habits and dispositions are crucial to doing and learning mathematics? What is the basis of mathematical reasoning? How is mathematical knowledge constructed in the discipline? How do mathematicians interact as a community over knowledge claims? These and other questions have guided our study of making mathematics reasonable, by and for children, in school.

In counterpoint, we assume that teaching demands a sensitivity and responsiveness to students' ideas, interests, lives, and trajectories. Teachers strive to hear their students, to work with them as they investigate and interpret their worlds. Respecting students means attending to who they are and what they bring as well as helping them grow beyond where they are now or where they think they can go. But attending to individual students' interests and proclivities is not enough. In school, teachers must be concerned with "covering" the mandated curriculum so that each student is prepared for the next grade and for the standardized tests used to chart his or her progress. And while many seek to redefine what "*covering the curriculum*" might mean (Lampert 1992, 2001), caring for students means also being responsible to current definitions of progress and learning (Delpit, 1985). Understanding teaching as centrally guided by students' ideas and thinking has led our work on mathematical reasoning to close examination of how students come to hold and believe in mathematical knowledge (Ball & Bass, 2000b).

Finally, the teaching in which we are interested aims to create a classroom community in which differences are valued; in which students learn to care about and respect one another; and in which commitments to a just, democratic, and rational society are embodied and learned (Dewey, 1916; Schwab, 1976). Care and respect for others includes listening to, hearing, and being able to represent others' ideas, even those with which one disagrees. Respect also means taking others' ideas seriously, appraising them critically, and evaluating their validity. In this work, we consider mathematical reasoning as producing more than individual conviction: as generating public knowledge that is usable by the collective.

A Framework for Mathematical Reasoning

The reasoning of justification in mathematics rests on two foundations. One foundation is a body of public knowledge on which to stand as a point of departure and that defines the granularity of acceptable mathematical reasoning within a given context or community. The second foundation of mathematical reasoning is language—symbols, terms, and other representations and their definitions—and rules of logic and syntax for their meaningful use in formulating claims and the networks of relationships used to justify them.

We first discuss the *base of public knowledge*, the term we use to refer to the knowledge on which claims and arguments are based within some context. Yackel and Cobb (1996) use the label *taken-as-shared* to refer to the meanings, norms, and ideas that are negotiated and used as common within a classroom. Edwards and Mercer (1987; 1989) also write about the development of common knowledge in teaching and learning and focus particularly on the discourse patterns whereby teachers establish such common knowledge. With Yackel and Cobb, we are interested in normative aspects of mathematics discussions specific to students' mathematical activity, such as agreements about what counts as mathematically different solutions or what counts as an acceptable mathematical explanation. And like Edwards and Mercer, we are interested in the development of common knowledge. In our framework, we focus further on the specific mathematical knowledge that is available for public use by a particular community in constructing mathematical claims and in seeking to justify those claims to others. This knowledge is of particular ideas, accepted procedures, defined terms, and methods of mathematical investigation and verification. This knowledge is already assumed or developed—part of the record of the children's prior experience or the class's past work. By identifying it as public, we seek to avoid implying that each member of the community knows it individually in the same way; ascertaining how mathematical knowledge is known individually is an empirical question beyond the scope of our analysis. We mean, rather, to call attention to the knowledge that can comfortably be assumed and used publicly without additional explanation. We contrast such knowledge with ideas or procedures that are not shared and must therefore be established before they can be used to justify claims in the collective discourse of a community.

This base of public knowledge is defined relative to a particular community of reasoners. For professional mathematicians, the base of public knowledge might consist of an axiom system for some mathematical structure (e.g., Euclidean geometry or group theory), simply admitted as given, plus a body of previously developed and publicly accepted mathematical knowledge derived from those axioms. We argue that this idea of a base of public knowledge is useful in understanding the work of a class of elementary school students as well, where this base of public knowledge comprises the expanding set of publicly established ideas and shared knowledge that can be used by the class in explanation or justification.

