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What Is Mathematical Modeling?

Mathematical models are representations of the world that people use to understand phenomena and make decisions. *Modelers* create these representations. *Modeling* is the process of creating the representations. When middle and high school students use their mathematical tools and ideas to make a decision, they are modeling. In this book, we focus on what teachers and students do while engaged in modeling in middle and high school classrooms, and how their work reflects four big ideas about mathematical modeling that are based in the disciplines of mathematics and statistics.[1] This book will guide you to becoming a teacher of modeling.

MODELING AND PROBLEM SOLVING

You already have habits of thinking about problems that form the foundation of modeling—all human beings do! Think of a time you were really invested in solving a problem: not necessarily a mathematics problem, but a problem you encountered in your day-to-day life. Maybe you decided to overhaul the layout of your classroom or incorporate new technology in your teaching. Maybe you tackled a problem at home, like training a dog or fixing a dripping faucet. It does not matter what the problem was, just that you were really engaged in solving it. What did your process look like?

When people solve problems, they often start by getting clear on what it is they are trying to accomplish. Maybe you asked yourself a question to guide your next steps: "How can I get this dog to heel?" or "What platform will help me communicate effectively with my students?" Once you had a sense of what you were trying to accomplish, you probably did a little research, especially if the problem was a new one for you. You watched videos online or checked out books from the library. Perhaps you contacted friends with more experience and asked for advice.

Armed with a little knowledge and some newfound courage, you dove into the task. You tried something, stepped back and thought about how well it was working, then stepped back in and tinkered with it a little bit.

[1] Throughout this book, we use the term *mathematics* to include statistics, because in secondary classrooms statistics content is taught as part of the mathematics curriculum.

Maybe you had to go back and do some more research or even had to hit the "reset" button more than once. After perseverance, your efforts ended in success and eventually you found yourself returning from a brisk walk with the dog to get a glass of water from a perfectly functioning faucet!

If you have ever engaged in a process like the one described here, you already know something about the process of mathematical modeling, because modeling has a lot in common with all problem-solving efforts. Applied mathematician Henry Pollak describes modeling this way:

> Step 1. You see something that is worth doing.
>
> Step 2. You do it.
>
> Step 3. You check to see if you have actually done it (Pollak 2016, p. viii).

Mathematical modeling follows a pattern we are all used to, one that helps us use our skills, resources, and creativity in situations we *actually care about*—in the words of Pollak, to do things that are worth doing!

How is mathematical modeling different from other problem-solving efforts? Modeling is different because of the kinds of questions we ask and the tools we pick up and use. When we ask, "How can I fix this leaky faucet?", our tool kit includes physical tools. We use wrenches and washers, pliers and putty. If we ask, "How can I get this dog to do what I say?", the term *tool kit* is more like a metaphor. The tools we need include a repertoire of behavioral techniques, a bag of treats, and quite a bit of patience.

In mathematical modeling, we ask questions that point us toward mathematics. Our tool kit includes strategies, representations, skills, ideas, and techniques that come from the discipline of mathematics. There are plenty of problems out there that require mathematical tools, and our classrooms are filled with students who want to know when and how they are going to use what they are learning. Mathematical modeling gives teachers a satisfying response to the age-old mathematics class question "When are we ever going to use this?" That response is "How about today?"

FOUR BIG IDEAS

We have found it useful to ground the classroom practice of modeling in four big ideas about mathematical modeling. These big ideas describe modeling whether it is being done in a classroom or a lab, whether the modeler is seven or seventy:

1. Modeling begins and ends outside the mathematical world.
2. Modeling deals with situations that are open and complex.
3. Modelers exercise judgment when investigating problems. These judgments stem from a set of values that may or may not be fully articulated, but are always present.
4. Modelers decide when a solution is good enough.

These big ideas each illuminate a fundamental aspect of mathematical modeling that makes it a powerful practice to incorporate in classrooms. Together, they capture the nature of modeling and reveal the exciting, compelling reasons to incorporate mathematical modeling in classroom mathematics instruction.

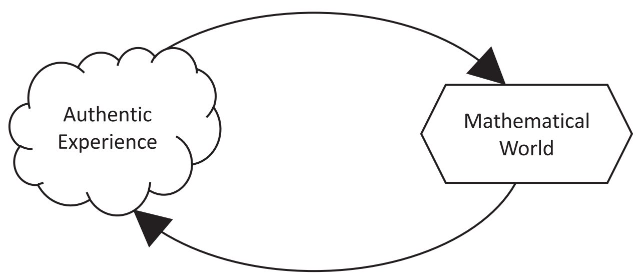
Big Idea 1: Modeling Begins and Ends outside the Mathematical World

Modeling questions are rooted in authentic, daily experiences. When we say *mathematical world,* we mean the intellectual space where mathematics is the tool that is used to solve problems. At first glance, situations that motivate modeling may not seem all that mathematical, but mathematical tools can help us understand a range of issues. Models help inform action in areas that range from combating disease to allocating resources, and from understanding human behavior to predicting the weather. In the secondary grades, students may not create models for something as complex as predicting the path of a hurricane, but they can use mathematical tools to answer questions about recommending the best passing period time between classes, deciding a fair way for a coach to select members of a school team, or predicting what size population a local water source will support. The point is that students do not have to wait until their mathematical kit is full of complex and highly sophisticated tools. They can use their existing mathematical knowledge to understand and address issues that arise in their own day-to-day experiences.

You may be thinking, "Well, I link students' experiences with mathematics all the time!" Perhaps you have launched a lesson on linear equations with a story about getting a subscription that has a base fee with additional fees for extra items, such as a family cell phone plan. Or maybe students learned about similarity and calculating triangle side lengths and then applied their learning to contexts that involved measuring heights of buildings. If you are doing these kinds of activities in your classroom, you have already started the practice of modeling. When teachers launch a task with an interesting situation, they may be connecting students' experiences with mathematics (moving from the cloud to the hexagon, as depicted in the top arrow of figure 1.1). When teachers ask students to apply what they have learned to an authentic situation, they are connecting mathematics to students' experiences (moving from the hexagon to the cloud, as depicted in the lower arrow of figure 1.1).

FIG. 1.1.

Modeling begins and ends outside the mathematical world.



Modeling involves moving in both directions. It begins with an authentic experience, and that experience gets translated into a question that points toward mathematics. The modeling process does not end until modelers link the mathematical work back to the original problem context.

Big Idea 2: Modeling Deals with Situations That Are Open and Complex

You may have noticed that in figure 1.1 we placed the words *Authentic Experience* in a cloud and *Mathematical World* in a hexagon. Problems that grow out of authentic experiences are often ill-defined. They do not have clear boundaries and rarely come with a prescribed solution strategy. Mathematics problems, even applied mathematics problems, are different. They tend to be tidier, with clear definitions and terms. This is not to say that mathematics problems and applied mathematics problems cannot be challenging—they often are! It is to say that moving from situations derived from day-to-day experiences involves making choices that remove some of the messiness from the problems human beings encounter in our daily lives.

In mathematics classrooms, teachers usually want to make problems clear. In fact, teachers work hard to remove any ambiguity from tasks given to students. Many people find mathematics satisfying for this reason. The problems modelers encounter do not really look like most textbook tasks. Modeling tasks are open, complex, and ambiguous. Modelers have to understand what the problem is, decide how to go about addressing it, and see if the solution meets the demands of the problem. They weigh approaches, make choices, and act creatively. Because modeling reflects the uncertainty of experiences, it is different from other kinds of mathematical problem-solving tasks.

Big Idea 3: Modelers Exercise Judgment

Because modeling involves making choices, it also involves exercising judgment. Modelers have to ask questions that focus on small slices of large problems, choose sound mathematical approaches, and decide what to include in, or exclude from, the model. They choose the mathematical approaches they believe are well suited to the problem, and not all modelers make the same decisions. In classrooms, students can see how other modelers understand the problem, discuss varied strategies, and allow the judgments others make to inform their work.

Judgments stem from values. A teacher could ask students to decide which meals are the best for a school fundraiser, but the word *best* raises the question, "Best for whom?" If students value having meals lots of students will enjoy, they might base their model on meals that are the most popular. If they value meals that are nutritious, students might decide to investigate the nutritional value of the meals. Those who think it is important to have a low-cost meal for the fundraiser might calculate the cost of the ingredients and supplies that they will need for the meal. When teachers make the values behind each approach visible, students have opportunities to understand how perspectives and experiences influence the ways problems are solved in mathematics.

All questions—including mathematical questions—have values embedded in them. It is easy to get so used to these embedded values that they are hard to recognize; they lie beneath the surface, unacknowledged. One element of modeling that we find particularly advantageous is that it draws these values to the surface, as the modeler has to acknowledge what is important—that is, what is valued—in the situation being modeled. Moving from issues that have consequences in the day-to-day experiences of people to mathematical problem statements requires modelers to make decisions about what is important, and decisions about what is important are tied to what people value. Because making decisions about what is important involves seeing situations from others' perspectives, modelers use empathy.

Big Idea 4: Modelers Decide When a Solution Is Good Enough

In mathematics classrooms, teachers are accustomed to being "done" with a problem when students find the correct solution. Sometimes teachers ask students to check their work or change their approach, but that is usually because of some sort of error along the way. Although modelers need to use sound mathematics, modeling problems do not have a single correct solution. So, how do modelers know when their solutions are good or even when they are done?

Solutions to modeling problems can be evaluated in a variety of ways. Modelers might ask whether the solution is enlightening, novel, or just plain useful. If the answer to any of these questions is no, the model can be revised. When students model in classrooms, they build skills that include and go beyond mathematical computation and application; when they reflect on their solutions, they ask questions like the following:

- Is the solution useful?
- Whose perspectives does the model consider?
- What do I notice about our different solutions?

These are challenging questions that force modelers to examine their mathematical decisions, reflect on complex situations, and consider their own and others' perspectives. They are also empowering questions, in that their answers rely on modelers' evaluation of their mathematical work and its relationship to the authentic experiences the model is intended to address.

MEETING TEACHERS OF MATHEMATICAL MODELING

In the chapters to come, you will meet seven middle school and high school teachers as they engage their students in mathematical modeling: Isaac, Julia, Ruby, Darius, Destiny, Lule, and Kiran. The classrooms described in each of these vignettes are based on classrooms in which we have seen modeling enacted or have taught modeling ourselves, even though the names and some of the details have been changed. As you read these vignettes, you will learn about the teachers' motivations for incorporating modeling in their classes and the choices they make as teachers of modeling. Each of them balances their goals for teaching students to embrace a modeling perspective with their goals for engaging students in robust study of mathematics content. As you read the vignettes, do not get caught up in thinking of them as tasks for you to enact in your own classroom. Instead, look for the features of modeling that the teachers' stories highlight. At the end of each chapter, we pose a set of reflective questions, and these will help you plan a modeling task that will suit your classroom.

Isaac (chapter 2) is an algebra teacher who wants to start the school year with a modeling task in order to instill habits of quantitative critical thinking among his students, and he identifies the school's recycling setup as a rich opportunity. He is the school adviser to the recycling club, and he chose a task about recycling because he is familiar with details about the context that students might want to know. He is also aware that because he is the adviser, he can give the students agency to enact the model and to revise it if necessary.

Julia (chapter 4) also teaches algebra, and she develops a task in which students investigate what is the best length to set for the school's passing period—the time that is allocated in the schedule for students to move from one period to the other. This task is very particular to her high school: It is a large, aging building, constructed over decades as additions were built in response to the town's increasing population, and the sprawling building is an ongoing source of student complaints. This task is just right for Julia's algebra class, and it might or might not suit your context—that's OK! You will see how Julia takes advantage of her context to build community in her classroom.

Ruby (chapter 5) is a precalculus teacher who wants to engage her students in a modeling task in which students study exponential decay, and she develops a task in which students predict the level of theobromine in a dog's circulatory system after the dog eats chocolate. She started by thinking about an applied mathematics task she has used—the exponential decay of medicines in the bloodstream—and turned this into a modeling task by describing to her students an authentic problem she and some of her students have faced. As you read this vignette, notice how Ruby negotiates the boundary between an applied mathematics problem and a modeling problem by keeping her focus on the big ideas of modeling.

In the next set of chapters, the vignettes each illustrate one of the big ideas about mathematical modeling. **Darius** (chapter 6) is an algebra teacher who designs a modeling task that engages students in understanding how to use rates. He develops a task whereby students study the flow of a creek that runs past the high school. **Destiny** (chapter 7) teaches sixth-grade mathematics and develops a task after overhearing her students discuss the potential locations for their upcoming field trip. **Lule** (chapter 8), another algebra teacher, is also a cross-country coach and found herself developing a mathematical model as she investigated ways to select members of the team to represent the school at the state race. She recognizes that her own authentic need for a model provides an opportunity to engage her students in modeling.

In chapter 10, you will meet **Kiran**, who teaches high school statistics and implements a task that lets students examine an issue of justice and use linear regression to investigate the situation. The students' use of linear regression gives him the opportunity to highlight differences between mathematical and statistical models.

In each vignette, look for ways that the big ideas of modeling are evident in these classrooms. In the final chapter of the book, we include commentary in the words of the teachers on whom we have based many of these characters.

GETTING READY TO BE A TEACHER OF MODELING

Modeling makes use of mathematics as a tool for empowerment, embracing the fact that mathematics does not happen in isolation. Modelers draw on daily experiences in conversation with mathematical experiences to explore and weigh in on issues that are meaningful to individuals, the community, or the broader world. Mathematical modeling is a process that can and should be accessed by everyone. Each of us has problem-solving strategies, mathematical tools, and personal insight that can shed light on a situation from different perspectives.

In the classroom, mathematical modeling helps to create and support a shared community rooted in access, equity, and empowerment. It allows teachers to draw on students' in- and out-of-school experiences and knowledge to investigate issues that are relevant and important to the class. Modeling allows for multiple approaches and mathematical perspectives and provides opportunities for each and every student to contribute to classroom progress. It provides ways for teachers to make meaningful connections among student solution strategies. In short, modeling promotes a classroom where students can see the intersection of their lives and their mathematical work, and through these experiences they can see mathematics as accessible and part of their world.

This book is divided into three parts. In the rest of part I, we discuss mathematical modeling broadly and invite you to think about its place in classrooms. Chapter 2 focuses on the practices and perspectives students develop as they engage in mathematical modeling. In chapter 3, we address how modeling can empower students and foster action-focused critical thinking. In chapter 4, we focus on mathematical learning communities, and discuss strategies for establishing a classroom culture that will support engagement in mathematical modeling. To finish part I, chapter 5 traces how mathematical modeling can play out in a classroom.

In part II, we explore each of the four big ideas and share specific strategies for making modeling a part of your classroom practice. At the beginning and end of each of these chapters, you will be invited to think about your students and their mathematical work, developing a modeling lesson as you read. You will be prepared to engage your students in a modeling lesson when you are finished with part II.

Finally, in part III, we highlight ways statistics is used in mathematical modeling, share ideas about how modeling might fit into your curriculum, describe ways technology can support mathematical modeling, and share teachers' reflections about how including modeling in their classrooms has empowered their students as individuals with a mathematical voice. We hope these testimonies, and the book as a whole, leave you feeling both empowered and excited to incorporate mathematical modeling into your teaching practice.

REFLECT AND DISCUSS

1. What made you pick up this book? What motivates you to explore mathematical modeling?
2. Describe a time you tackled a complex task. What strategies did you use?
3. Which of the four big ideas is most exciting to you? Why?