Effective Teaching and Learning

An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically.

The teaching of mathematics is complex. It requires teachers to have a deep understanding of the mathematical knowledge that they are expected to teach (Ball, Thames, and Phelps 2008) and a clear view of how student learning of that mathematics develops and progresses across grades (Daro, Mosher, and Corcoran 2011; Sztajn et al. 2012). It also requires teachers to be skilled at teaching in ways that are effective in developing mathematics learning for all students. This section presents, describes, and illustrates a set of eight research-informed teaching practices that support the mathematics learning of all students. Before turning to these teaching practices, however, we must be clear about the mathematics learning such teaching must inspire and develop and the inextricable connection between teaching and learning.

The learning of mathematics has been defined to include the development of five interrelated strands that, together, constitute mathematical proficiency (National Research Council 2001):

1. Conceptual understanding
2. Procedural fluency
3. Strategic competence
4. Adaptive reasoning
5. Productive disposition

Conceptual understanding (i.e., the comprehension and connection of concepts, operations, and relations) establishes the foundation, and is necessary, for developing procedural fluency (i.e., the meaningful and flexible use of procedures to solve problems).

Strategic competence (i.e., the ability to formulate, represent, and solve mathematical problems) and adaptive reasoning (i.e., the capacity to think logically and to justify one's thinking) reflect the need for students to develop mathematical ways of thinking as a basis for solving mathematics problems that they may encounter in real life, as well as within mathematics and other disciplines. These ways of thinking are variously described as "processes" (in NCTM's [2000] Process Standards), "reasoning habits" (NCTM 2009), or "mathematical practices" (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010). In this publication, in alignment with the Common

Core State Standards for Mathematics (CCSSM), we refer to them as "mathematical practices," which represent what students are doing as they learn mathematics (see fig. 1).

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Fig. 1. Standards for Mathematical Practice (NGO Center and CCSSO 2010, pp. 6–8)

The fifth strand identified on the preceding page, productive disposition, is "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (National Research Council 2001, p. 131). Students need to recognize the value of studying mathematics and believe that they are capable of learning mathematics through resolve and effort (Schunk and Richardson 2011). This conviction increases students' motivation and willingness to persevere in solving challenging problems in the short term and continuing their study of mathematics in the long term. Interest and curiosity evoked throughout the study of mathematics can spark a lifetime of positive attitudes toward the subject.

Student learning of mathematics "depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum" (Ball and Forzani 2011, p. 17). Ball and other researchers (e.g., Ball et al. 2009; Grossman, Hammerness, and McDonald 2009; Lampert 2010; McDonald, Kazemi, and Kavanagh 2013) argue that the profession of teaching needs to identify and work together toward the implementation of a common set of high-leverage practices that underlie effective teaching. By "high-leverage practices," they mean "those practices at the heart of the work of teaching that are most likely to affect student learning" (Ball and Forzani 2010, p. 45).

Although effective teaching of mathematics may have similarities with productive teaching in other disciplines (Duit and Treagust 2003; Hlas and Hlas 2012), each discipline requires focused attention on those teaching practices that are most effective in supporting student learning specific to the discipline (Hill et al. 2008; Hill, Rowan, and Ball 2005). Research from both cognitive science (Mayer 2002; Bransford, Brown, and Cocking 2000; National

Research Council 2012a) and mathematics education (Donovan and Bransford 2005; Lester 2007) supports the characterization of mathematics learning as an active process, in which each student builds his or her own mathematical knowledge from personal experiences, coupled with feedback from peers, teachers and other adults, and themselves. This research has identified a number of principles of learning that provide the foundation for effective mathematics teaching. Specifically, learners should have experiences that enable them to—

- engage with challenging tasks that involve active meaning making and support meaningful learning;
- connect new learning with prior knowledge and informal reasoning and, in the process, address preconceptions and misconceptions;
- acquire conceptual knowledge as well as procedural knowledge, so that they can meaningfully organize their knowledge, acquire new knowledge, and transfer and apply knowledge to new situations;
- construct knowledge socially, through discourse, activity, and interaction related to meaningful problems;
- receive descriptive and timely feedback so that they can reflect on and revise their work, thinking, and understandings; and
- develop metacognitive awareness of themselves as learners, thinkers, and problem solvers, and learn to monitor their learning and performance.

Mathematics Teaching Practices

Eight Mathematics Teaching Practices provide a framework for strengthening the teaching and learning of mathematics. This research-informed framework of teaching and learning reflects the learning principles listed above, as well as other knowledge of mathematics teaching that has accumulated over the last two decades. The list on the following page identifies these eight Mathematics Teaching Practices, which represent a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics.

Obstacles

Dominant cultural beliefs about the teaching and learning of mathematics continue to be obstacles to consistent implementation of effective teaching and learning in mathematics classrooms (Handal 2003; Philipp 2007). Many parents and educators believe that students should be taught as they were taught, through memorizing facts, formulas, and procedures and then practicing skills over and over again (e.g., Sam and Ernest 2000). This view perpetuates the traditional lesson paradigm that features review, demonstration, and practice and is still pervasive in many classrooms (Banilower et al. 2006; Weiss and Pasley 2004). Teachers, as well

Mathematics Teaching Practices
Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

as parents, are often not convinced that straying from these established beliefs and practices will be more effective for student learning (Barkatsas and Malone 2005; Wilken 2008).

In sharp contrast to this view is the belief that mathematics lessons should be centered on engaging students in solving and discussing tasks that promote reasoning and problem solving (NCTM 2009; National Research Council 2012a). Teachers who hold this belief plan lessons to prompt student interactions and discourse, with the goal of helping students make sense of mathematical concepts and procedures. However, the lack of agreement about what constitutes effective mathematics teaching constrains schools and school systems from establishing coherent expectations for high-quality, productive teaching of mathematics (Ball and Forzani 2011).

Teachers' beliefs influence the decisions that they make about the manner in which they teach mathematics, as indicated in the table at the right. Students' beliefs influence their

Beliefs about teaching and learning mathematics	
Unproductive beliefs	**Productive beliefs**
Mathematics learning should focus on practicing procedures and memorizing basic number combinations.	Mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse.
Students need only to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems.	All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures.
Students can learn to apply mathematics only after they have mastered the basic skills.	Students can learn mathematics through exploring and solving contextual and mathematical problems.
The role of the teacher is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve mathematics problems.	The role of the teacher is to engage students in tasks that promote reasoning and problem solving and facilitate discourse that moves students toward shared understanding of mathematics.
The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests.	The role of the student is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others.
An effective teacher makes the mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused.	An effective teacher provides students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics.

perception of what it means to learn mathematics and their dispositions toward the subject. As the table summarizes, the impact of these beliefs on the teaching and learning of mathematics may be unproductive or productive. It is important to note that these beliefs should not be viewed as good or bad. Instead, beliefs should be understood as unproductive when they hinder the implementation of effective instructional practice or limit student access to important mathematics content and practices.

Overcoming the obstacles

Teaching mathematics requires specialized expertise and professional knowledge that includes not only knowing mathematics but knowing it in ways that make it useful for the work of teaching (Ball and Forzani 2010; Ball, Thames, and Phelps 2008). Mathematics teaching demands subject-specific understanding and insight so that teachers can skillfully carry out their work in mathematics classrooms. Some of the work of mathematics teaching includes finding an example or task to make a specific mathematical point, linking mathematical representations to underlying ideas and other representations, and evaluating students' mathematical reasoning and explanations. This work also requires teachers to be able to unpack mathematical topics that they know well and to reexamine these through the eyes of learners, as well as to be able to work with many learners simultaneously in classrooms, each with unique backgrounds, interests, and learning needs.

The following discussion and illustrations of the eight Mathematics Teaching Practices support the incorporation of the productive beliefs identified above into the daily professional work of effective teachers of mathematics. This framework offers educators within schools and across districts a common lens for collectively moving toward improved instructional practice and for supporting one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students.

Establish Mathematics Goals to Focus Learning

> *Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.*

Effective mathematics teaching begins with a shared understanding among teachers of the mathematics that students are learning and how this mathematics develops along learning progressions. This shared understanding includes clarifying the broader mathematical goals that guide planning on a unit-by-unit basis, as well as the more targeted mathematics goals that guide instructional decisions on a lesson-by-lesson basis. The establishment of clear goals not only guides teachers' decision making during a lesson but also focuses students' attention on monitoring their own progress toward the intended learning outcomes.

Discussion

Mathematics goals indicate what mathematics students are to learn and understand as a result of instruction (Wiliam 2011). In fact, "formulating clear, explicit learning goals sets the stage for everything else" (Hiebert et al. 2007, p. 57). Goals should describe what mathematical concepts, ideas, or methods students will understand more deeply as a result of instruction and identify the mathematical practices that students are learning to use more proficiently. Teachers need to be clear about how the learning goals relate to and build toward rigorous standards, such as the Common Core State Standards for Mathematics. The goals that guide instruction, however, should not be just a reiteration of a standard statement or cluster but should be more specifically linked to the current classroom curriculum and student learning needs, referring, for example, to particular visual representations or mathematical concepts and methods that students will come to understand as a result of instruction.

Learning goals situated within mathematics learning progressions (Daro, Mosher, and Corcoran 2011) and connected to the "big ideas" of mathematics (Charles 2005) provide a stronger basis for teachers' instructional decisions. Learning progressions or trajectories describe how students make transitions from their prior knowledge to more sophisticated understandings. The progressions also identify intermediate understandings and link research on student learning to instruction (Clements and Sarama 2004; Sztajn et al. 2012). Both teachers and students need to be able to answer crucial questions:

- What mathematics is being learned?
- Why is it important?
- How does it relate to what has already been learned?
- Where are these mathematical ideas going?

Situating learning goals within the mathematical landscape supports opportunities to build explicit connections so that students see how ideas build on and relate to one another and come to view mathematics as a coherent and connected discipline (Fosnot and Jacob 2010; Ma 2010).

The mathematical purpose of a lesson should not be a mystery to students. Classrooms in which students understand the learning expectations for their work perform at higher levels than classrooms where the expectations are unclear (Haystead and Marzano 2009; Hattie 2009). Although daily goals need not be posted, it is important that students understand the mathematical purpose of a lesson and how the activities contribute to and support their mathematics learning. Goals or essential questions motivate learning when students perceive the goals as challenging but attainable (Marzano 2003; McTighe and Wiggins 2013). Teachers can discuss student-friendly versions of the mathematics goals as appropriate during the lesson so that students see value in and understand the purpose of their work (Black and Wiliam 1998a; Marzano 2009). When teachers refer to the goals during instruction, students become more focused and better able to perform self-assessment and monitor their own learning (Clarke, Timperley, and Hattie 2004; Zimmerman 2001).

A clear grasp of the mathematics frames the decisions that teachers make as they plan mathematics lessons, make adjustments during instruction, and reflect after instruction on the progress that students are making toward the goals. In particular, by establishing specific goals and considering how they connect with the broader mathematical landscape, teachers are better prepared to use the goals to make decisions during instruction (Hiebert et al. 2007). This includes facilitating meaningful discourse, ensuring connections among mathematical ideas, supporting students as they struggle, and determining what counts as evidence of students' learning (Seidle, Rimmele, and Prenzel, 2005). The practice of establishing clear goals that indicate what mathematics students are learning provides the starting point and foundation for intentional and effective teaching.

Illustration

Establishing clear goals begins with clarifying and understanding the mathematical expectations for student learning. Figure 2 presents an excerpt from a session in which two teachers, Ms. Burke and Mr. Miller, together with their math coach, engage in a collaborative planning session to discuss and clarify the mathematics learning goals for their second-grade students. Notice how the teachers begin by describing what the students will be doing in the lesson, rather than what they will be learning. Of course, teachers need to attend to the logistics of a lesson, but they must also give sufficient attention to establishing a detailed understanding of the mathematics learning goals. Consider how the math coach intentionally shifts the conversation to a discussion of the mathematical ideas and learning that will be the focus of instruction.

Two classes of second-grade students are currently working on understanding and solving addition and subtraction problems set in real-world situations. The following conversation develops among two teachers and their math coach in a planning session. The teachers have selected three story problems to give meaning to subtraction and serve as a focus for one of the lessons:

- Morgan wants to buy the next book in her favorite series when it is released next month. So far, she has saved $15. The book will cost $22. How much more money does Morgan need to save so that she can buy the book? (Problem type: Add to/Change Unknown)
- George and his dad are in charge of blowing up balloons for the party. The package had 36 balloons in it. After blowing up many balloons, George's dad noticed that the package still contained 9 balloons. How many balloons had they blown up? (Problem type: Take from/Change Unknown)
- Lou and Natalie are preparing to run a marathon. Lou ran 43 training miles this week. Natalie ran 27 miles. How much farther did Lou run than Natalie? (Problem type: Compare/Difference Unknown)

Ms. Burke: I think we should have the students work together in small groups to solve the word problems.

