Chapter 1
Algebra

The word *algebra* can be traced back to the year 830, when an astronomer named Mohammed ibn Musa al-Khowârizmi wrote a treatise entitled *Al-jabr w'al muqâbala.* Originally from the Arabic *al-jabr*, algebra meant "restoring" (that is, balancing) an equation, and *al-muqâbala* meant "simplification," as in combining like terms (Kline 1972). Although algebra has taken on various nonmathematical meanings throughout the centuries, today we find it as an important component of the curriculum. The development of algebraic thinking begins informally as early as the primary grades (Curcio and Schwartz 1997; NCTM 2008), and as students progress through the grades, a formal treatment of algebra occurs through pattern recognition, generalizations, variables, and functions.

The high school algebra standards are partitioned into four domains: (1) seeing structure in expressions, (2) arithmetic with polynomials and rational expressions, (3) creating equations, and (4) reasoning with equations and inequalities. As mentioned in the Preface, we employ two approaches in this book to develop mathematical problem-solving skills and support the Common Core State Standards. First, we present rich problems that give a starting point for lessons; and second, we provide a series of expressions and questions that are designed to allow mathematical ideas to emerge. In this chapter, you will find tasks to support all four domains of the algebra standards: seeing structure in expressions (tasks 1.1 and 1.2), arithmetic with polynomials and rational expressions (tasks 1.3 and 1.4), creating equations and modeling (task 1.5), and reasoning with equations and inequalities (tasks 1.6, 1.7, and 1.8). The eight Standards for Mathematical Practice (MP) (as listed on page vi) are interwoven throughout these domains. For each task, we will discuss those standards that are most relevant to the problem at hand.

Seeing Structure in Expressions

The Seeing Structure in Expressions domain of the high school algebra standards encourages students to solve problems using the structural attributes of mathematical expressions. An examination of the relationships among the graphic, algebraic, and concrete area models of quadratic expressions of the form $ax^2 + bx + c$, and of related equations $ax^2 + bx + c = 0$ or $y = ax^2 + bx + c$, provides an opportunity for deep understanding of their underlying mathematical structure. For example, when a quadratic expression is factorable, it can be modeled as a rectangle using algebra tiles (fig. 1.1), and the graph of the associated parabola will intersect the x-axis at rational points (fig. 1.2). If it is a perfect square trinomial, it can be modeled as a square using algebra tiles (fig. 1.3), and its associated parabola will be tangent to the x-axis (fig. 1.4). If it is not factorable, it cannot be modeled by a rectangle (fig. 1.5), and its associated parabola will either not intersect the x-axis or will intersect at irrational values (fig. 1.6).