Take a rudimentary example. Early in September one day, in Ball's third-grade class, the children were working on this problem: "Write number sentences for 10." In the early stages of their work, most students were writing simple equations: 4 + 6 = 10, 3 + 7 = 10, and 8 + 2 = 10. Pressing them gently into more complex solutions, Ball gave the children the following challenge (transcript of class, 18 September 1989):

> I wonder if someone can think of a number sentence that uses more than two numbers here. Just so we have a bunch of ideas of how we could do this. Who can make a number sentence that equals 10 but has more than two numbers adding up to 10?

Quickly, Tembe began, "One plus one plus one plus one—…"

Ball wrote on the board as Tembe rattled off numbers: "One plus one plus one plus one plus one plus one plus one. Plus three."

"Why does that equal ten?" asked Ball, who then called on Harooun to justify Tembe's claim. Harooun, repeating, said, "That's just one plus one plus one plus one.…" Ball asked the students, "How do we know that that equals ten?" Riba, eager to respond, explained, "Because one plus one plus one plus one plus one plus one plus one and plus three equals ten." Ball, still not satisfied, said, "You're just sort of reading it. How could you prove it to somebody who wasn't sure?"

Riba: Because I counted it.

Ball: What did you count? What did you find out?

Riba: There's one and the next one is two and the next one is three, next one is four, next one is five, next one is six, next one is seven, next one is—seven, and then three more, eight, nine, ten."

In this simple segment, Tembe's offered solution, the first that included more than two terms, was not something automatically presumed to be within the common knowledge of the class, and Ball asked for justification. If this exchange had taken place in a fifth-grade class, one would likely presume differently. A young child's initial sense of addition comes from counting, which is adding one at a time. Adding many terms at once or adding two numbers larger than one in a single step are higher-order operations, not only for young children but also mathematically. When Riba is called on to explain why the string of terms equals 10, she first just recites the equation. When Ball presses her, "But how could you prove it to somebody who wasn't sure?" Riba replies that she "counted," and Ball then encourages her further to make this counting public. In response, Riba expands her explanation:

> There's one and the next one is two and the next one is three, next one is four, next one is five, next one is six, next one is seven, next one is—seven, and then three more, eight, nine, ten.

Perhaps it is through this counting that Riba first proves to herself that Tembe's formulation was valid. Her teacher is requiring her to make this reasoning public to persuade the class as well. Her teacher then publicly validates Riba's work, underlining early in the year a standard for explanation and justification that is more than simple restating of the assertion:

> Do you see the difference in Riba's second explanation? Did you see how she really showed us how it equals ten? The first time you just read it. And the second time you explained it. That was really nice.

Here the teacher does more than praise Riba. She points explicitly to Riba's work—the mathematical explanation she has constructed—and comments on the difference between repeating a statement and explaining it.

A process of reasoning typically consists of a sequence of steps, each of which has the form of justifying one claim by invocation of another, to which the first claim is logically reduced. This process, which merely transforms one claim into another, is not a vicious circle, because the reduced claim is typically of a more elementary or accessible nature and, in a finite number of such steps, one arrives at a claim that requires no further warrant because of being part of the base of publicly shared knowledge and therefore universally persuasive within a particular community of reasoners. Thus, the base of public knowledge both constrains and enables the stepping-stones of an argument. In that sense, publicly shared knowledge defines the granularity of acceptable mathematical reasoning within a given context. In the example above, the addition represented in the equation $1 + 1 + 1 + 1 + 1 + 1 + 1 + 3 = 10$ was not, at that moment, presumed to be part of the base of established knowledge of that class, and so Riba was pressed to reduce the claim to an iterated counting, keeping track of the total as she counted. At that point, Tembe's assertion was sufficiently reduced to a level that relied on knowledge common to the class—counting by ones—and required no further justification.

The crucial issue is how to justify a mathematical claim. One way to justify a claim is to state the claim and to undergird its truth by the sheer force of authority. Students often receive mathematical knowledge in school that is justified by little else than the textbook's or the teacher's assertion. By default, the book has epistemic authority: Teachers explain assignments to pupils by saying, "This is what they want you to do here," and the right answers are found in the answer key. According to Davis (1967), learning to "play by the rules" often involves a "suspension of sense-making" in school mathematics. But that route is the antithesis of warranting claims through a process of mathematical reasoning.