Mr. Miller: I agree, and they could take turns reading the problems, and then everyone could draw diagrams or use cubes to solve them, and then they could compare their answers.

Fig. 2. Collaborative planning session focused on clarifying mathematics goals for a lesson on problem situations for subtraction

Math Coach:	OK, that's what you want the students to do. So now let's talk more about what is it that you want your students to learn as a result of this lesson.
Ms. Burke:	We want them to better understand these different types of word problems and be able to solve them.
Math Coach:	OK. So, let's list some of the indicators that would show they understand.
Mr. Miller:	They would be able to use cubes or draw diagrams to show what is happening in the problem, explain what they did and why, and be able to get the right answer.
Ms. Burke:	I also want them to write an equation that models each situation. Some of the equations might be $15 + \square = 22$, $36 = \square + 9$ or $36 - \square = 9$, and $43 - 27 = \square$ or $43 = 27 + \square$.
Mr. Miller:	Then if we have time in this lesson, or maybe the next day, we want the students to compare the different problems and equations and be able to explain how these relate to addition and subtraction, even though the contexts seem so different.
Math Coach:	Can you say a little more about why you picked these three problems for this lesson?
Mr. Miller:	Each word problem is about a different situation that gives meaning to subtraction. One problem is about finding an unknown addend, one is about subtraction as taking away, and the other is about finding the difference when comparing two amounts.
Ms. Burke:	We are hoping that the students get better at thinking about the relationships among the quantities in each context and how this relates to addition and subtraction. And they need to be able to work with these harder problem types and not just the easy take-away word problems [*i.e., Take from/Result Unknown*].
Math Coach:	Let me see if I can summarize this for us. Your learning goals for these lessons are for the students to represent and solve word problems by using diagrams or objects and equations, compare how the problem situations are similar and different, and explain how the underlying structure in each problem relates to addition and subtraction.
Ms. Burke:	Yes, and in their explanations, I want to hear them talk about what each number means in the problem, so in this lesson they know the total amount and one of the parts or addends, and they need to find the other unknown addend.

Note: Classification of problem types is based on CCSSM Glossary, Table 1 (NGA Center and CCSSO 2010, p. 88).

Fig. 2. *Continued*

As a result of the planning conversation, the teachers have a more precise understanding of the addition and subtraction concepts that they hope will surface during the lesson. For example, they expect their students to connect math drawings and equations and compare the mathematical structures of the various types of problem situations. At the beginning of the lesson, they discuss with students the goal and importance of understanding different kinds of word problems by using math drawings and writing equations. During instruction, the teachers are attentive to ensuring that students are not just finding the answers to the word problems but are able to explain how each problem relates to addition and subtraction and how that relationship is reflected in their drawings and equations. This in turn will compel students to focus on the how these problem situations relate to addition and subtraction and why that is an important aspect in their learning of mathematics.

Teacher and student actions

Effective teaching requires a clear understanding of what students need to accomplish mathematically. Clear learning goals focus the work of teaching and student learning. Teachers need to establish clear and detailed goals that indicate what mathematics students are learning, and they need to use these goals to guide decision making during instruction. Students also need to understand the mathematical purpose of a lesson. Teachers should help students understand how specific activities contribute to and support the students' learning of mathematics as appropriate during instruction. Students can then gauge and monitor their own learning progress. The actions listed in the table below provide guidance on what teachers and students do in establishing and using goals to focus learning in the mathematics classroom.

Establish mathematics goals to focus learning **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Establishing clear goals that articulate the mathematics that students are learning as a result of instruction in a lesson, over a series of lessons, or throughout a unit. Identifying how the goals fit within a mathematics learning progression. Discussing and referring to the mathematical purpose and goal of a lesson during instruction to ensure that students understand how the current work contributes to their learning. Using the mathematics goals to guide lesson planning and reflection and to make in-the-moment decisions during instruction.	Engaging in discussions of the mathematical purpose and goals related to their current work in the mathematics classroom (e.g., What are we learning? Why are we learning it?) Using the learning goals to stay focused on their progress in improving their understanding of mathematics content and proficiency in using mathematical practices. Connecting their current work with the mathematics that they studied previously and seeing where the mathematics is going. Assessing and monitoring their own understanding and progress toward the mathematics learning goals.

Implement Tasks That Promote Reasoning and Problem Solving

> *Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.*

Effective mathematics teaching uses tasks as one way to motivate student learning and help students build new mathematical knowledge through problems solving. Research on the use of mathematical tasks over the last two decades has yielded three major findings:

1. Not all tasks provide the same opportunities for student thinking and learning. (Hiebert et al. 1997; Stein et al. 2009)
2. Student learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature. (Boaler and Staples 2008; Hiebert and Wearne 1993; Stein and Lane 1996)
3. Tasks with high cognitive demands are the most difficult to implement well and are often transformed into less demanding tasks during instruction. (Stein, Grover, and Henningsen 1996; Stigler and Hiebert 2004)

To ensure that students have the opportunity to engage in high-level thinking, teachers must regularly select and implement tasks that promote reasoning and problem solving. These tasks encourage reasoning and access to the mathematics through multiple entry points, including the use of different representations and tools, and they foster the solving of problems through varied solution strategies.

Furthermore, effective teachers understand how contexts, culture, conditions, and language can be used to create mathematical tasks that draw on students' prior knowledge and experiences (Cross et al. 2012; Kisker et. al. 2012; Moschkovich 1999, 2011) or that offer students a common experience from which their work on mathematical tasks emerges (Boaler 1997; Dubinsky and Wilson 2013; Wager 2012). As a result of teachers' efforts to incorporate these elements into mathematical tasks, students' engagement in solving these tasks is more strongly connected with their sense of identity, leading to increased engagement and motivation in mathematics (Aguirre, Mayfield-Ingram, and Martin 2013; Boaler 1997; Hogan 2008; Middleton and Jansen 2011).

Discussion

Mathematical tasks can range from a set of routine exercises to a complex and challenging problem that focuses students' attention on a particular mathematical idea. Stein and colleagues

(Stein, Grover, and Henningsen 1996; Stein and Smith 1998) have developed a taxonomy of mathematical tasks based on the kind and level of thinking required to solve them. Smith and Stein (1998) show the characteristics of higher- and lower-level tasks and provide samples in each category; figure 3 reproduces their list of the characteristics of tasks at four levels of cognitive demand, and figure 4 provides examples of tasks at each of the levels.

Levels of Demands

Lower-level demands (memorization):
- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced

Lower-level demands (procedures without connections):
- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used
- Are focused on producing correct answers instead of on developing mathematical understanding
- Require no explanations or explanations that focus solely on describing the procedure that was used

Higher-level demands (procedures with connections):
- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

Higher-level demands (doing mathematics):
- Require complex and nonalgorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships
- Demand self-monitoring or self-regulation of one's own cognitive processes
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task
- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required

These characteristics are derived from the work of Doyle on academic tasks (1988) and Resnick on high-level-thinking skills (1987), the *Professional Standards for Teaching Mathematics* (NCTM 1991), and the examination and categorization of hundreds of tasks used in QUASAR classrooms (Stein, Grover, and Henningsen 1996; Stein, Lane, and Silver 1996).

Fig. 3. Characteristics of mathematical tasks at four levels of cognitive demand. From Smith and Stein (1998).

Lower-Level Demands

Memorization

What is the rule for multiplying fractions?

Expected student response:

You multiply the numerator times the numerator and the denominator times the denominator.

or

You multiply the two top numbers and then the two bottom numbers.

Procedures without Connections

Multiply:

$$\frac{2}{3} \times \frac{3}{4}$$

$$\frac{5}{6} \times \frac{7}{8}$$

$$\frac{4}{9} \times \frac{3}{5}$$

Expected student response:

$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$$

$$\frac{5}{6} \times \frac{7}{8} = \frac{5 \times 7}{6 \times 8} = \frac{35}{48}$$

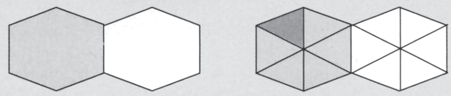
$$\frac{4}{9} \times \frac{3}{5} = \frac{4 \times 3}{9 \times 5} = \frac{12}{45}$$

Higher-Level Demands

Procedures with Connections

Find 1/6 of 1/2. Use pattern blocks. Draw your answer and explain your solution.

Expected student response:



First you take half of the whole, which would be one hexagon. Then you take one-sixth of that half. So I divided the hexagon into six pieces, which would be six triangles. I only needed one-sixth, so that would be one triangle. Then I needed to figure out what part of the two hexagons one triangle was, and it was 1 out of 12. So 1/6 of 1/2 is 1/12.

Doing Mathematics

Create a real-world situation for the following problem:

$$\frac{2}{3} \times \frac{3}{4}.$$

Solve the problem you have created without using the rule, and explain your solution.

One possible student response:

For lunch Mom gave me three-fourths of a pizza that we ordered. I could only finish two-thirds of what she gave me. How much of the whole pizza did I eat?

I drew a rectangle to show the whole pizza. Then I cut it into fourths and shaded three of them to show the part Mom gave me. Since I only ate two-thirds of what she gave me, that would be only two of the shaded sections.

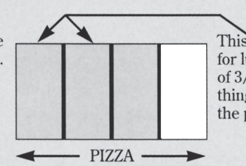


Fig. 4. Sample tasks for four levels of cognitive demand. From Smith and Stein (1998).

From the perspective of this taxonomy, mathematical tasks are viewed as placing *higher-level* cognitive demands on students when they allow students to engage in active inquiry and exploration or encourage students to use procedures in ways that are meaningfully connected with concepts or understanding. Tasks that encourage students to use procedures, formulas, or algorithms in ways that are not actively linked to meaning, or that consist primarily of memorization or the reproduction of previously memorized facts, are viewed as placing *lower-level* cognitive demands on students. Consider figure 5, which shows two tasks, both of which might be used in an algebra unit that includes analyzing and solving pairs of simultaneous equations.

Task A: Smartphone Plans	Task B: Solving systems of equations
You are trying to decide which of two smartphone plans would be better. Plan A charges a basic fee of \$30 per month and 10 cents per text message. Plan B charges a basic fee of \$50 per month and 5 cents per text message. How many text messages would you need to send per month for plan B to be the better option? Explain your decision. (Adapted from Illustrative Mathematics Illustrations: www.illustrativemathematics.org/illustrations/469.)	Solve each of the following systems: $-4x - 2y = -12$ $4x + 8y = -24$ $x - y = 11$ $2x + y = 19$ $8x + y = -1$ $-3x + y = -5$ $5x + y = 9$ $10x - 7y = -18$

Fig. 5. Comparison of tasks with different cognitive demand

Task A is a high-level task, since no specific pathway has been suggested or is implied, and students could use several different approaches to enter and solve the task (e.g., guess and check, make a table, graph equations to find the point of intersection, solve a system of two linear equations by using algebra). Further, students must put forth effort to determine and enact a course of action and justify the reasonableness and accuracy of their solutions. By contrast, task B is a low-level task because it is likely that students are expected to use a specific memorized procedure that leaves little or no ambiguity about what they need to do. The mathematics that students can learn in doing a high-level task is significantly different from the mathematics that they learn from low-level tasks. Over time, the cumulative effect of the use of mathematics tasks is students' implicit development of ideas about the nature of mathematics—about whether mathematics is something that they personally can make sense of and how long and how hard they should have to work to solve any mathematical task.

It is important to note that not all tasks that promote reasoning and problem solving have to be set in a context or need to consume an entire class period or multiple days. What is critical is that a task provide students with the opportunity to engage actively in reasoning, sense making, and problem solving so that they develop a deep understanding of mathematics. Take, for example, the task on exponential functions in figure 6, which calls on students to analyze functions by using visual representations. In working on this task, students explore what happens to the graph of the function when the values of *a* change, and through their use of representations, they generalize the behavior of the function.