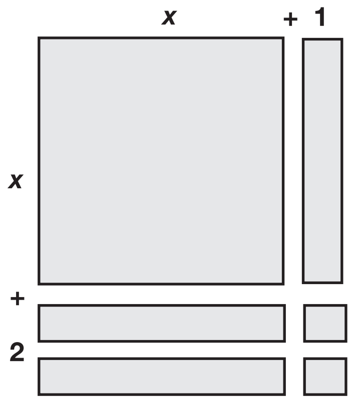

Fig. 1.1. How we can factor $x^2 + 3x + 2$ into $(x + 1)(x + 2)$

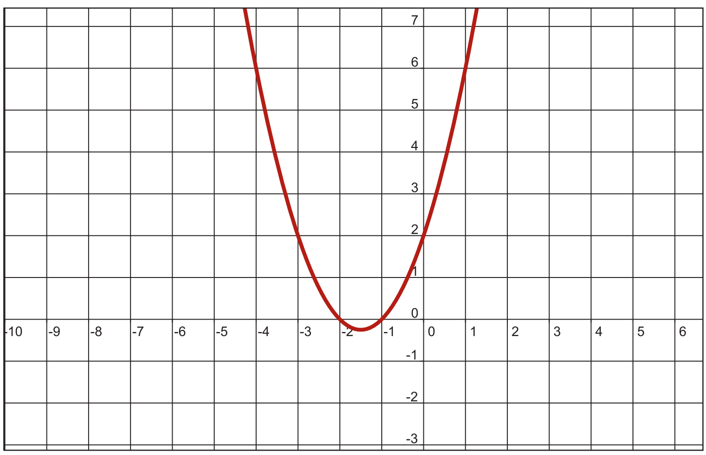

Fig. 1.2. The graph of $y = x^2 + 3x + 2$

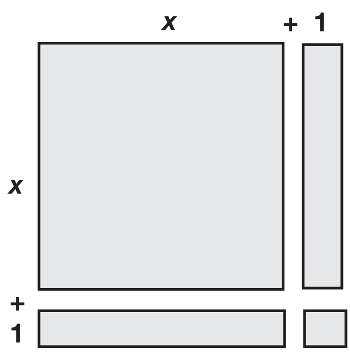

Fig. 1.3. How we can factor $x^2 + 2x + 1$ into $(x + 1)(x + 1)$ or $(x + 1)^2$

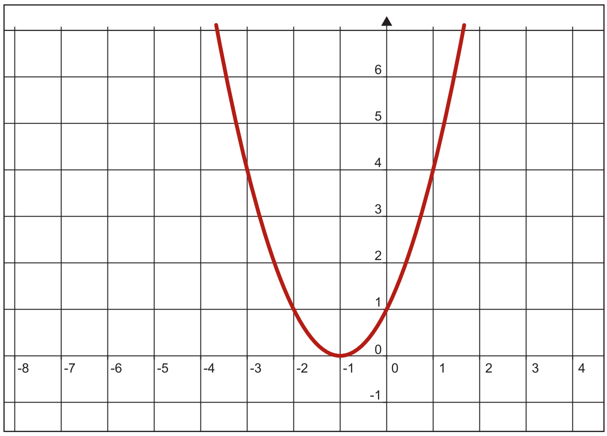

Fig. 1.4. The graph of $y = x^2 + 2x + 1$

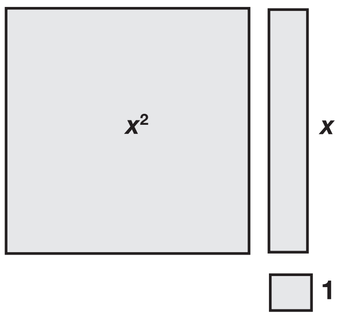

Fig. 1.5. The pieces of $x^2 + x + 1$ cannot be combined to make a rectangle, so $x^2 + x + 1$ is not factorable.

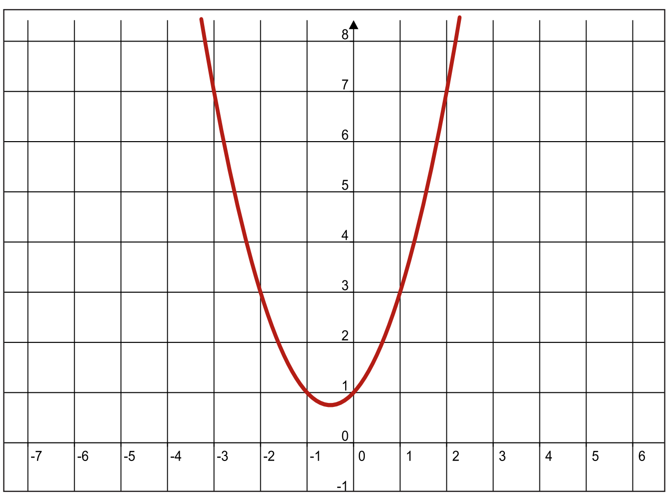

Fig. 1.6. The graph of $y = x^2 + x + 1$

Task 1.1 is intended as an introduction to these ideas, and it addresses standard A-SSE.2: "Use the structure of an expression to identify ways to rewrite it" (CCSSI 2010, p. 64). Specifically, students are asked to make observations about the resulting polynomial and its coefficients when carefully selected binomials are multiplied, leading to insight about factoring certain polynomial expressions. We encourage using the language of "distributive property" or "distribution" to describe the procedure of multiplying binomials rather than the commonly used but easily misinterpreted FOIL (First, Outer, Inner, Last) acronym. The task includes five sets of examples so that teachers may use a cooperative learning approach, but teachers may decide that fewer sets are necessary. If a cooperative learning approach is used, groups can each be given a different set, and teachers can help students compile and compare their results.

Task 1.1

(*a*) Multiply each pair of factors. How are the terms in the resulting polynomial related to the terms in the factors? Make at least two conjectures.