The base of public knowledge consists of knowledge of certain facts and concepts; of the meanings of mathematical terms and expressions; and of procedures and resources for calculation, for problem solving, and even for reasoning. The base of public knowledge is always present, in both latent and active forms, although it may be tacit and only implicit in the discourse of the community, whether mathematicians or third graders.

Whether a particular piece of knowledge is in fact commonly shared is an empirical question, one that a teacher must often assess. Did everyone in the class understand and agree that Riba's elaborated explanation satisfactorily proved Tembe's claim? And, further, how many children were not already adequately convinced by his initial statement? These questions are difficult to resolve fully; they are also not completely within the teacher's or the students' view as a class discussion proceeds. Still, reasoning within a community depends on the presumption of common knowledge and shared established methods. This presumption is most often represented by the use of knowledge already used and established publicly. Arguments that do not build on publicly shared knowledge are unlikely to produce grounded conviction in others. At the same time, the process of reasoning can in fact help build and extend a group's common knowledge. As claims are proved, and ideas developed, the claims may become part of the legacy of public knowledge on which subsequent claims may depend and build.

In our analysis, mathematical language is the foundation of mathematical reasoning that is complementary to the base of publicly shared knowledge. Language is used here expansively, comprising the entire linguistic infrastructure that supports mathematical communication with its requirements for precision, clarity, and economy of expression. Language is essential for mathematical reasoning and for communicating about mathematical ideas, claims, explanations, and proofs. Language is a medium in which mathematics is enacted, used, and created.

In our framework, language includes the nature and role of definitions in mathematics; the nature of, and rules for, manipulating symbolic notation; and the conceptual compression afforded by timely use of such notation. Definitions and terms play a crucial role: Not simply delivered names to be memorized, definitions and terms originate in, and emerge from, new ideas and concepts and develop through active investigation and reflection. Definitions and terms facilitate reasoning about those new ideas by naming and specification. Decisions about what to name, when to name it, and how to specify that which is being named are important components of mathematical sensibility and discrimination central to the construction of mathematical knowledge. Using symbolic and other representations to encode ideas, as well as decoding ideas represented in symbolic or other forms, are essential communicative tools for the construction of mathematical knowledge. Precise language is also needed to articulate the correspondences between equivalent representations of the same mathematical entity or concept. Notation can be used to compress ideas into forms that, when done skillfully, can reduce computation and manipulation to manageable proportions; how and when to do this is an important skill of mathematical representation useful in reasoning.

Mathematical language is central to constructing mathematical knowledge; it provides resources with which claims are developed, made, and justified. Lampert writes,

> Mathematical discourse is about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions. These definitions and assumptions are not given, but are negotiated in the process of figuring out what is true. (1990, p. 42)

Some disagreements stem from divergent or unreconciled uses of terminology, whereas others are rooted in substantive and conflicting mathematical claims (Crumbaugh, 1998; Lampert, 1998). The ability to distinguish between issues of terminology and issues of mathematical claims requires sensitivity to the nature and role of language in mathematics. We return to this aspect of language in the examples analyzed in the next section of this chapter.

Developing Mathematical Reasoning in a Third-Grade Class

In this section, we take a close look at two classroom episodes in which elementary school students are learning to engage in mathematical reasoning. These classroom episodes are based on Ball's third-grade class records for the school year from 1989 to 1990 (see Lampert & Ball, 1998). We have chosen two excerpts of instruction from the same classroom, one less than 5 months after the first. In fact, the first episode is from the first day of mathematics class in September, and the second, from a day late in January. Our purpose in choosing these two segments is to examine the evolution of the reasoning of justification. What were the students and teacher doing as they sought to reason about mathematics in September, and in what ways were those approaches the same or different in January, less than 5 months later? What do the students seem to be learning? Comparing these two points in time helps establish that mathematical reasoning is something that students can learn to do and, hence, that teachers can teach. Our analysis leads to the final section of the chapter, in which we turn to considering what approaches teachers might adopt to help make mathematics reasonable in school.

In the first episode, early in September, the students are working on an arithmetic problem that has multiple, but finitely many—six—solutions. The task is to find all the solutions and then to show that all solutions have been found. In the later episode, in January, students are working on a conjecture about addition of odd numbers: that an odd number plus an odd number equals an even number. The task is to determine whether this conjecture is true. In both instances, the mathematical work for students calls for justification: How do you know that you have found all the answers? Can you prove that this statement is always true? What the students do in January entails more complex reasoning in that it concerns a claim about all the infinitely many pairs of odd numbers; this work shows substantial development of reasoning skills and sensibility since September.

September: How Do You Know That You Have Them All?

In the first regular mathematics period of the school year, the students are working on the following problem:

> I have pennies, nickels, and dimes in my pocket. Suppose I pull out two coins. How much money might I have?

In setting up the task, Ball reads the problem with the students and they try an example together. She asks the class, "I'm going to pull out two coins. How much money could I pull out? Like this, like I'm not even looking and I'm going to reach in and pull out one coin and reach in and pull out another. How much money might I have?" "Ten cents," proposes Lucy.

Ball asks for more explanation: "How could I pull out 10 cents?"

Lucy: Two 5s.

Ball: What do other people think about that? If I pulled out two nickels, would I have 10 cents?

Students: Yeah.

Ball: How do you know that? How do you know that that would be 10 cents? Ofala?

Ofala: Because five plus five is ten.

The students work on the task for about 20 minutes while Ball circulates, noting what different students are doing and asking and answering an occasional question. A few times during class, Ball calls the group together to share bits of work or to discuss how the work is proceeding. We zoom in on the class discussion of solutions to the problem because it affords a close look at the group's early efforts to reason mathematically, as individuals and as a group.

About 20 minutes before the end of class, Ball brings them back together. She elicits solutions to the problem from different students. The class discusses and verifies each proposed solution. Ball records the answers on the board in a list:

15¢
20¢
6¢
11¢
2¢
10¢

Ball then reads the solutions off the board:

Ball: We have 15 cents, 20 cents, 6 cents, 11 cents, 2 cents, and 10 cents. Any more? Look at your lists, and see if you have anything that we didn't put on the board. (pause) No, Ofala? You don't have anything else in your notebook? (pause) Jeannie, do you have anything else in your notebook? Riba, do you? Does anybody? How many different possible answers did we find for this problem? How many answers did we come up with here?

Latifa: Six.

Latifa is right. They have come up with six solutions for the problem. However, Ball does not affirm Latifa's answer. Instead, she presses a bit, pushing for justification.

Ball: Six answers. How do we know that we have them all, though? How do we know there isn't a seventh one? Or an eighth one that we didn't find yet?

Ofala, noting a constraint in the given statement of the problem, says she thinks that six is all they can make because they cannot use quarters. Mei comments, "I think we have them all 'cause we had lots of them already and all the people had six."

Lisa speaks next, but her voice is almost inaudible. Ball interrupts to tell her to "talk so that other people can hear you," and directs the rest of the class to listen to what Lisa is saying. Lisa declares, more audibly, "We can only pick up two coins, and if we pick up seven, then we would be picking up three or four." Ball, seeking to understand Lisa's argument, asks for clarification, "So to get a seventh answer, we'd have to pick up three coins?" Lisa assents, and Ball asks her how she knows. Lisa says something about a nickel and a penny, "and if we add another penny, it will be seven and three coins." She seems to talking about a solution for seven cents, but, because it requires three coins, she may be trying to show that the production of a seventh solution—Ball's original question—would require more than two coins.

At this juncture, Sheena raises her hand. She says that she has been working on the question, and she keeps trying to find more solutions but keeps "getting the same answers." Ball repeats her idea to the class: "If you keep picking them up, you'll get the same answers? How many people solved this problem by picking up coins until they got the same ones again?… How many people reached into the box or onto their pile and kept picking them up until they started to repeat?" Many students raise their hands. "What's another way to do the problem?" she asks, looking around.

Mei offers another method: "You first think of what you can make from, you can make out of nickels, dimes, and pennies; and then you take, and then you write them down and you think about it some more until you're—(pause) then you'll get them all."

The children seem reasonably satisfied that they have found all the solutions. They believe they have found them all because they cannot find any more. This argument is empirical or inductive, not deductive. It is the kind of empirical reasoning that can increase confidence in a scientific hypothesis, but it is not a mathematical proof. Still, the children find it persuasive, and they do not seem to conceive any method by which one could confirm more definitively that they have all solutions. Ball ends class by urging them to "think very hard before math tomorrow" to see whether they can find any more answers to this problem. By not affirming the completeness of the solution—with six answers—she seeks to maintain some need to show more firmly that they found all the answers, something more than trying many solutions or being personally convinced.

At this early stage, the children have not witnessed or been formally introduced to the notion of mathematical proof. But we see here that the imperative to mathematically justify is being seeded in their work, before they even know what a proof is or looks like. The challenges to justify their own conjectures serve to motivate them to construct, through reflection and analysis, some of the intellectual architecture of mathematical reasoning.

January: Can We Prove That Betsy's Conjecture Is Always True?

We revisit the class almost five months later. The students have been working on problems that involve patterns with sums. For example, they have worked on such problems as the following:

> Erasers cost 2¢, and pencils cost 7¢. How many different combinations of erasers and pencils can you buy if you want to spend exactly 30¢?

The numbers in this problem have been chosen deliberately so that the children might notice that an even number of pencils must be bought if the total has to be 30—an even number—because 7 is an odd number. Indeed, solving such problems has generated a series of conjectures about even and odd numbers:

Even + even = even
Even + odd = odd
Odd + odd = even

Ball has helped the children formulate these conjectures and understand them and has challenged the children to see whether they could prove that these statements are always true. The students have generated long lists of examples in their notebooks, seeking to confirm the conjectures or to find examples that do not work and therefore would show that a conjecture is not (always) true. They are working in small groups on a particular conjecture.

In the course of this work, something new happens. For the first time, the thought occurs to some of the children that to prove such a conjecture is something they have not done before and that doing so presents challenges they have not previously appreciated. Jeannie and Sheena, who are working on the conjecture that an odd number plus an odd number equals an even number, report on their work:

Jeannie: Me and Sheena were working together, but we didn't find one that didn't work. We were trying to prove that… you can't prove that Betsy's conjecture (*odd + odd = even*) always works. Because, um, there's, um, like numbers go on and on forever, and that means odd numbers and even numbers go on forever, so you couldn't prove that *all* of them work.

Ofala protests. She declares, looking closely into her notebook, that she has tried "almost eighteen" of them and even some special cases, and they have all "worked," so she thinks that "it can always work."

Mei then offers a very different kind of objection. "I think it could always work because with those conjectures [motioning to several previously discussed and widely agreed-on conjectures posted above the chalkboard], we haven't even tried them with all the numbers there is, so why do you say that those work? We haven't tried those with all the numbers that there ever could be."

Ball asks whether this statement means that she is disagreeing with Jeannie or agreeing with her. "I disagree," replies Mei, emphatically. Ball asks her to clarify what she is saying, and she repeats and amplifies her point that the class has already agreed to accept other conjectures even though they were not able to check them with every number. But Jeannie and Sheena are not to be deterred. Jeannie says that she never said that those other conjectures were true all the time. "Then why didn't you disagree when everyone agreed with those conjectures?" presses Mei. Sheena explains that they had never thought about any of this before, and now they see that a problem exists because numbers go on forever. Class ends with this issue unresolved, and Ball tells the students to think some more about what Jeannie and Sheena are claiming: "What do other people think? Do you think we can't prove that it's always true, or do you think that we can prove that it's always true?"

A few days later, the class takes a big step. Betsy, together with a few classmates, presents a proof of the conjecture, which they first illustrate with 7 + 7:

> What we figured out how it's always true is that we would have seven dots, or lines, plus seven lines (draws fourteen hash marks on the board, seven at a time)

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