Using your graphing calculator, investigate the changes that occur in the graph of $y = a^x$ for different values of a, where a is any real number. Explain what happens in the following cases:

(1) $a > 1$

(2) $a = 1$

(3) $0 < a < 1$

(4) $a = 0$

(5) $a < 0$

Fig. 6. An algebra task requiring students to use graphical representations to analyze exponential functions

This task promotes problem solving because students are positioned to explore the situation without being told in advance what to expect. Through reasoning about this task, they are likely to determine the general shape of the graph of the function (e.g., when $a > 1$, the graph starts out "flat" and close to the x-axis and then shoots up; when $0 < a < 1$, the graph shows a rapidly shrinking function), what occurs at 0 and 1, and the difference between a growth function and a decay function. Extending this discussion to the case of $a < 0$ provides an important opportunity for students to learn why exponential functions are restricted to $a \geq 0$.

Tasks engaging students in reasoning and problem solving are not limited to middle and high school content. Consider the task in figure 7, in which students in kindergarten–grade 1 decompose the number 10 into pairs in more than one way.

There are 10 cars in the parking lot. Some of the cars are red and some of the cars are black. How many red cars and how many black cars could be in the parking lot?

Think of as many different combinations of cars as you can.

Show your solutions in as many ways as you can with cubes, drawings, or words, and write an equation for each solution.

Fig. 7. A task for K–grade 1 on number pairs that make 10. Adapted from the North Carolina Department of Public Instruction; http://commoncoretasks.ncdpi.wikispaces.net/First+Grade+Tasks.

In this problem, students identify one or more combinations that equal 10, using drawings, cubes, or other tools (e.g., fingers, ten frame, Rekenrek) as needed to support their problem solving and explaining. This is a high-level task for most kindergarten and first-grade students because they have not yet learned these combinations, and they can use a variety of strategies (e.g., trial and error, counting up to 10 from a selected number, decomposing 10 into two sets) to determine the combinations that will work. Through the process of solving this task, students may recognize similar combinations (e.g., $4 + 6 = 6 + 4$) and begin to see number patterns (e.g., $1 + 9$, $2 + 8$, $3 + 7$; as one number gets bigger by 1, the other number gets smaller by 1).

In determining the level of task, it is important to consider the prior knowledge and experiences of the students who will be engaged in the task. Tasks may begin as high-level tasks for students who are initially learning about the underlying mathematics (e.g., systems of linear equations, behaviors of functions, number combinations). Eventually, as students solidify their understanding of the underlying mathematics, these tasks may become more routine experiences for them. Students then need tasks that further extend these mathematical ideas in ways that continue to deepen understanding and strengthen mathematical reasoning and problem solving.

Illustration

Although selecting tasks that promote reasoning and problem solving is a critical first step, giving the task to students does not guarantee that students will actually engage in the task at a high level. Consider the comparison that figure 8 presents in the implementation of task A, Smartphone Plans, shown in figure 5, in two algebra classrooms.

Note that although Ms. Carson uses a task that could promote reasoning, as soon as she sees students struggling, she provides them with a pathway for solving the task. By taking over the thinking for her students, Ms. Carson removes their opportunity to engage deeply and meaningfully with the mathematics and leaves them simply to apply a specific procedure.

By contrast, when Ms. McDonald sees her students struggling to figure out what to do, she provides suggestions that will help them make progress on the task without giving them a specific pathway to follow. This is the approach that NCTM (2000 p. 19) has long advocated:

> Teachers must decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge.

As a result of the way in which Ms. McDonald orchestrates the lesson, students have the opportunity to consider different strategies and engage in mathematical problem solving at a

As students in two algebra classrooms begin working with their partners on the Smartphone Plans task, it becomes evident that students are struggling to get started.

Ms. Carson's classroom

Ms. Carson calls the class together and tells the students that they first need to write equations for both smartphone plans. She writes $y = mx + b$ on the board and asks students what m and b would be for each phone plan. Once they have established the two equations, she goes to the board and creates a table that contains three columns: x (number of text messages), y_1 (cost for plan A), and y_2 (cost for plan B). She suggests that they begin with 0 text messages and then increment the x values in the table by 10. The students resume their work with their partners and easily complete the table, identifying (400, 70) as the point of intersection of the two equations.

Ms. McDonald's classroom

Ms. McDonald poses questions to students as she walks around the room. When she notices students struggling to get started, she asks them how much it will cost to send one text message in each plan. This question enables her to be sure that the students understand the relationship among the number of messages, the cost per message, and the basic fee. She asks them which plan will cost more for a specific number of messages and to consider whether this plan's cost will always be higher. Then she leaves the partners to discuss ways to use this information to solve the problem. As the students continue working, she observes different approaches, hears debates on whether the answer is 400 messages or 401 messages, and plans how to sequence the whole-class discussion to analyze and compare the varied strategies.

Fig. 8. A look inside two algebra classrooms at the implementation of Smartphone Plans (task A in fig. 5)

high level of cognitive demand. Moreover, and most important, the students are challenged to deepen their understanding of linear equations and what the point of intersection means, both graphically and contextually.

Teacher and student actions

For students to learn mathematics with understanding, they must have opportunities to engage on a regular basis with tasks that focus on reasoning and problem solving and make possible multiple entry points and varied solution strategies. The actions listed in the table on the next page provide a summary of what teachers and students need to do when implementing such tasks in the mathematics classroom. It is important to note that tasks that focus on learning and applying procedures do have a place in the curriculum and are necessary for developing fluency. Such tasks, however, should not dominate instruction and preempt the use of tasks that promote reasoning. Instead, these tasks should build on and emerge from these sense-making and problem-solving experiences.

Implement tasks that promote reasoning and problem solving Teacher and student actions	
What are *teachers* doing?	**What are *students* doing?**
Motivating students' learning of mathematics through opportunities for exploring and solving problems that build on and extend their current mathematical understanding. Selecting tasks that provide multiple entry points through the use of varied tools and representations. Posing tasks on a regular basis that require a high level of cognitive demand. Supporting students in exploring tasks without taking over student thinking. Encouraging students to use varied approaches and strategies to make sense of and solve tasks.	Persevering in exploring and reasoning through tasks. Taking responsibility for making sense of tasks by drawing on and making connections with their prior understanding and ideas. Using tools and representations as needed to support their thinking and problem solving. Accepting and expecting that their classmates will use a variety of solution approaches and that they will discuss and justify their strategies to one another.

Use and Connect Mathematical Representations

> *Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.*

Effective mathematics teaching includes a strong focus on using varied mathematical representations. NCTM (2000) highlighted the important role of mathematical representations in the teaching and learning of mathematics by including the Process Standard for Representation in *Principles and Standards for School Mathematics*. Representations embody critical features of mathematical constructs and actions, such as drawing diagrams and using words to show and explain the meaning of fractions, ratios, or the operation of multiplication. When students learn to represent, discuss, and make connections among mathematical ideas in multiple forms, they demonstrate deeper mathematical understanding and enhanced problem-solving abilities (Fuson, Kalchman, and Bransford 2005; Lesh, Post, and Behr 1987).

Discussion

The general classification scheme for types of representations shown in figure 9 indicates important connections among contextual, visual, verbal, physical, and symbolic representational forms (Lesh, Post, and Behr 1987). Tripathi (2008) noted that using these "different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective that makes the picture (concept) richer and deeper" (p. 439). Students, especially young learners, also benefit from using physical objects or acting out processes during problem solving (National Research Council 2009).

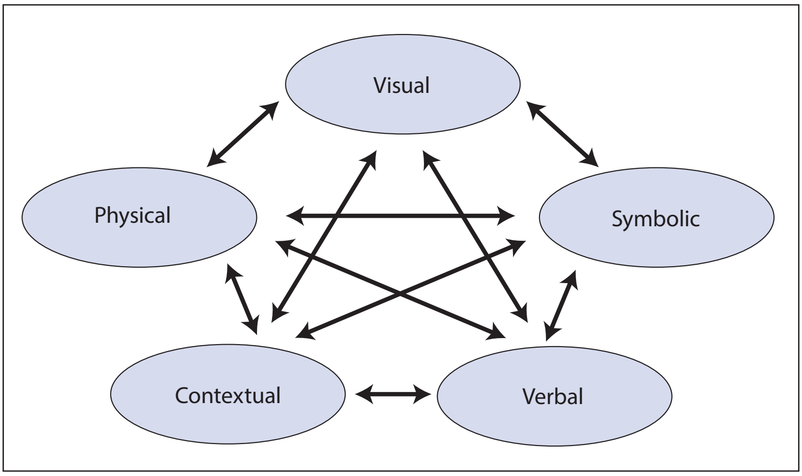


Fig. 9. Important connections among mathematical representations

According to the National Research Council (2001), "Because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas" (p. 94). The depth of understanding is related to the strength of connections among mathematical representations that students have internalized (Pape and Tchoshanov 2001; Webb, Boswinkel, and Dekker 2008). For example, students develop understanding of the meaning of the fraction $7/4$ (symbolic form) when they can see it as the quantity formed by "7 parts of size one-fourth" with a tape diagram or on a number line (visual form), or measure a string that has a length of 7-fourths yards (physical form).

Visual representations are of particular importance in the mathematics classroom, helping students to advance their understanding of mathematical concepts and procedures, make sense of problems, and engage in mathematical discourse (Arcavi 2003; Stylianou and Silver 2004). Visuals support problem solving as students consider relationships among quantities when they sketch diagrams or make tables and graphs. The visual representations also support discourse because the diagrams or drawings leave a trace of student problem solving that can be displayed, critiqued, and discussed. Math drawings and other visual supports are of particular importance for English language learners, learners with special needs, or struggling learners, because they allow more students to participate meaningfully in the mathematical discourse in the classroom (Fuson and Murata 2007). The visuals assist students in following the reasoning of their classmates and in giving voice to their own explanations as they gesture to parts of their math drawings and other visual representations.

Students' understanding is deepened through discussion of similarities among representations that reveal underlying mathematical structures or essential features of mathematical ideas that persist regardless of the form (Zimba 2011). For example, fractions are composed of the iteration of unit fractions, a structure that can be identified and discussed when students use paper strips as fraction models, draw tape diagrams or number lines, or use symbols. Likewise, the addition of fractions has a structure that is similar to that of the addition of whole numbers, in that all addition involves combining same-sized units, such as adding tens to tens or twelfths to twelfths. Mathematical structure can also be emphasized and discussed by asking students to translate or alternate directionality among the various representations, such as by linking symbols back to contexts (e.g., describing a real-world situation for 3×29 or $y = 3x + 5$), making a table of values for a given ratio, or making a graph based on the information in a table (Greeno and Hall 1997).

Success in solving problems is also related to students' ability to move flexibly among representations (Huinker 2013; Stylianou and Silver 2004). Students should be able to approach a problem from several points of view and be encouraged to switch among representations until they are able to understand the situation and proceed along a path that will lead them to a solution. This implies that students view representations as tools that they can use to help them solve problems, rather than as an end in themselves. If, by contrast, algebra tiles or base-ten blocks, for instance, are not used meaningfully, students may view use of the physical objects as the goal instead of reaching an understanding of how the tiles allow them to make sense of polynomials or how the base-ten blocks show the structure of the base-ten number system.

Illustration

Students' representational competence can be developed through instruction. Marshall, Superfine, and Canty (2010, p. 40) suggest three specific strategies:

1. Encourage purposeful selection of representations.
2. Engage in dialogue about explicit connections among representations.
3. Alternate the direction of the connections made among representations.

Consider the lesson presented in figure 10, and focus on how the teacher, Mr. Harris, uses these strategies with his third-grade students as they represent and solve a problem involving setting up chairs for a band concert.

The third-grade class is responsible for setting up the chairs for the spring band concert. In preparation, they have to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area.

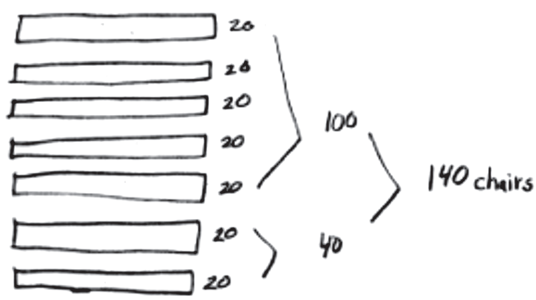
Mr. Harris explains to his students that they need to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. Next he asks the students to consider how they might represent the problem: "Before you begin working on the task, think about a representation you might want to use and why, and then turn and share your ideas with a partner."

The students then set to work on the task. Most sketch equal groups or decompose area models. Two students cut an array out of grid paper. A few students make a table or T-chart, listing the number of rows with the corresponding number of chairs. Some students use symbolic approaches, such as repeated addition or partial products.