Set 1: $(x + 3)(x + 4)$; $(x + 3)(x - 4)$; $(x - 3)(x + 4)$; $(x - 3)(x - 4)$

Set 2: $(x + 1)(x + 5)$; $(x + 1)(x - 5)$; $(x - 1)(x + 5)$; $(x - 1)(x - 5)$

Set 3: $(x + 3)(x + 3)$; $(x + 3)(x - 3)$; $(x - 3)(x + 3)$; $(x - 3)(x - 3)$

Set 4: $(x + 10)(x + 8)$; $(x + 10)(x - 8)$; $(x - 10)(x + 8)$; $(x - 10)(x - 8)$

Set 5: $(2x + 3)(x + 5)$; $(2x + 3)(x - 5)$; $(2x - 3)(x + 5)$; $(2x - 3)(x - 5)$

(*b*) Test your conjectures by writing another set of binomials. See whether you can accurately predict the product of each pair of binomials in your set.

(*c*) What factors were multiplied to result in each of the polynomials below? Explain how the observations you made in part (*a*) would help you to figure this out.

$x^2 + 6x - 7$

$x^2 - 6x - 7$

$2x^2 + 15x + 7$

When assigning the problem in task 1.1, teachers might consider using a recording sheet so that students can make comparisons within and across examples. The design of this problem reveals the structure of expressions by allowing for a comparison of the binomials to the resulting trinomial. Students must carefully examine the structure of the terms of the binomials and the trinomial to accurately determine their relationship. Students should notice that when multiplying $(x + a)(y + b)$ the x-coefficient of the resulting polynomial is $a + b$ and the constant is ab. Teachers and students can discuss the more complicated relationship that results when the leading coefficient is not equal to one.

This approach can be extended to polynomials that are the difference of two perfect squares and trinomials that are squares of binomials. For example, several sets of binomials of the form $(x + a)(x + a)$; $(x + a)(x - a)$; $(x - a)(x + a)$; and $(x - a)(x - a)$ might be given to students, leading to factoring such polynomials as $x^2 - a^2$ or $x^2 + 2ax + a^2$.

A more concrete approach is presented in task 1.2 below, where students are asked to "make rectangles" out of the concrete representations of polynomials, revealing the relationship between factorable quadratics and the existence of a concrete rectangular model of a quadratic. Students should have some familiarity with the structure of algebra tiles before attempting this task. Note that we have only included positive coefficients because it is often counterproductive to spend time on the somewhat cumbersome representation of negative terms using a concrete area model.

Task 1.2

Using algebra tiles, form a rectangle to represent each of the following expressions, and then answer the questions that follow.

(1) $x^2 + 7x + 10$

(2) $x^2 + 11x + 10$

(3) $x^2 + 7x + 12$

(4) $x^2 + 7x + 16$

(5) $x^2 + 11x + 5$

(*a*) In which cases were you able to model the trinomial using a rectangle? Explain why you think certain trinomials did not "work."

(*b*) Factor each trinomial above. If a trinomial is not able to be factored, explain why.

(*c*) Create a trinomial that can be modeled using a rectangle, and create a trinomial that cannot be modeled using a rectangle. Make a conjecture about the factors of each of your trinomials.

(*d*) Create a trinomial that can be modeled using a square. What do you notice about the factors of this trinomial?

STANDARDS *for Mathematical Practice—Tasks 1.1 and 1.2*

Several of the CCSS Standards for Mathematical Practice (in particular, standards 1, 4, 6, and 7) are addressed by these two problems, which first ask students to conjecture, then to test, and eventually to apply what they have discovered.

MP.1

The structure and nature of these problems allow students to make sense of the mathematics, and through the use of concrete models, persevere in solving them. Students analyze given information, make and test conjectures, and consider analogous problems of their own making, thereby "monitor[ing] and evaluat[ing] their progress" in solving each problem (CCSSI 2010, p. 6).

MP.4

Although students are not modeling a real-life situation, task 1.2 requires that they model a mathematical expression using an area model, which allows for insight into the underlying structure of the mathematics. The structure of this task allows students to "analyze relationships mathematically to draw conclusions," in this case regarding the structure of the expressions (CCSSI 2010, p. 7).

MP.6

When multiplying the binomials in task 1.1, students must make sure their calculations are correct, and thus the problem supports the sixth Standard for Mathematical Practice, "Attend to precision" (CCSSI 2010, p. 7). If the factors are not multiplied carefully, erroneous conjectures will result. Indeed, there is a self-checking feature built into the problem. If students make an error and a product does not seem to follow the same pattern as the others, it is hoped that they would find the error.