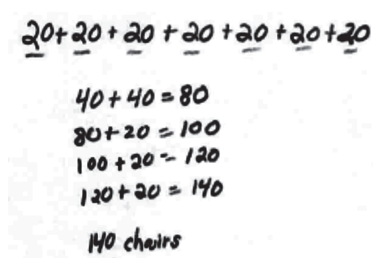
A few students change representations as they work. Dominique starts to draw tally marks but then switches to using a table. When Mr. Harris asks her why, she explains that she got tired of making all those marks. Similarly, Jamal starts to build an array with connecting cubes but then switches to drawing an array. These initial attempts are valuable, if not essential, in helping each of these students make sense of the situation.

As the students work, the teacher poses purposeful questions to press them to consider critical features of their representations: "How does your drawing show 7 groups?" "Why are you adding all those twenties?" "How many twenties are you adding, and why?"

Before holding a whole-class discussion, Mr. Harris has the students find a classmate who used a different representation and directs them to take turns explaining and comparing their work as well as their solutions. For example, Jasmine, who drew the diagram shown below on the left, compares her work with Kenneth, who used equations, as shown on the right. Then Mr. Harris has the students repeat the process, finding another classmate and holding another share-and-compare session.



Jasmine's drawing



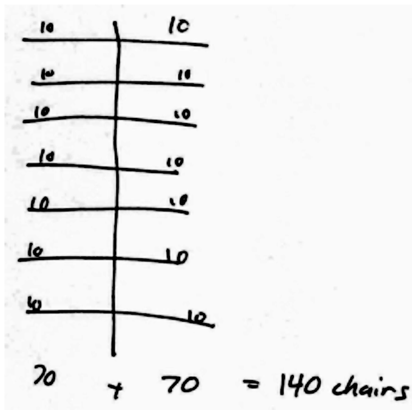
Kenneth's equations

Mr. Harris begins the whole-class discussion by summarizing the goal for the lesson as understanding how the different representations are related to the operation of multiplication. He first asks students to identify and explain how different visual representations show both the number of equal groups and the amount in each group as a structure of

Fig. 10. A third-grade lesson emphasizing mathematical representations to solve a task on setting up chairs for a band concert

multiplication. This prompts the students to compare diagrams with equal groups, arrays, and area models and discuss how they are similar and different. The students comment that it is easy to see the number of chairs in each row in some of the diagrams but not in others. Mr. Harris then writes 7 × 20 on the board and asks the students to explain how the expression matches each of the diagrams.

Finally, Mr. Harris has the students discuss and compare the representations of those students who considered the aisle and worked with tens rather than with twenties, such as Amanda, whose work is shown below. He asks them to take this final step, knowing that this informal experience and discussion of the distributive property will be important in subsequent lessons.



Amanda's work with tens

Fig. 10. *Continued*

Mr. Harris selects the task about the chairs for the band concert to focus on a problem situation that can be represented with arrays. The goal for the lesson is for students to understand how the structure of multiplication is evident within and among different representations. He chooses the numbers purposefully to build his students' conceptual understanding of multiplying one-digit whole numbers by multiples of 10, using strategies based on place value and properties of operations. He allows students to select and discuss their choices to represent the problem situation. Mr. Harris pays close attention to what students are doing, and the questions that he poses as they work and during the whole-class discussion help his students make explicit connections among the representations in ways that further their understanding of the central mathematical ideas of the lesson.

Teacher and student actions

Effective teaching emphasizes using and making connections among mathematical representations to deepen student understanding of concepts and procedures, support mathematical discourse among students, and serve as tools for solving problems. As students use and make connections among contextual, physical, visual, verbal, and symbolic representations, they grow in their appreciation of mathematics as a unified, coherent discipline. The teacher and student actions listed in the table below provide a summary of what teachers and students do in using mathematical representations in teaching and learning mathematics.

Use and connect mathematical representations **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Selecting tasks that allow students to decide which representations to use in making sense of the problems. Allocating substantial instructional time for students to use, discuss, and make connections among representations. Introducing forms of representations that can be useful to students. Asking students to make math drawings or use other visual supports to explain and justify their reasoning. Focusing students' attention on the structure or essential features of mathematical ideas that appear, regardless of the representation. Designing ways to elicit and assess students' abilities to use representations meaningfully to solve problems.	Using multiple forms of representations to make sense of and understand mathematics. Describing and justifying their mathematical understanding and reasoning with drawings, diagrams, and other representations. Making choices about which forms of representations to use as tools for solving problems. Sketching diagrams to make sense of problem situations. Contextualizing mathematical ideas by connecting them to real-world situations. Considering the advantages or suitability of using various representations when solving problems.

Facilitate Meaningful Mathematical Discourse

Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Effective mathematics teaching engages students in discourse to advance the mathematical learning of the whole class. Mathematical discourse includes the purposeful exchange of ideas through classroom discussion, as well as through other forms of verbal, visual, and written communication. The discourse in the mathematics classroom gives students opportunities to share ideas and clarify understandings, construct convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives (NCTM 1991, 2000).

Discussion

Discourse that focuses on tasks that promote reasoning and problem solving is a primary mechanism for developing conceptual understanding and meaningful learning of mathematics (Michaels, O'Connor, and Resnick 2008). According to Carpenter, Franke, and Levi (2003, p. 6),

> Students who learn to articulate and justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers develop a deep understanding that is critical to their future success in mathematics and related fields.

Although discourse provides important opportunities for students to learn what mathematics is *and* how one does it, creating a culture of discourse in the mathematics classroom also presents challenges. Teachers must determine how to build on and honor student thinking while ensuring that the mathematical ideas at the heart of the lesson remain prominent in class discussions (Engle and Conant 2002). For example, in orchestrating a class discussion of student approaches to solving a task, the teacher must decide what approaches to share, the order in which they should be shared, and the questions that will help students make connections among the different strategies and the key disciplinary ideas that are driving the lesson. Such discussions can easily become little more than elaborate show-and-tell sessions (Wood and Turner-Vorbeck 2001) in which it is not clear what each solution adds to students' developing understanding or how it advances the mathematical storyline of the lesson. Smith and Stein (2011) describe five practices for effectively using student responses in whole-class discussions:

1. *Anticipating* student responses prior to the lesson
2. *Monitoring* students' work on and engagement with the tasks
3. *Selecting* particular students to present their mathematical work
4. *Sequencing* students' responses in a specific order for discussion
5. *Connecting* different students' responses and connecting the responses to key mathematical ideas

Students must also have opportunities to talk with, respond to, and question one another as part of the discourse community, in ways that support the mathematics learning of all students in the class. Hufferd-Ackles, Fuson, and Sherin (2004) describe a framework for moving toward a classroom community centered on discourse. They examine how teachers and students proceed through levels in shifting from a classroom in which teachers play the leading role in pursuing student mathematical thinking to one in which they assist students in taking on important roles. The framework describes growth in five components (Hufferd-Ackles, Fuson, and Sherin 2004):

1. How the teacher supports student engagement
2. Who serves as the questioner and what kinds of questions are posed
3. Who provides what kinds of explanations
4. How mathematical representations are used
5. How much responsibility students share for the learning of their peers and themselves

Figure 11 shows a table developed by Hufford-Ackles, Fuson, and Sherin (2014) to describe the levels of classroom discourse through which teachers and their students advance.

Illustration

Mr. Donnelly and his seventh-grade students are studying proportional relationships and their use to solve real-world and mathematical problems. As part of this work, Mr. Donnelly wants his students to be able to identify multiplicative relationships between quantities and recognize three strategies for solving such problems—scaling up, scale factor, and unit rate. He has selected the Candy Jar task, shown in figure 12, for the lesson, since it is aligned with his goals, provides opportunities for high-level reasoning, and offers multiple entry points. Figure 13 shows Mr. Donnelly's lesson on the Candy Jar task.

Suppose you have a new candy jar with the same ratio of Jolly Ranchers (JR) to jawbreakers (JB) as shown in the picture, but it contains 100 Jolly Ranchers.

How many jawbreakers do you have?

Justify your answer.

Note: In the picture, Jolly Ranchers are represented by 5 rectangles, and jawbreakers are shown by 13 circles.

Fig. 12. The Candy Jar task. Adapted from Smith and colleagues (2005).

	Teacher role	Questioning	Explaining mathematical thinking	Mathematical representations	Building student responsibility within the community
Level 0	Teacher is at the front of the room and dominates conversation.	Teacher is only questioner. Questions serve to keep students listening to teacher. Students give short answers and respond to teacher only.	Teacher questions focus on correctness. Students provide short answer-focused responses. Teacher may give answers.	Representations are missing, or teacher shows them to students.	Culture supports students keeping ideas to themselves or just providing answers when asked.
Level 1	Teacher encourages the sharing of math ideas and directs speaker to talk to the class, not to the teacher only.	Teacher questions begin to focus on student thinking and less on answers. Only teacher asks questions.	Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in an explanation. Students provide brief descriptions of their thinking in response to teacher probing.	Students learn to create math drawings to depict their mathematical thinking.	Students believe that their ideas are accepted by the classroom community. They begin to listen to one another supportively and to restate in their own words what another student has said.
Level 2	Teacher facilitates conversation between students, and encourages students to ask questions of one another.	Teacher asks probing questions and facilitates some student-to-student talk. Students ask questions of one another with prompting from teacher.	Teacher probes more deeply to learn about student thinking. Teacher elicits multiple strategies. Students respond to teacher probing and volunteer their thinking. Students begin to defend their answers.	Students label their math drawings so that others are able to follow their mathematical thinking.	Students believe that they are math learners and that their ideas and the ideas of their classmates are important. They listen actively so that they can contribute significantly.
Level 3	Students carry the conversation themselves. Teacher only guides from the periphery of the conversation. Teacher waits for students to clarify thinking of others.	Student-to-student talk is student initiated. Students ask questions and listen to responses. Many questions ask "why" and call for justification. Teacher questions may still guide discourse.	Teacher follows student explanations closely. Teacher asks students to contrast strategies. Students defend and justify their answers with little prompting from the teacher.	Students follow and help shape the descriptions of others' math thinking through math drawings and may suggest edits in others' math drawings.	Students believe that they are math leaders and can help shape the thinking of others. They help shape others' math thinking in supportive, collegial ways and accept the same support from others.

Fig. 11. Levels of classroom discourse. From Hufford-Ackles, Fuson, and Sherin (2014), table 1.

Mr. Donnelly monitors his students as they work in small groups on the Candy Jar task, providing support as needed and taking note of their strategies. He notices that students who use the scaling up strategy do so in different ways. Some use a table that shows a constant increase of 5 Jolly Ranchers and 13 jawbreakers (see solution 1 below), some use a ratio table that contains different multiples of 5 and 13, and some even draw pictures of candy jars. He decides to have the groups who created solutions 1, 2, and 3, shown below, present their work (in this order), since these groups used the strategies that he is targeting (i.e., scaling up, scale factor, and unit rate). This sequencing reflects the sophistication and frequency of strategies (i.e., most groups used a version of the scaling up strategy, and only one group used the unit rate strategy).

Solution 1. Scaling up reasoning

Student explanation: "I started with 5 Jolly Ranchers (JR) and 13 jawbreakers (JB), and I just kept adding on 5 JR and 13 JB every time until I got to 100 JR. Then I saw that I had 260 JB."

JR	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
JB	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260

Solution 2. Scale factor reasoning

Student explanation: "You had to multiply the five Jolly Ranchers by 20 to get 100, so you'd also have to multiply the 13 jawbreakers by 20, getting 260."

(× 20)

5 JR ⟶ 100 JR

13 JB ⟶ 260 JB

(× 20)

Solution 3. Unit rate reasoning

Student explanation: "Since the ratio is 5 Jolly Ranchers (JR) for 13 jawbreakers (JB), you could give each JR that you have 2 JB. That would use up 10 of them, and then you still have 3 JB that have to be shared. So to distribute the 3 JB to the 5 JR, that would be $3 \div 5 = 0.6$ of a JB, so putting that together would give the ratio of 1 JR to 2.6 JB. So then you just multiply 100 by 2.6."

(× 100)

1 JR ⟶ 100 JR

2.6 JB ⟶ 260 JB

(× 100)

During the discussion, Mr. Donnelly asks the presenters to explain what they did and why, and he invites other students to consider whether the approach makes sense and to ask questions. He makes a point of labeling each of the three strategies, asking students which one is most efficient in solving this particular task, and he poses questions that help students make connections among the strategies and with the key ideas that he is targeting.

Fig. 13. Mr. Donnelly's implementation of the Candy Jar task. Solutions adapted from Smith and colleagues (2005).

Specifically, he wants students to see that the scale factor is the same as the number of entries in the table used for scaling up. In other words, it would take 20 candy jars with the same number of Jolly Ranchers and jawbreakers as the original jar to make the new candy jar. Mr. Donnelly then will have his students compare this result with the unit rate, which is the factor that relates the number of Jolly Ranchers and the number of jawbreakers in each column of the table in solution 1 (e.g., $5 \times 2.6 = 13$, just as $55 \times 2.6 = 143$, just as $100 \times 2.6 = 260$).

Toward the end of the lesson, Mr. Donnelly places the solution shown below as solution 4 on the document camera in the classroom and asks students to decide whether or not this is a viable approach to solving the task and to justify their answers.

Solution 4. Incorrect additive reasoning

Student explanation: "100 Jolly Ranchers is 95 more than the 5 I started with. So I will need 95 more jawbreakers than the 13 I started with."

$$5 \text{ JR} + 95 \text{ JR} = 100 \text{ JR}$$

$$13 \text{ JB} + 95 \text{ JB} = 108 \text{ JB}$$

Mr. Donnelly gives the students five minutes to write a response, and he collects their responses as they leave the room to go to the next class. He expects their responses to give him some insight into whether they are coming to understand that for ratios to remain constant, their numerators and denominators must grow at a rate that is multiplicative, not additive.

Fig. 13. *Continued*

Mr. Donnelly keeps close track of what his students are doing as they explore the task (*monitoring*) so that he is positioned to make strategic choices regarding which solutions to highlight during the whole-class discussion (*selecting*) and in what order (*sequencing*). He selects three groups to present their work—each of which used one of the strategies that he has targeted in his goal for the lesson. By making deliberate choices about what to focus on during the whole-class discussion, he is able to use the discussion time to engage students productively in a thoughtful consideration of a small number of approaches and the connections among them (*connecting*). His decision to end the class by asking students to write individual critiques of a response that uses incorrect additive reasoning gives him a way of assessing the extent to which his students understand that the relationship between the types of candies is multiplicative, not additive.

Mr. Donnelly facilitates rather than directs the discussion. By building on the work produced by students, he positions them as "authors" of the mathematics and engages them in rich discourse about an important set of ideas related to ratios and proportional relationships. Although he asks questions and provides information (e.g., labels for the strategies) to ensure that the mathematics learning goals are met, he does so in a way that gives the students ownership of their learning. Mr. Donnelly is clearly in charge of the lesson, but he offers guidance mostly under the radar, so that it does not impinge on students' growing mathematical authority.

Teacher and student actions

Mathematical discourse among students is central to meaningful learning of mathematics. Teachers carefully prepare and purposefully facilitate discourse, such as whole-class discussions that build on student thinking and guide the learning of the class in a productive disciplinary direction. Students are active members of the discourse community as they explain their reasoning and consider the mathematical explanations and strategies of their classmates. The actions listed in the table below provide some guidance on what teachers and students do as they engage in meaningful discourse in the mathematics classroom.

Facilitate meaningful mathematical discourse **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Engaging students in purposeful sharing of mathematical ideas, reasoning, and approaches, using varied representations. Selecting and sequencing student approaches and solution strategies for whole-class analysis and discussion. Facilitating discourse among students by positioning them as authors of ideas, who explain and defend their approaches. Ensuring progress toward mathematical goals by making explicit connections to student approaches and reasoning.	Presenting and explaining ideas, reasoning, and representations to one another in pair, small-group, and whole-class discourse. Listening carefully to and critiquing the reasoning of peers, using examples to support or counterexamples to refute arguments. Seeking to understand the approaches used by peers by asking clarifying questions, trying out others' strategies, and describing the approaches used by others. Identifying how different approaches to solving a task are the same and how they are different.

Pose Purposeful Questions

> *Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.*

Effective mathematics teaching relies on questions that encourage students to explain and reflect on their thinking as an essential component of meaningful mathematical discourse. Purposeful questions allow teachers to discern what students know and adapt lessons to meet varied levels of understanding, help students make important mathematical connections, and support students in posing their own questions. However, merely asking questions is not enough to ensure that students make sense of mathematics and advance their reasoning. Two critical issues must be considered—the types of questions that teachers ask and the pattern of questioning that they use.

Discussion

Researchers have created a variety of frameworks to categorize the types of questions that teachers ask (e.g., Boaler and Brodie 2004; Chapin and O'Connor 2007). Though the categories differ across frameworks, commonalities exist among the types of questions. For example, the frameworks generally include questions that ask students to recall information, as well as questions that ask students to explain their reasoning. Figure 14 displays a set of question types that synthesizes key aspects of these frameworks that are particularly important for mathematics teaching. Although the question types differ with respect to the level of thinking required in a response, all of the question types are necessary in the interactions among teachers and students. For example, questions that gather information are needed to establish what students know, while questions that encourage reflection and justification are essential to reveal student reasoning.

Question type		Description	Examples
1	Gathering information	Students recall facts, definitions, or procedures.	When you write an equation, what does the equal sign tell you? What is the formula for finding the area of a rectangle? What does the interquartile range indicate for a set of data?
2	Probing thinking	Students explain, elaborate, or clarify their thinking, including articulating the steps in solution methods or the completion of a task.	As you drew that number line, what decisions did you make so that you could represent 7 fourths on it? Can you show and explain more about how you used a table to find the answer to the Smartphone Plans task? It is still not clear how you figured out that 20 was the scale factor, so can you explain it another way?

Fig. 14. A framework for types of questions used in mathematics teaching

Question type		Description	Examples
3	Making the mathematics visible	Students discuss mathematical structures and make connections among mathematical ideas and relationships.	What does your equation have to do with the band concert situation? How does that array relate to multiplication and division? In what ways might the normal distribution apply to this situation?
4	Encouraging reflection and justification	Students reveal deeper understanding of their reasoning and actions, including making an argument for the validity of their work.	How might you prove that 51 is the solution? How do you know that the sum of two odd numbers will always be even? Why does plan A in the Smartphone Plans task start out cheaper but become more expensive in the long run?

Fig. 14. *Continued*

While the *types* of questions that teachers ask are important, so are the *patterns* of questions that they use during teacher-student interactions (Walsh and Sattes 2005). In the Initiate-Response-Evaluate (I-R-E) pattern, the teacher starts by asking a question to gather information, generally with a specific response in mind; a student responds; and then the teacher evaluates the response (Mehan 1979). It is not uncommon for teachers to allocate less than five seconds for a student to respond, and to take even less time to consider the answer themselves. This pattern of questioning generally affords very limited opportunities for students to think and provides teachers with no access to whether or how students are making sense of mathematics. Other questioning patterns involve more than asking recall questions. Two of these patterns of questioning are *funneling* and *focusing* (Herbel-Eisenmann and Breyfogle 2005; Wood 1998).

The funneling pattern of questioning involves using a set of questions to lead students to a desired procedure or conclusion, while giving limited attention to student responses that veer from the desired path. The teacher has decided on a particular path for the discussion to follow and leads the students along that path, not allowing students to make their own connections or build their own understanding of the targeted mathematical concepts. The I-R-E pattern is closely akin to funneling, though higher-level questions may be part of the funneling pattern.

In contrast, a focusing pattern of questioning involves the teacher in attending to what the students are thinking, pressing them to communicate their thoughts clearly, and expecting them to reflect on their thoughts and those of their classmates. The teacher who uses this pattern of questioning is open to a task being investigated in multiple ways. On the basis of content knowledge related to the topic and knowledge of student learning, the teacher plans questions and outlines key points that should become salient in the lesson.

Illustration

Figure 15 shows two high school teachers' implementation of the Coin Circulation task in their classrooms. They choose the task because it provides an opportunity for students to summarize, represent, and interpret data and to further understand and evaluate random processes underlying statistical investigations. In particular, students represent data with plots on the real number line and interpret differences in shape, center, and spread in the context of the data set.

Students in two high school classrooms are investigating how much money it costs to mint coins. As part of the investigation, the students decide that it would be helpful to determine the approximate number of years that a coin stays in circulation. Rather than studying all different types of coins, the students choose to collect data about the ages of pennies. This sets the stage for the students to investigate the question, "How many years does a penny stay in circulation?"

The teachers' mathematics learning goals for the task are for students to collect data, analyze the data, and reach a conclusion, as well as to identify the limitations of this investigation with respect to its sampling method. Specifically, the teachers want the students to recognize that the results do not generalize to a larger population.

Both teachers ask all the students to bring in pennies. The goal for each class is to bring in the equivalent of about one roll for every two or three students. Small groups sort their pennies by the year of minting and determine the age of each coin. The data from the entire class is recorded in a table on the board. The small groups then create dot plots and box plots similar to those shown below, based on the age of the coins.

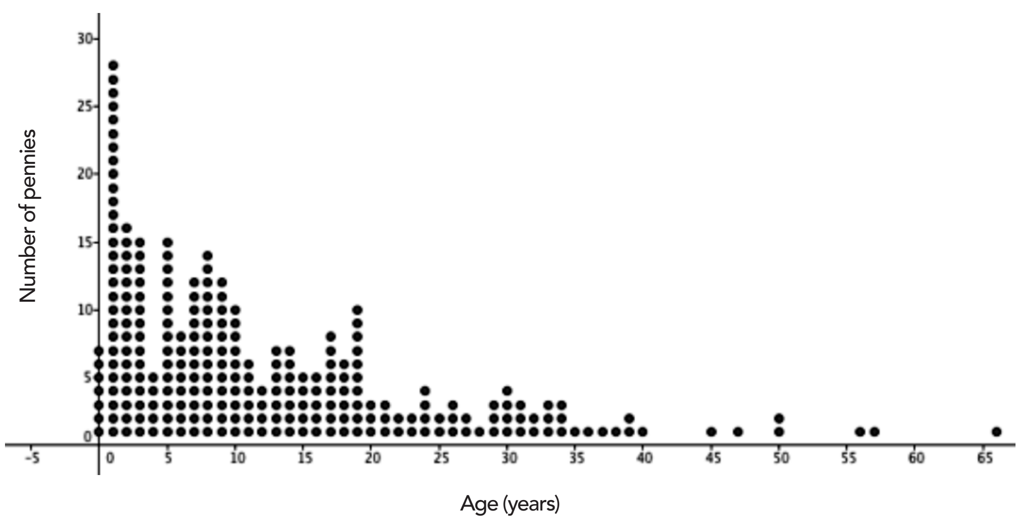


Fig. 15. The Coin Circulation task

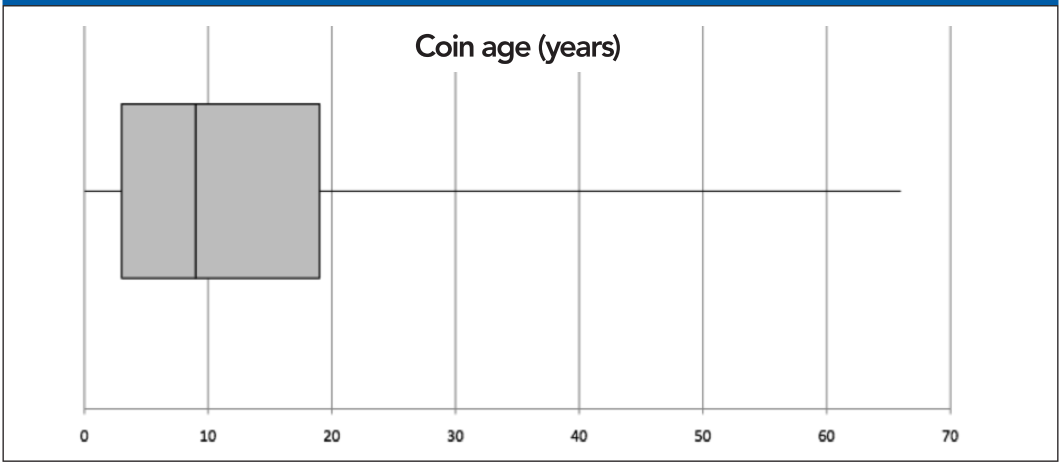


Fig. 15. *Continued*

Although the task engages the students in both classrooms in reasoning and problem solving, the teachers use different questioning patterns. The excerpts from the whole-class discussions shown in figure 16 illustrate the two teachers' questioning patterns.

Questioning pattern: Funneling	Questioning pattern: Focusing
T: What do you notice about the graph? [*waits briefly*] Do you see a pattern in the data? [*waits briefly again*] What are the measures of center for the pennies? *S1:* The mean is about 12.9 years, and the median is about 9 years. *T:* What does the box plot tell us about the variability of the data? *S2:* It has a long tail on one side. *T:* That may be true, but what about the interquartile range—the IQR? What does it tell us? *S3:* Where most of the pennies occur. *T:* Is that really what the IQR tells us? What does each part of the box plot stand for?	*T:* What things do you notice or wonder about the age of pennies? *S1:* It doesn't seem like many of them are very old. *T:* What about the graph makes you say that? *S1:* There's a big mound for newer pennies. *T:* Is there anything else that you notice? *S2:* I found the interquartile range and saw that most pennies are from 3 to 19 years old. *T:* Explain to us what the interquartile range tells us. *S2:* It is where most of the pennies occur. *T:* What do you mean by "most of the pennies"?

Fig. 16. A comparison of questioning patterns on the Coin Circulation task in two classrooms. (*T* is *Teacher*, *S1* is *Student 1*, and so on.)

Questioning pattern: Funneling	Questioning pattern: Focusing
S4: Each part is 25 percent. *T:* Yes, so what else? *S5:* The middle is 50 percent of the pennies and is from 3 to 19 years old. *T:* Good. What can we say about pennies on the basis of this information? *S6:* That most of them are about 10 years old. *T:* But since these are pennies, what does that tell us about all coins? *S7:* That coins will be about 10 years old. *T:* Well, 10 years is for pennies, but this wouldn't necessarily be the same for, say, quarters. Why not?	*S2:* Well, I mean the middle 50 percent. I thought the graph made it hard to tell where things really were. It doesn't look normal, so I couldn't use the middle 68 percent thing we talked about. *T:* I'm not sure I understand. Can someone else comment on what she's saying? *S3:* She means that since there's a tail, the graph isn't like the normal curves we studied. If it were, we could approximate where the most likely ages are—like 68 percent of the data would be within one standard deviation of the mean. [*More discussion follows, and the students determine that 75 percent of the pennies are not more than 19 years old.*] *T:* Would I be correct if I said that a fifty-cent piece would probably be no more than 19 years old? *S4:* Yes, because these coins were a random sample, and that means we can generalize. *S5:* But we looked at pennies, so we can't generalize to quarters. People use pennies more. *T:* What do you mean by that? *S5:* Pennies may wear out. We don't know about other coins from our sample, because quarters would be a different population.

Fig. 16. *Continued*

In the funneling example, the teacher wants students to look at the measures of center and the dispersion of the data. The dialogue demonstrates a reliance on gathering-of-information questions. Some recall of information is necessary so that the teacher knows the baseline of the students' thinking. But questions that probe for understanding need to be part of a questioning pattern that advances student reasoning. As this funneling dialogue moves forward, the teacher has the students look at the center and spread of the data to draw a conclusion and finally asks a higher-level question: "What can we say about pennies on the basis of this information?" Because the students have not been given an opportunity to think deeply enough about what the data tells them about the circulation of pennies, they can give only superficial responses to this question. This example illustrates a far-too-common pattern of questioning, in which the teacher initially uses a probing question but allows little wait time and immediately follows up with questions that become more directed toward one particular answer.

By contrast, the focusing example illustrates how the teacher purposefully blends all four types of questions. Some questions have been planned in advance of the lesson, along with consideration of possible student responses. Other questions are formulated on the spot, in response to student statements and actions during the lesson. Throughout the dialogue, the teacher strives to include questions that push students to clarify their ideas and make the mathematics visible, with the aim of deepening students' mathematical understanding in alignment with the intended learning goals.

Teacher and student actions

In effective teaching, teachers use a variety of question types to assess and gather evidence of student thinking, including questions that gather information, probe understanding, make the mathematics visible, and ask students to reflect on and justify their reasoning. Teachers then use patterns of questioning that focus on and extend students' current ideas to advance student understanding and sense making about important mathematical ideas and relationships. The teacher and student actions listed in the table below provide a summary of using questions purposefully in the mathematics classroom.

Pose purposeful questions **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Advancing student understanding by asking questions that build on, but do not take over or funnel, student thinking. Making certain to ask questions that go beyond gathering information to probing thinking and requiring explanation and justification. Asking intentional questions that make the mathematics more visible and accessible for student examination and discussion. Allowing sufficient wait time so that more students can formulate and offer responses.	Expecting to be asked to explain, clarify, and elaborate on their thinking. Thinking carefully about how to present their responses to questions clearly, without rushing to respond quickly. Reflecting on and justifying their reasoning, not simply providing answers. Listening to, commenting on, and questioning the contributions of their classmates.

Build Procedural Fluency from Conceptual Understanding

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Effective mathematics teaching focuses on the development of *both* conceptual understanding *and* procedural fluency. Major reports have identified the importance of an integrated and balanced development of concepts and procedures in learning mathematics (National Mathematics Advisory Panel 2008; National Research Council 2001). Furthermore, NCTM (1989, 2000) and CCSSM (NGA Center and CCSSO 2010) emphasize that procedural fluency follows and builds on a foundation of conceptual understanding, strategic reasoning, and problem solving.

Discussion

When procedures are connected with the underlying concepts, students have better retention of the procedures and are more able to apply them in new situations (Fuson, Kalchman, and Bransford 2005). Martin (2009, p. 165) describes some of the reasons that fluency depends on and extends from conceptual understanding:

> To use mathematics effectively, students must be able to do much more than carry out mathematical procedures. They must know which procedure is appropriate and most productive in a given situation, what a procedure accomplishes, and what kind of results to expect. Mechanical execution of procedures without understanding their mathematical basis often leads to bizarre results.

Fluency is not a simple idea. Being fluent means that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of the operations to the eventual use of general methods as tools in solving problems. This sequence is beneficial whether students are building toward fluency with single- and multi-digit computation with whole numbers or fluency with, for example, fraction operations, proportional relationships, measurement formulas, or algebraic procedures.

Computational fluency is strongly related to number sense and involves so much more than the conventional view of it encompasses. Developing students' computational fluency extends far beyond having students memorize facts or a series of steps unconnected to understanding (Baroody 2006; Griffin 2005). A rush to fluency, however, undermines students' confidence and interest in mathematics and is considered a cause of mathematics anxiety (Ashcraft 2002; Ramirez et al. 2013). Further, early work with reasoning strategies is related to algebraic reasoning. As students learn how quantities can be taken apart and put back together in different ways (i.e., decomposition and composition of numbers), they establish a basis for understanding properties of the operations. Students need this early foundation for meaningful learning of more formal algebraic concepts and procedures throughout elementary school and into middle and high school (Carpenter, Franke, and Levi 2003; Griffin 2003; Common Core State Standards Writing Team 2011).

In meaningful learning of basic number combinations (i.e., addition and subtraction within 20 and multiplication and division within 100), students progress through well-documented phases toward fluency (Baroody 2006; Baroody, Bajwa, and Eiland 2009; Carpenter et al. 1999). Students begin by using objects, visual representations, and verbal counting, and then they progress to reasoning strategies using number relationships and properties. For example, to solve $8 + 4$, a first grader might count on from 8 early in the school year, whereas later in the year the same student might reason that since $8 + 2$ is 10, then $8 + 4$ must be 2 more than 10, or 12. A third grader might initially use repeated addition to solve 4×6 and then progress to reason that 2 sixes are 12, so 4 sixes must be double that amount, which is 24. This approach supports students, over time, in knowing, understanding, and being able to use their knowledge of number combinations meaningfully in new situations.

Learning procedures for multi-digit computation needs to build from an understanding of their mathematical basis (Fuson and Beckmann 2012/2013; Russell 2000). For example, consider the work in figure 17 by David and Anna, two fourth graders, on a multiplication problem, $57 \times 4 = \square$, and their explanations of what they have done.

David's solution	Anna's solution
$\begin{array}{r} {}^{+2} \\ 57 \\ \times\quad 4 \\ \hline 288 \end{array}$	4×57 $4 \times 50 = 200$ $4 \times 7 = 28$ $200 + 28 = 228$
I multiplied 7 and 4 and got 28. I put down the 8 and carried the 2. Then I added the 2 and the 5 and got 7 and multiplied it by 4 and got 28. I put down the 28 and got 288.	I did it in parts. First I multiplied 4 x 50 and got 200. Then I multiplied 4 and 7 and got 28. Then I just added those two parts together to get the answer.

Fig. 17. David's and Anna's solutions to a multiplication problem.
Adapted from Russell (2000).

David's faulty application of the multiplication algorithm leads to an incorrect answer that he should have recognized as too large (i.e., a reasonable answer must be less than 4×60). Anna's solution, by contrast, shows her understanding that 57 can be partitioned into tens and ones, that each quantity can be multiplied by 4 (an application of the distributive property), and that those new quantities can then be combined.

Similarly, a high school student who does not understand the distance formula,

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

may have trouble accurately recalling it and applying it appropriately to problem situations. By contrast, a student who understands that the formula is an application of the Pythagorean theorem (i.e., the distance between two points can be thought of as the hypotenuse of a right triangle) can use an understanding of this underlying relationship to solve a problem involving the distance between two points correctly (Martin 2009).

Clearly, students need procedures that they can use with understanding on a broad class of problems. This raises questions regarding *how* students can move most effectively toward fluency with general methods or algorithms, as well as what defines an algorithm. Fuson and Beckmann (2012/2013) argue that a standard algorithm is defined by its mathematical approach and not by the way in which the steps in the approach are recorded. They suggest that variations in written notation are not only acceptable but indeed valuable in supporting students' understanding of the base-ten system and properties of the operations. They also emphasize the importance of understanding, explaining, and visualizing: "Standard algorithms are to be understood and explained and related to visual models before there is any focus on fluency" (p. 28).

For example, as figure 18 illustrates, the conventional algorithm for multi-digit multiplication is difficult to understand, whereas the three alternative methods shown are more transparent with respect to the central mathematical features of place-value meanings and properties of the operations (Fuson 2003). The diagrams show the multiplication of tens and ones and the relative size (in area) of the partial products. The accessible algorithm shows a clear record of the four pairs of numbers that are multiplied. This progression also supports students in establishing a basis from which to apply and extend these understandings to operations with rational numbers and algebraic expressions.

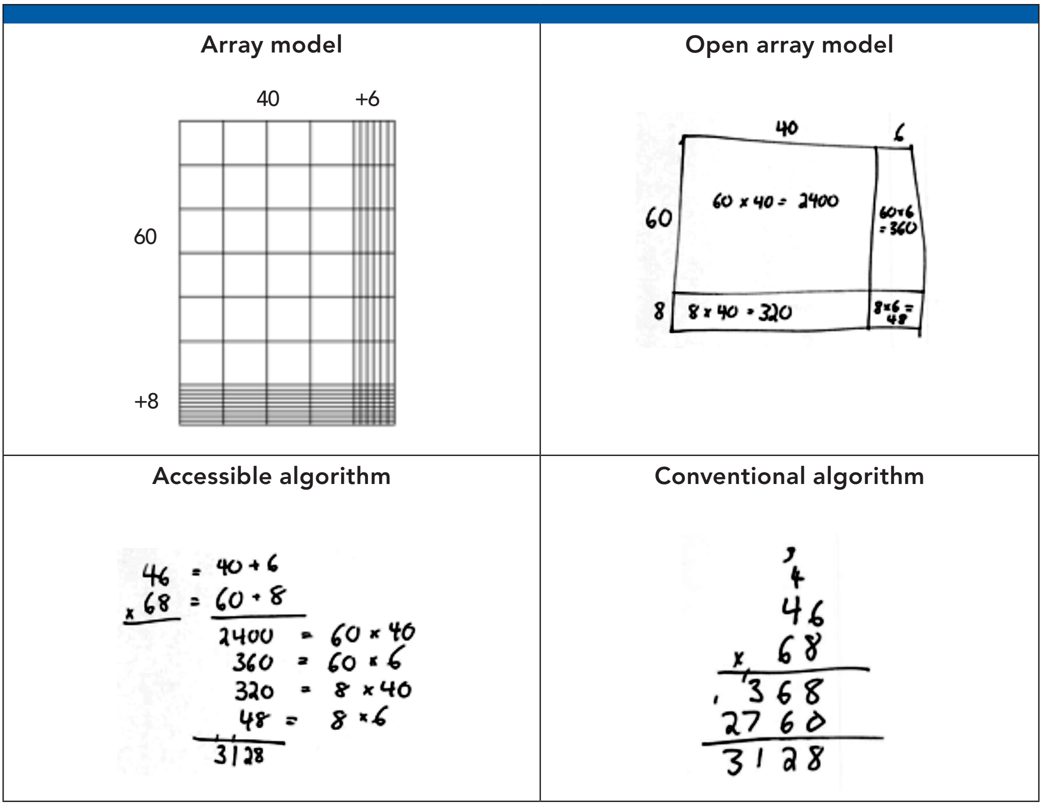


Fig. 18. Methods for multi-digit multiplication, using 68 × 46.
Adapted from Fuson (2003, p. 303).

In moving to fluency, students also need opportunities to rehearse or practice strategies and procedures to solidify their knowledge. However, giving students too many practice problems too soon is an ineffective approach to fluency. Students need opportunities to practice on a moderate number of carefully selected problems after they have established a strong conceptual foundation and the ability to explain the mathematical basis for a strategy or procedure. At that point, providing students with practice on a small number of problems, "spacing" or distributing these over time, and including feedback on student performance support learning outcomes (Pashler et al. 2007; Rohrer 2009; Rohrer and Taylor 2007).

Similarly, practice with basic number combinations should occur after students can explain and justify their use of efficient reasoning strategies. A word of caution is important in regard to timed tests. The premature and overuse of such tests may hinder students' mathematical proficiency and lower their confidence in themselves as learners of mathematics (Boaler 2012; Seeley 2009). Practice with basic number combinations should focus on solidifying students' use of an efficient strategy for specific number combinations (Rathmell 2005; Thornton 1978). Isaacs and Carroll (1999) suggest that practice be brief, engaging, purposeful, and distributed. For example, practice can target specific strategies, such as making a ten for addition or doubling a known fact for multiplication, and can be embedded in problem-solving tasks and games (Crespo, Kyriakides, and McGee 2005).

Illustration

Mr. Donnelly's use of the lesson featuring the Candy Jar task, illustrated in figure 13, is an important step in building his students' fluency in solving problems that involve proportional relationships. Mr. Donnelly helps his students understand that the ratios need to remain constant and that they can use different approaches to preserve this constant multiplicative relationship between the numerator and the denominator. Over time, Mr. Donnelly will need to discuss the efficiency of some strategies over others (e.g., using the scale factor is usually more efficient than scaling up by using a table), and he will need to provide examples of problems that specific strategies would be particularly useful in solving. Ultimately, Mr. Donnelly will want to give his students problems in which neither the unit rate nor the scale factor are integers (e.g., $5/13 = 127/x$) and ask students to devise methods for finding the missing value. Students might generate either of the approaches shown in figure 19, the scale factor method and the unit rate method.

Consider the reasoning that underlies each of these methods and how each is clearly grounded in an understanding of ratio concepts and multiplicative relationships. Mr. Donnelly could then ask his students to consider the generalizability of these approaches as another step toward fluency in solving problems involving proportional relationships.

Teacher and student actions

Effective teaching not only acknowledges the importance of both conceptual understanding and procedural fluency but also ensures that the learning of procedures is developed over time, on a strong foundation of understanding and the use of student-generated strategies in solving problems. This approach supports students in developing the ability to understand and explain their use of procedures, choose flexibly among methods and strategies to solve contextual and mathematical problems, and produce accurate answers efficiently. The actions identified in the table at the right summarize what teachers and students are doing in mathematics classrooms to build procedural fluency from conceptual understanding and problem-solving experiences.

$$\frac{5}{13} = \frac{127}{x}$$

Scale factor method	Unit rate method
JR: $5n = 127$ $n = 25.4$ ← scale factor JB: $13 \cdot 25.4 = 330.2$	$5n = 13$ Unit rate $= 2.6$ So 1 JR $= 2.6$ JB $127 \cdot 2.6 = 330.2$ jaw breakers in the new jar
Student explanation: "The original jar contained 5 Jolly Ranchers, but the new jar contains 127 Jolly Ranchers, so 5 times some number is 127. So, 127 ÷ 5 = 25.4. So, this is the factor that I need to use because the new jar has to have 25.4 times more Jolly Ranchers. Since the original jar had 13 jawbreakers and I need to keep the same ratio, I needed to multiply 13 by the same scale factor, so 13 × 25.4 = 330.2 jawbreakers in the new jar."	**Student explanation:** "The ratio is 5 Jolly Ranchers for every 13 jawbreakers, so 5 times some number is 13. If I distribute the 13 jawbreakers equally among the 5 Jolly Ranchers, 13 ÷ 5 = 2.6, which gives the ratio of 1 Jolly Rancher for every 2.6 jawbreakers, so 2.6 is the unit rate. Since I have 127 Jolly Ranchers, or units, in the new jar, I have to multiply this by the unit rate, so 127 × 2.6 = 330.2 jawbreakers. "Well, 330.2 is the exact answer. But since jawbreakers have to be whole numbers, the answer to problem is 330 jawbreakers."

Fig. 19. Student approaches to the Candy Jar task, leading to general methods

Build procedural fluency from conceptual understanding
Teacher and student actions

What are *teachers* doing?	What are *students* doing?
Providing students with opportunities to use their own reasoning strategies and methods for solving problems.	Making sure that they understand and can explain the mathematical basis for the procedures that they are using.
Asking students to discuss and explain why the procedures that they are using work to solve particular problems.	Demonstrating flexible use of strategies and methods while reflecting on which procedures seem to work best for specific types of problems.
Connecting student-generated strategies and methods to more efficient procedures as appropriate.	Determining whether specific approaches generalize to a broad class of problems.

Build procedural fluency from conceptual understanding Teacher and student actions, *continued*	
What are *teachers* doing?	**What are *students* doing?**
Using visual models to support students' understanding of general methods. Providing students with opportunities for distributed practice of procedures.	Striving to use procedures appropriately and efficiently.

Support Productive Struggle in Learning Mathematics

Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Effective mathematics teaching supports students in struggling productively as they learn mathematics. Such instruction embraces a view of students' struggles as opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions. In contrast to productive struggle, unproductive struggle occurs when students "make no progress towards sense-making, explaining, or proceeding with a problem or task at hand" (Warshauer 2011, p. 21). A focus on student struggle is a necessary component of teaching that supports students' learning of mathematics with understanding (Hiebert and Grouws 2007). Teaching that embraces and uses productive struggle leads to long-term benefits, with students more able to apply their learning to new problem situations (Kapur 2010).

Discussion

In comparisons of mathematics teaching in the United States and in high-achieving countries, U.S. mathematics instruction has been characterized as rarely asking students to think and reason with or about mathematical ideas (Banilower et al. 2006; Hiebert and Stigler 2004). Teachers sometimes perceive student frustration or lack of immediate success as indicators that they have somehow failed their students. As a result, they jump in to "rescue" students by breaking down the task and guiding students step by step through the difficulties. Although well intentioned, such "rescuing" undermines the efforts of students, lowers the cognitive demand of the task, and deprives students of opportunities to engage fully in making sense of the mathematics (Reinhart 2000; Stein et al. 2009). As teachers plan lessons, key components for them to consider are the student struggles and misconceptions that might arise. Thinking about these in advance allows teachers to plan ways to support students productively without removing the opportunities for students to develop deeper understanding of the mathematics.

Mathematics classrooms that embrace productive struggle necessitate rethinking on the part of both students and teachers. Students must rethink what it means to be a successful learner of mathematics, and teachers must rethink what it means to be an effective teacher of mathematics. Figure 20 summarizes one such effort to redefine success in the mathematics classroom (Smith 2000), including expectations for students in regard to what it means to know and do mathematics, and actions for teachers with respect to what they can do to support students' learning, including acknowledging and using struggles as opportunities to learn.

Expectations for students	Teacher actions to support students	Classroom-based indicators of success
Most tasks that promote reasoning and problem solving take time to solve, and frustration may occur, but perseverance in the face of initial difficulty is important.	Use tasks that promote reasoning and problem solving; explicitly encourage students to persevere; find ways to support students without removing all the challenges in a task.	Students are engaged in the tasks and do not give up. The teacher supports students when they are "stuck" but does so in a way that keeps the thinking and reasoning at a high level.
Correct solutions are important, but so is being able to explain and discuss how one thought about and solved particular tasks.	Ask students to explain and justify how they solved a task. Value the quality of the explanation as much as the final solution.	Students explain how they solved a task and provide mathematical justifications for their reasoning.
Everyone has a responsibility and an obligation to make sense of mathematics by asking questions of peers and the teacher when he or she does not understand.	Give students the opportunity to discuss and determine the validity and appropriateness of strategies and solutions.	Students question and critique the reasoning of their peers and reflect on their own understanding.
Diagrams, sketches, and hands-on materials are important tools to use in making sense of tasks.	Give students access to tools that will support their thinking processes.	Students are able to use tools to solve tasks that they cannot solve without them.
Communicating about one's thinking during a task makes it possible for others to help that person make progress on the task.	Ask students to explain their thinking and pose questions that are based on students' reasoning, rather than on the way that the teacher is thinking about the task.	Students explain their thinking about a task to their peers and the teacher. The teacher asks probing questions based on the students' thinking.

Fig. 20. Redefining student and teacher success. Adapted from Smith (2000, p. 382).

Teachers greatly influence how students perceive and approach struggle in the mathematics classroom. Even young students can learn to value struggle as an expected and natural part of learning, as demonstrated by the class motto of one first-grade math class: “If you are not struggling, you are not learning” (Carter 2008, p. 136). Teachers must accept that struggle is important to students’ learning of mathematics, convey this message to students, and provide time for them to try to work through their uncertainties. Unfortunately, this may not be enough, since some students will still simply shut down in the face of frustration, proclaim “I don’t know,” and give up. Dweck (2006) has shown that students with a fixed mindset—that is, those who believe that intelligence (especially math ability) is an innate trait—are more likely to give up when they encounter difficulties because they believe that learning mathematics should come naturally. By contrast, students with a growth mindset—that is, those who believe that intelligence can be developed through effort—are likely to persevere through a struggle because they see challenging work as an opportunity to learn and grow.

The fixed mindset appears to be more prevalent in mathematics than in other subject areas (Dweck 2008). Mindsets, however, can be changed when students realize that they are in control of how they approach and view their own abilities to learn (Blackwell, Trzesniewski, and Dweck 2007). It is important to note that even students who have always gotten good grades may have a fixed mindset. These higher-achieving students are often concerned about how smart they appear to be, so they prefer tasks that they can already do well and try to avoid tasks in which they may make mistakes. Dweck (2008, p. 8) offers important words of caution:

> For the last few decades many parents and educators have been more interested in making students feel good about themselves in math and science than in helping them achieve. Sometimes this may take the form of praising their intelligence or talent and sometimes this may take the form of relieving them of the responsibility of doing well, for example, by telling them they are not a “math person.” Both of these strategies can promote a fixed mindset.

A key message from this research is that teachers must acknowledge and value students for their perseverance and effort in reasoning and sense making in mathematics and must provide students with specific descriptive feedback on their progress related to these efforts (Clarke 2003; Hattie and Timperley 2007). This behavior by teachers may include giving feedback to students that values their efforts at trying varied strategies in solving problems, their willingness to ask questions about specific aspects of the task, or their attempts to be precise in explanations and use of mathematical language. For example, if students need to be more precise in their written or verbal explanations, the teacher could provide feedback that details how their explanations either are, or are not, precise. The result will be the development of students who are more likely to embrace difficulties and uncertainties as natural opportunities in solving problems and maintain engagement and persistence in their mathematics learning. (For an example of a warm-up routine that engages students in an eighth-grade classroom in productive struggle, view “My Favorite No: Learning from Mistakes” [https://www.teachingchannel.org/videos/class-warm-up-routine].)

Illustration

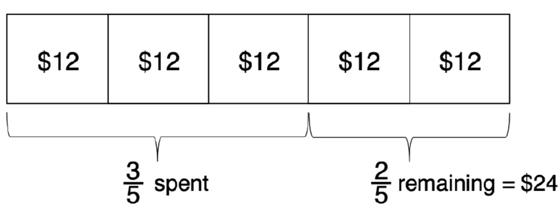
Figure 21 illustrates how two teachers, Ms. Flahive and Ms. Ramirez, present a real-world task related to fractions to two classes of fifth-grade students. In both classrooms, some students are immediately at a loss, upset, and vocal about their feeling that they don't know what to do. The two teachers respond to their students' discomfort in different ways.

Ms. Flahive and Ms. Ramirez teach fifth grade and plan their lessons collaboratively. Their current instructional unit focuses on fractions. They have selected the Shopping Trip task shown below because they think it will be accessible to their students yet provoke some struggle and challenge, since a solution pathway is not straightforward. The mathematics goal for students is to draw on and apply their understanding of how to build non-unit fractions from unit fractions and to use visual representations to solve a multi-step word problem:

> **Shopping Trip Task**
>
> Joseph went to the mall with his friends to spend the money that he had received for his birthday. When he got home, he had \$24 remaining. He had spent $3/5$ of his birthday money at the mall on video games and food. How much money did he spend? How much money had he received for his birthday?

When Ms. Flahive and Ms. Ramirez present the problem in their classrooms, both teachers see students struggling to get started. Some students in both classrooms immediately raise their hands, saying, "I don't get it," or "I don't know what to do."

Ms. Flahive is very directive in her response to her students. She tells them to draw a rectangle and shows them how to divide it into fifths to represent what Joseph had spent and what he had left. She then guides her students step by step until they have labeled each one-fifth of the rectangle as worth \$12, as shown below. Finally, she tells the students to use the information in the diagram to figure out the answers to the questions.



Ms. Ramirez approaches her students' struggles very differently. After she sees them struggling, she has them stop working on the problem and asks all the students to write down two things that they know about the problem and one thing that they wish they knew because it would help them make progress in solving the problem. Then Ms. Ramirez initiates a short class discussion in which several ideas are offered for what to do next. Suggestions include drawing a tape diagram or number line showing fifths, or just picking a number, such as \$50 and proceeding through trial and error. Ms. Ramirez encourages the students to consider the various ideas that have been shared as they continue working on the task.

Fig. 21. Two teachers' responses to students' struggles to solve a multi-step word problem involving fractions

Ms. Flahive wants the students to be successful in figuring out the answer, so she begins to direct their work. Ms. Ramirez resists the temptation to step in but instead supports the students in considering what they know and what they need to figure out. As a result of these different approaches by the teachers to supporting struggling students, the students have very different opportunities to learn. Ms. Flahive's students learn that if you struggle and are vocal about your confusion, the teacher will ultimately tell you what to do; Ms. Ramirez's students learn that if you struggle and are at an impasse, the teacher will provide some assistance—but in the end you have to figure things out for yourself.

Teacher and student actions

Effective mathematics teaching uses students' struggles as valuable opportunities to deepen their understanding of mathematics. Students come to realize that they are capable of doing well in mathematics with effort and perseverance in reasoning, sense making, and problem solving. Teachers provide supports for students, individually and collectively, to work through uncertainties as they grapple with representing a mathematical relationship, explaining and justifying their reasoning, or finding a solution strategy for a mathematical problem. The table below summarizes teacher and student actions that embrace struggle as a natural aspect of learning in the mathematics classroom.

Support productive struggle in learning mathematics **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Anticipating what students might struggle with during a lesson and being prepared to support them productively through the struggle.	Struggling at times with mathematics tasks but knowing that breakthroughs often emerge from confusion and struggle.
Giving students time to struggle with tasks, and asking questions that scaffold students' thinking without stepping in to do the work for them.	Asking questions that are related to the sources of their struggles and will help them make progress in understanding and solving tasks.
Helping students realize that confusion and errors are a natural part of learning, by facilitating discussions on mistakes, misconceptions, and struggles.	Persevering in solving problems and realizing that is acceptable to say, "I don't know how to proceed here," but it is not acceptable to give up.
Praising students for their efforts in making sense of mathematical ideas and perseverance in reasoning through problems.	Helping one another without telling their classmates what the answer is or how to solve the problem.

Elicit and Use Evidence of Student Thinking

Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Effective mathematics teaching elicits evidence of students' current mathematical understanding and uses it as the basis for making instructional decisions. This attention to both eliciting and using evidence is an essential component of formative assessment (Wiliam 2007a). Leahy and colleagues (2005) noted that "teachers using assessment for learning continually look for ways in which they can generate evidence of student learning, and they use this evidence to adapt their instruction to better meet their students' learning needs" (p. 23). A focus on evidence includes identifying indicators of what is important to notice in students' mathematical thinking, planning for ways to elicit that information, interpreting what the evidence means with respect to students' learning, and then deciding how to respond on the basis of students' understanding (Jacobs, Lamb, and Philipp 2010; Sleep and Boerst 2010; van Es 2010).

Discussion

A focus on evidence begins with a clear understanding of what counts as an indicator of students' mathematical thinking (Chamberlin 2005; Sherin and van Es 2003) and requires that teachers attend to more than just whether an answer is or is not correct (Crespo 2000). One source for identifying critical indicators of student thinking is learning trajectories that describe how students' mathematical understanding develops over time (Clements and Sarama 2004; Sztajn et al. 2012). Another source for defining what counts as evidence is common patterns of reasoning that appear in students' thinking, including common difficulties, mistakes, and misconceptions (Swan 2001).

For example, in planning for the task about chairs for the band concert, presented in figure 10, Mr. Harris creates a list of key indicators to notice in his students' work. Specifically, he plans to look for strategies that decompose groups or use the distributive property. He also plans to listen to learn whether students are precise in using concept-based language in discussing their reasoning, such as breaking apart and putting together groups.

The gathering of evidence should neither be left to chance nor occur sporadically. Preparation of each lesson needs to include intentional and systematic plans to elicit evidence that will provide "a constant stream of information about how student learning is evolving toward the desired goal" (Heritage 2008, p. 6). Waiting until the quiz on Friday or the unit test to find out whether students are making adequate progress is too late. Rather, it is important to identify and address potential learning gaps and misconceptions when it matters most to students, which is during instruction, before errors or faulty reasoning becomes consolidated and more difficult to remediate.

Teachers can identify strategic points in each lesson and then plan ways to “check in” on student thinking. One approach is to use high-level tasks to reveal students’ thinking and reasoning. For example, tasks that require students to explain, represent, and justify mathematical understanding and skills provide stronger evidence of their understanding for ongoing assessment and instructional decisions. Another approach is to carefully construct key questions, prior to teaching, to draw out specific understandings, conceptual gaps, or common errors, with the goal of making them visible and accessible for examination and discussion (Bray 2013; Swan 2001; Schifter 2001). For example, in the “focusing” pattern of questions that figure 16 shows for the Coin Circulation task, the teacher asked, “Would I be correct if I said a fifty-cent piece would probably be no more than 19 years old?” The teacher has prepared this question to elicit students’ understanding of the relationship between random samples and generalizability. The teacher might have also elicited useful evidence from more students by having them turn and talk with a partner about the question prior to the whole-class discussion or having all the students respond to it in writing and handing in their responses for further analysis after the lesson.

Finally, teachers must consider how to interpret and respond to what students say, draw, build, or write, as well as attend to the absence of specific evidence. Jacobs and Ambrose (2008) provide several suggestions for ways that teachers might respond to student thinking. For example, to support students, teachers can ask students to restate a problem in their own words, change the problem to use easier numbers, or, when students are unsuccessful with a specific strategy, remind them of other strategies or tools that they have used in the past. To extend student thinking, teachers can have students compare and contrast strategies, try a more advanced strategy to solve the same problem, or solve similar problems with numbers strategically selected to promote more sophisticated strategies. Although there is no single best way to respond to student thinking, the response that the teacher gives should be intended to help students deepen their conceptual understanding while moving them forward, toward procedural fluency and advanced mathematical reasoning.

Illustration

Figure 22 illustrates ways in which how Ms. Lewis, a first-grade teacher, elicits and uses evidence of student thinking. Having noticed that some students seem unsure of the meaning of the equal sign as a symbol of equality, Ms. Lewis wonders whether this uncertainty might be prevalent among her other students as well. Her learning goal for the lesson that the figure shows is to help her students understand more clearly that the equal sign indicates that quantities or expressions “have the same value.” Ms. Lewis observes her students’ different solutions and strategies in their work and probes some of the students’ thinking to learn more about their reasoning, and she uses this information to make adjustments to her instruction.

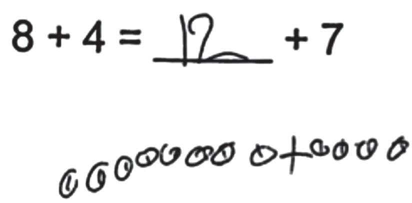
Consider how Ms. Lewis uses the evidence of her students’ thinking throughout the lesson to adjust her instruction in ways that support students in engaging in mathematical discourse about equality and the meaning of the equal sign. In particular, notice how Gabe's thinking about the "number of the day" routine influences both the decisions of the teacher and the reasoning of the students. Then note how Ms. Lewis uses a writing prompt to gather further evidence on what each student understands by the end of the lesson.

Ms. Lewis begins the lesson by asking all the students to work on their own to solve the problem 8 + 4 = ☐ + 7. As the students work, she takes note of the different solutions and strategies in their work and probes some of the students' thinking to learn more about their reasoning.

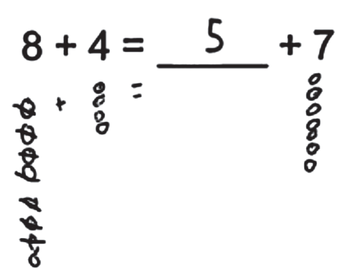
Ms. Lewis notices several different answers, including 12, 5, 19, 11, and 6, so she asks the students to find someone in the class with an answer that is different from their own and compare and discuss their solutions. The conversation is lively as students wonder how there can possibly be so many different answers and whether any of them is even correct. Some students even change their answers as a result of their conversations.

After a few minutes, Ms. Lewis asks the students to bring their papers to the rug so that they can discuss the work as a class. Ms. Lewis asks Maddie to share her work first (shown below on the left). Maddie explains that she didn't know what to do with the 7. The class affirms that the sum of 8 and 4 is 12, and they agree that this fact seems to be an important thing to know in solving the problem.

Gabe presents his work next (shown below on the right). He explains that he thought the total had to be the same on both sides of the equal sign, so he used his drawing to figure out that 5 will make both sides total 12. Ms. Lewis asks him to explain why he thought it might be true that both sides have to have the same total. He said that he thought about how they sometimes write equations that only have one number on the left, like 5 = 2 + 3, or when they write the "number of the day" in different ways without using an equal sign at all. The teacher asks the other students to comment on these ideas. Alex adds that they write the number of the day in different ways to name that number, and he suggests that this case might be something like that. Ms. Lewis asks all the students to turn and talk with a partner about how this task might relate to their previous work when 12 was the number of the day.



Maddie's work



Gabe's work

Fig. 22. Ms. Lewis's eliciting and use of student thinking on the meaning of the equal sign

After some more whole-class discussion, Ms. Lewis asks all the students to return to their seats and take out a piece of paper. She asks them to make up a similar problem on their own and use it to complete this sentence starter, "The equal sign means that ________." The students find partners to review their work, then they make revisions to it, and finally the teacher collects the work to analyze it further and consider her next instructional steps.

Fig. 22. *Continued*

Teacher and student actions

Effective teaching involves finding the mathematics in students' comments and actions, considering what students appear to know in light of the intended learning goals and progression, and determining how to give the best response and support to students on the basis of their current understandings. Teachers also use the evidence gathered after the instructional session to reflect on the lesson and student progress and then identify next steps in planning future lessons and designing interventions. The actions in the table below summarize what teachers and students are doing in mathematics classrooms that use evidence of student thinking to assess, support, and extend learning.

Elicit and use evidence of student thinking **Teacher and student actions**	
What are *teachers* doing?	**What are *students* doing?**
Identifying what counts as evidence of student progress toward mathematics learning goals. Eliciting and gathering evidence of student understanding at strategic points during instruction. Interpreting student thinking to assess mathematical understanding, reasoning, and methods. Making in-the-moment decisions on how to respond to students with questions and prompts that probe, scaffold, and extend. Reflecting on evidence of student learning to inform the planning of next instructional steps.	Revealing their mathematical understanding, reasoning, and methods in written work and classroom discourse. Reflecting on mistakes and misconceptions to improve their mathematical understanding. Asking questions, responding to, and giving suggestions to support the learning of their classmates. Assessing and monitoring their own progress toward mathematics learning goals and identifying areas in which they need to improve.

Moving to action

Although the important work of teaching is not limited to the eight Mathematics Teaching Practices discussed in this chapter, this core set of research-informed practices is offered as a framework for strengthening the teaching and learning of mathematics. The next steps involve educators in collectively and collaboratively supporting one another in moving toward improved instruction through the lens of these core teaching practices. Effective teaching of mathematics begins with teachers clarifying and understanding the mathematics that students need to learn and how it develops along learning progressions. The establishment of clear goals supports the selection of tasks that promote reasoning and problem solving while developing conceptual understanding and procedural fluency. With effective teaching, the classroom is rich in mathematical discourse among students in using and making connections among mathematical representations as they compare and analyze varied solution strategies. The teacher carefully facilitates this discourse with purposeful questioning. Teachers acknowledge the value of productive struggle in learning mathematics, and they support students in developing a disposition to persevere in solving problems. They guide their teaching and learning interactions by evidence of student thinking so that they can assess and advance student reasoning and sense making about important mathematical ideas and relationships.