MP.7

These tasks support development of mathematically proficient students by requiring them to "look closely to discern a pattern or structure" in the sets of binomials and their products in task 1.1, thus addressing the seventh Standard for Mathematical Practice, "Look for and make use of structure" (CCSSI 2010, p. 8). Task 1.2 requires that students examine how the coefficients relate to whether the trinomial can be modeled by a rectangle. As recommended by this standard, "They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects" (CCSSI 2010, p. 8). The structure of the tasks requires students to compare equivalent expressions in different forms (e.g., a trinomial can be expressed as the product of two binomials).

Arithmetic with Polynomials and Rational Expressions

This domain of the high school algebra standards requires that students perform arithmetic operations on polynomial and rational expressions. The following problem addresses standard A-APR.7: "Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions" (CCSSI 2010, p. 65). Specifically, task 1.3 requires that students interpret a real-life situation that leads to a mathematical model in which two rational expressions need to be combined using an arithmetic operation. Students are required to express the result in simplest form to ensure that the arithmetic operation is performed on the rational expression.

Task 1.3

Ms. A's class has decided to buy her a gift. They found the perfect item at the store, and it costs G dollars. The students in the class agree to split the cost equally. The next day, ten students decide that they cannot contribute. Write an algebraic expression for the additional amount the other students had to contribute to buy the same gift.

(*a*) What quantities do we know? How might they be represented? Which need to be represented by a variable?

(*b*) Decide which quantities need to be represented by an algebraic expression, and write the expressions.

(*c*) Write an expression for the difference in price. Make sure it is in simplest form.

A teacher may elect to give the above problem without the scaffolding provided in parts (*a*), (*b*), and (*c*). Students might be asked to think about the questions that could be asked about this problem in order to solve it. This would require careful monitoring of students as they work, with teachers prompting student thought through questioning, including a discussion of the minimum number of students that must be in the class. Another modification of the problem would be to give a numerical value for the cost of the gift, the number of students in the class, or both. This simpler version of the problem could precede either version of the problem as given above. If a teacher elects to give specific values, students could be given, or be encouraged to construct, a recording table or chart to organize their findings.

The algebra standards also require that students perform arithmetic operations on polynomials. The problem that follows has an open-ended approach that allows many—in fact, infinitely many—answers to the questions, and it encourages students to critique the reasoning of others. Task 1.4 addresses cluster A-APR, "Perform arithmetic operations on polynomials" (CCSSI 2010, p. 64).

Task 1.4

The perimeter of a rectangle is represented by $4x + 24$. In response to the question, "What is a possible representation of the area of this rectangle?" Mark says, "$x^2 + 36$," and Sarah says, "No, $x^2 + 12x + 36$." What assumptions are both Mark and Sarah making about the rectangle? Comment on their answers. Other students are also discussing the problem. Alex says the area is $24x$. Anna says that she thinks the area is $x^2 + 12x + 27$. Where do you think they are getting these answers? Can they all be correct?

This problem has many correct answers and can lead to a rich discussion of the relationship between perimeter and area alongside the discussion of polynomials. An extension of this problem and/or an opportunity for differentiated instruction is to have students try to find different possible areas that have not yet been mentioned, including those that meet specific requirements, such as another that is a monomial (another is actually not possible), binomial, and trinomial. It is more challenging to find three different areas that are trinomials than ones that are binomials. Furthermore, teachers can change the initial given perimeter in ways that make the problem more or less challenging. For example, an initial perimeter of $3x + 7$ might be very challenging when students realize that rational expressions must be involved. Finally, students might be given an area that is impossible given the constraints of the problem: "Is the area $3x^2 + 72x$ possible? Why or why not?"

The problem can be extended by using various figures, such as right triangles or right trapezoids. The added challenge for these problems is that not all arrangements of sides "make sense," and that some measures are not used when calculating area.

STANDARDS *for Mathematical Practice—Tasks 1.3 and 1.4*

In order to solve the problem in task 1.3 successfully, students must carefully analyze the information given and make decisions regarding the approach they will use, including whether to use specific information (e.g., a numerical quantity for the price or number of students) to make sense of the general form. In the problem, students are told that the price of the gift can be represented by G, but need to define a variable themselves to represent the number of students in the class. For the purposes of this discussion, the number of students in the class will be represented by N. The problem in task 1.3 supports several of the Standards for Mathematical Practice (i.e., standards 1, 3, 4, and 7).

MP.1 and MP.4

Similar to tasks 1.1 and 1.2 in the previous section, task 1.3 supports the goal for mathematically proficient students to "make sense of problems and persevere in solving them" (CCSSI 2010, p. 6), by creating a situation where students need to explain "to themselves the meaning of a problem and look for entry points to its solution" (CCSSI 2010, p. 6). This is particularly true in the less scaffolded approach to the problem. Students must also be comfortable with modeling each student's contribution to the cost of the gift. With all of the students participating, each contribution is G/N, and with ten fewer students participating the cost per student becomes $G/(N-10)$. The additional amount each student would need to pay is thus modeled by the difference $G/(N-10) - G/N$. When expressing the difference in simplest form, students must find a common denominator in order to combine the fractions. Teachers can assign a specific value for either one of the variables in the problem, G or N. For example, if we allow the number of students in the class to be, say, $N = 25$, then the difference is $G/15 - G/25$, allowing for finding a common denominator without variables, but still requiring the manipulation of the variable in the numerator.

If, however, teachers elect to give a specific value for the price of the gift, this would allow for a numerical value in the numerator of each fraction, while requiring students to find the common denominator involving variables. For example, if the price of the gift is, say, $G = 50$, and there are N students, the difference is $50/(N-10) - 50/N$, requiring students to find the common denominator of $N - 10$ and N. The rational expressions above are all mathematical models of the situation, addressing the fourth standard for mathematical practice. What begins as a real-life situation, one that many students may have once been in themselves, leads to a mathematical model that is a rational expression.

MP.3

In order to respond successfully to this question, students must first analyze the problem, realize that Mark and Sarah are assuming that the rectangle is a square, identify the error in Mark's multiplication, and then construct a response critiquing the incorrect answer and explaining the correct one. While Mark and Sarah's assumption that the rectangle is square is not incorrect, it certainly is not a requirement, and students must realize that other rectangles must be considered, as in the responses of Alex and Anna. Students must then realize not only that the answers provided by Alex and Anna are both mathematically correct, but that there are many (in fact, infinitely many) additional possible expressions of the area of the rectangle. In solving this problem students should be constructing viable arguments and critiquing the reasoning of others as emphasized in MP.3, part of which states: "Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is" (CCSSI 2010, p. 7). In explaining that both answers are correct, and that there are other correct ones, students develop mathematical proficiency.

MP.7

This task supports the development of mathematically proficient students "who look for and make use of structure" in mathematics (CCSSI 2010, p. 8). In comparing how to add the rational expressions in the task to adding fractions with numerical values encountered in earlier grades, students must make use of the related but more complicated structure of the rational expressions, including finding a common denominator. In particular, finding the common denominator when the denominators are numerical values (i.e., when the number of students in the class is a specific value) can be compared to doing so when the number of students is represented by a variable, allowing for a structural comparison of finding a common denominator when dealing with constants to finding one when working with variables in the denominator.

Task 1.4 not only supports the development of mathematically proficient students in a similar way to task 1.3, but it also provides an opportunity for students to "construct viable arguments and critique the reasoning of others" (CCSSI 2010, p. 6).

Creating Equations

This domain of the high school algebra standards addresses students' ability to "create equations that describe numbers or relationships" (CCSSI 2010, p. 65). Task 1.5 addresses standards A-CED.2 and A-CED.3 which challenge students to "Create equations in two or more variables to represent relationships between quantities," and "Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context" (CCSSI 2010, p. 65). Specifically, task 1.5 asks students to interpret a situation involving a business that is modeled by a quadratic function. Students are required to analyze the constraints of the problem, determine appropriate domain and range values for the function, and, if using technology, to determine an appropriate window to graph the function. A recording sheet might be provided to students, or students might be encouraged to determine their own way to organize the data. Note that the numbers are quite large by design, so that the problem is realistic.

Task 1.5

Mr. Yates makes and sells 1,000 of his new JPads per week at a cost of 350 dollars per unit. Because the demand is high, he has decided to raise the price of the JPad, but he is only considering raises in five-dollar increments. Market research has shown that for each five-dollar rise in the price, ten fewer customers are expected to buy the JPad. Thus, if the price is 355 dollars per JPad (an increase of only one five-dollar increment) only 990 customers are expected to buy the item (ten fewer than 1,000). If the price of the JPad is set at 360 dollars (going up two five-dolllar increments), then only 980 customers will buy it. Assuming that the market research is correct: