

Mathematical Notations and Procedures of Recent Immigrant Students

MATHEMATICS IS OFTEN REFERRED TO AS A universal language. Compared with the differences in language and culture faced by students who are recent immigrants, the differences in mathematical notation and procedures seem to be minor. Nevertheless, immigrant students confront noticeable differences between the way that mathematical ideas are represented in their countries of origin and the manner that they are represented in the United States. If not addressed, the differences in notation and procedures can add to the difficulties that immigrants face during their first years in a new country.

Exposing teachers to these differences expands their repertoires and gives them an appreciation of their students' previous experiences and struggles. For example, the statement $59 : 8 = 7 + 3 : 8$ might cause a teacher in the United States to pause until he or she realizes that the colon can also denote division; the statement is another way to express the conversion of $59/8$ to $7 \frac{3}{8}$. In the same way, an immigrant student might hesitate when faced with unfamiliar notation. Confusion may also arise when parents who were schooled in other countries try to help their children by using procedures that are different from those taught in the United States (Ron 1998).

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Teachers are often unaware that an immigrant student is confused or has a question, because the traditions of many of these students discourage questioning the teacher; students may need several weeks before they feel confident enough to ask questions. The teacher can help to include immigrant students by introducing alternative algorithms with the preface “In other countries, you might see this problem done in the following way.”

As teachers encounter algorithms taught in other countries, they realize that the algorithms that they have learned are just some of the possible ways to compute answers. This realization can help teachers become more accepting when students deviate from the procedures or algorithms taught in class and use their own procedures.

This article describes some differences in representations of mathematical concepts and procedures that recent immigrants from Latin America face in schools in the United States. We do not attempt to describe all the differences that immigrants face in school mathematics. Rather, our purpose is to help students and teachers be aware of those differences and use them to the advantage of the students. Of course, immigrants from other parts of the world will also bring their own notations and procedures, and the teacher can include those conventions, too.

Most of the differences reported here were collected over a period of three years by the first author from her students, who included recent immigrants. The second author, schooled in Mexico, provided additional examples. We first discuss notational differences, then examine algorithmic differences; we finish with some considerations for success of recent immigrants in their new school systems.

Notational Differences

Numbers

The same symbols for numbers are used in Latin America and the United States, but some differences exist in the way that numbers are written, in the names that are read for numbers, in the use of the decimal point, and in the separation of digits in large numbers.

In writing numerals, immigrant students may put a crosshatch in the number 7 to distinguish it from 1. Most people in the United States recognize this character, although some say that the 7 looks like a handwritten F. The problem for immigrants is that they cannot always distinguish a 1 or a 7 written by someone from the United States. See **table 1** for a summary of differences in handwritten numerals.

We can also find differences in the ways in which numbers are read in Latin America and the United States. Both systems are identical through the mil-

TABLE 1
Differences in Handwritten Numerals

UNITED STATES	LATIN AMERICA	COMMENTS
7	7	Sometimes the numeral 7 is drawn from the bottom up.
8	8	The numeral 8 is often drawn from the bottom up.
4	4	The numeral 4 is sometimes drawn from the bottom up. Students may confuse 4s and 9s.
9	9	The numeral 9 may resemble a lowercase g, particularly when written by Cuban students.

lions; they differ, however, at the point that textbooks in the United States identify as *billions*. Both a student in the United States and one from Latin America will read the number 782,621,751 as “782 million, 621 thousand, 751.” Students from Latin America, and some other countries, such as Great Britain, however, will not designate *billions* until a number has at least thirteen digits. The number 23,500,000,000,000 is read by a student in the United States as “23 trillion, 500 billion” and by a Latin American as “23 billion, 500 thousand million” (Serralde et al. 1993). A student schooled in the United States will read 10,782,621,751 as “10 billion, 782 million, 621 thousand, 751.” In some students’ countries of origin, the number is read as “10 mil 782 *millones*, 621 mil, 751”; or it is read as “10 thousand 782 million, 621 thousand, 751.”

In the United States, people separate numbers into groups of three digits by commas. In some countries of Latin America, the point is used to separate such groups. Consequently, the number 10,752,101 is equivalent to 10.752.101 (Secada 1983). Other textbooks (Serralde et al. 1993) leave a space between groups of three digits and write 10 753 101. In Mexico, a third convention is used (Secretaría de Educación Pública 1993): Millions are separated by an apostrophe, and multiples of thousands are separated by commas, as in 10'752,101. The semicolon is also used in Mexico to separate millions from thousands, as in, for example, 1;958,201.

In Mexico, negative numbers can be expressed in two ways: with a preceding negative sign (–2) or with a bar over the number (2̄). Some students who are accustomed to this last notation have problems with the bar for repeating decimal fractions used in the United States. For example, the repeating decimal 0.3333333 . . . is designated as 0.3̄. Some Mexican textbooks indicate a repeating decimal with an arc, for example, 0.3̂ (Beristáin and Campos 1993).

Measurement

Immigrants, most of whom are accustomed to the metric system, in which all measures are interrelated, find the units of measure used in the United States confusing. Divisions in the metric system are based on powers of ten, but the English system does not have consistent subdivisions; for example, 12 inches is equal to 1 foot and 3 feet is equal to 1 yard. Immigrant students from Latin America may also find that textbooks in the United States use linear metric measures (such as the decameter and the hectometer) that are seldom used in their countries of origin.

Symbols

A dot or point may be used in decimal notation and to represent multiplication in the United States. Differences in placement of the point may create several problems for immigrant students. In designating decimal fractions, the point rests on the line and between the numbers. The number 2.54, for example, refers to 2 whole parts and 54 hundredths. Some

countries in Latin America use the comma to separate the fractional parts from the whole. The decimal number 2.54 would be written as 2,54 (Secada 1983). In the United States, the point is also used to indicate multiplication. In that use, it is placed in the middle of the line and between the given numbers; $2 \bullet 54$ means 2 times 54. A bolder or larger raised point, such as in $2 \bullet 54$, is also used in Mexico to indicate multiplication (Almaguer et al. 1994). In other countries, the dot used to multiply is placed in the lower position and between the given numbers; the notation 2.54 indicates the product of 2 and 54, not a decimal fraction. Because of these differences, students may interpret the notation 3.789 in a variety of ways. Students may assume that the notation refers to

the number three thousand, seven hundred, eighty-nine, deleting the point in the belief that it is merely a “spacer” between thousands. Others may assume that the notation calls for multiplying 3 and 789 or that it is the decimal expression of 3 whole parts and 789 thousandths of another.

Four different symbols are used to denote division in Mexican textbooks, namely, \div , $/$, $:$, and $\overline{\hspace{1cm}}$. All four symbols are used in the United States, but some confusion may arise with the use of the colon. In the

United States, the colon is seen primarily in ratios and proportions, but in Latin America, it is also used to designate division. The division of the fractions

$$\frac{3}{4} \text{ and } \frac{3}{5}$$

might be written as

$$\frac{3}{4} : \frac{3}{5};$$

the division of 16 by 2 can be written as $16 : 2 = 8$. Similarly, the equation $4 : 5 = 8 : x$ has the solution $x = 10$ (Serralde et al. 1993).

Another notation difference is that used in angles. Textbooks in the United States might note angles in one of several ways:

$$\angle \alpha \quad \angle ABC \quad \angle 1$$

Some Mexican textbooks write the same angles as follows:

$$\hat{\alpha} \quad \widehat{ABC} \quad \hat{1}$$

In addition, some immigrant students may try to write the angle symbol as the teacher directs, but they make the symbol in the manner that they were previously taught and place it on the side. The result often looks like this:

$$<\alpha \quad <ABC \quad <1$$

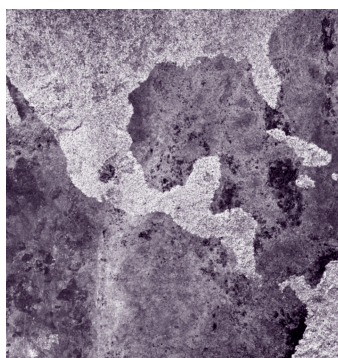
Summary of Notational Differences

THE CONFUSION IS APPARENT WHEN STUDENTS begin to work with inequalities. The symbols look the same to them as those for angles do. Students may wonder why they are doing operations on angles, whereas the teacher may ask why the students are writing the angles as if they were inequalities.

Teachers need to point out notational differences before the beginning of any lesson. Doing so not only validates the experiences of the students but also reminds them that understanding differing systems enriches their knowledge and establishes the idea that mathematics can be examined and represented in different ways.

Algorithmic Differences

IMMIGRANT STUDENTS OFTEN LEARN DIFFERENT algorithms for operations. Frequently, teachers know only the algorithms that they were taught and are unaware of alternative methods. In some coun-



Point out notational differences before the lesson begins

tries, intermediate steps in a procedure are computed mentally. Many recent immigrants take pride in the ability to compute quickly and accurately in their heads. As a result, an immigrant student may merely write the answer to a problem and omit the intermediate steps in his or her written work. Teachers who are unfamiliar with the student's background might wrongly assume that the student has copied another's work.

Subtraction

One subtraction algorithm taught in Mexico is based on the fact that the difference will not change if the same number is added to both subtrahend and minuend. For example, if 10 is added to both numbers, as ten units in the minuend and as a unit in the tens place in the subtrahend, the difference remains the same (see **fig. 1**). This method transforms the problem $42 - 19$ into $(40 + 12) - (20 + 9)$, which is solved by two easy subtraction steps, $12 - 9$ and $40 - 20$. This algorithm eliminates the need to "borrow" from the column to the left when the number at the top of a column is smaller than the number below. Rather, the student adds ten units to one column in the number on top and, in the next step, adds one unit to the column to the left of the number below to compensate (see **table 2**). The reaction of many teachers in the United States is that the student does subtraction backward.

Division

Division is done in a similar manner in the United States and Mexico, but more steps are written in long division, the version taught in the United States. In the Mexican division algorithm, numbers are multiplied and subtracted mentally, and these steps are not written down. Often, immigrant students take pride in being able to do intermediate steps mentally. Some of them consider writing all the steps in long division to be an exercise for younger children: "*Nomás para los niños en la primaria, maestra*" (Only for the children in the elementary school). The long-division algorithm for $126 \div 3$ is illustrated in the left column of **figure 2**. In the shorter version, the subtractions are done mentally. The steps are illustrated in the middle column of **figure 2**. The third column of **figure 2** shows another way of writing the steps that students learn in such countries as Honduras and Cuba.

Parentheses and distributive property

Students in the United States are taught to do all work inside the parentheses before any other op-

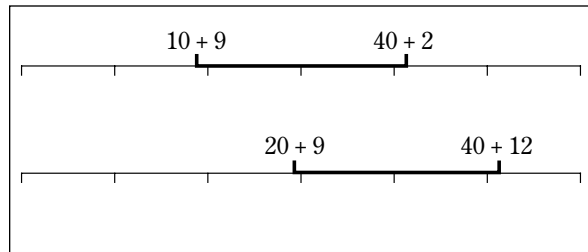


Fig. 1 Constant difference

TABLE 2

Subtraction Algorithm Based on Missing Addend

WRITTEN FORM	THOUGHT PROCESS
$\begin{array}{r} 542 \\ -269 \\ \hline 3 \end{array}$	9 from 12? 3. (Notice that ten units were added mentally to the upper number to convert the 2 into a 12.) Write the 3, and add 1 (ten) mentally to the 6 (tens). (Notice that we add 1 ten to the lower number in a different column.)
$\begin{array}{r} 542 \\ -269 \\ \hline 73 \end{array}$	7 (tens) from 14 (tens)? 7 (tens). Write down the 7 (tens), and mentally add 1 (hundred) to the 2 (hundreds) in the next column.
$\begin{array}{r} 542 \\ -269 \\ \hline 273 \end{array}$	3 (hundreds) from 5 (hundreds)? 2 (hundreds). Write down the 2 (hundreds) to complete the problem.

LONG DIVISION	SHORT DIVISION	ANOTHER FORM OF SHORT DIVISION
$\begin{array}{r} 42 \\ 3 \overline{)126} \\ \underline{-12} \\ 06 \\ \underline{-6} \\ 0 \end{array}$	$\begin{array}{r} 42 \\ 3 \overline{)126} \\ 06 \\ 0 \end{array}$	$\begin{array}{r} 126 \overline{)3} \\ 06 42 \\ 0 \end{array}$

Fig. 2 Three ways to divide 126 by 3

erations. The expression $2(3 + 5 - 2)$ is evaluated by doing $3 + 5 - 2$ first, then multiplying the resulting 6 by 2. In Mexican textbooks, the expression is evaluated by using the distributive property, $2(3 + 5 - 2) = 2 \cdot 3 + 2 \cdot 5 - 2 \cdot 2 = 6 + 10 - 4 = 12$. The two procedures are mathematically equivalent, but to a learner, they can be confusing if the connection between the two is not made explicit.

Prime factorization

Textbooks in the United States generally use a factor tree to find prime factors systematically. Mexican textbooks use a vertical line to accomplish the

140		2
70		2
35		5
7		7
1		

Fig. 3 Factorization in Mexico

12	18	2	(prime common factor of 12 and 18)
6		2	(prime factor of 12)
3	9	3	(prime common factor of 12 and 18)
1	3	3	(prime factor of 18)
	1		

Fig. 4 Prime factors of 12 and 18

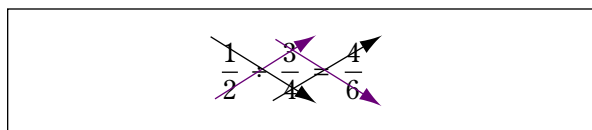


Fig. 5 Mexican algorithm for division of fractions

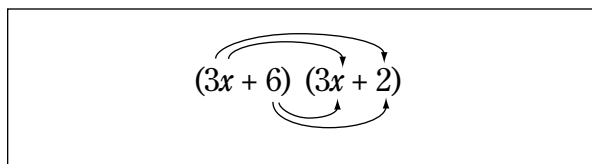


Fig. 6 Order for multiplication of terms

same process (see **fig. 3**). The first prime is 2; 140 divided by 2 is 70; 70 divided by 2 is 35; 35 is not divisible by 2 nor by the next prime, 3; therefore, the next consecutive prime, 5, is tried. The final prime factor is 7. All prime factors appear on the column to the right. In contrast, in a factor tree, prime factors appear only at the end of each branch. Often, students have trouble keeping track of the final factors because they are spread all over the tree.

Common denominators

To change fractions to obtain common denominators, Mexican textbooks show both denominators decomposed into primes. The lowest common denominator is found by multiplying all the common prime factors and the prime factors that appear in at least one of the two denominators. **Figure 4** shows the steps for finding the prime factors of 12 and 18 to obtain the least common multiple of the two numbers, which is $2 \times 2 \times 3 \times 3$, or 36.

Division of fractions

The most common algorithm in the United States to divide fractions is to invert the second fraction, then multiply. In Mexico, a common algorithm for a fraction division problem, such as

$$\frac{1}{2} \div \frac{3}{4}$$

is to cross-multiply, as shown in **figure 5**. Teachers can help students see how the two algorithms are equivalent. In this method, the numerator of the first fraction is multiplied by 4 and its denominator is multiplied by 3. This approach is equivalent to multiplying the first fraction by $4/3$, the inverse of the second fraction.

Algebraic equations

The equation $x + 35 = 75$ is solved in the United States by subtracting 35 from both sides to arrive at $x = 40$. Some Mexican students write the original problem the same way, but their thought processes are different. They ask themselves, “What number and 35 add up to 75?” The answer, of course, is that 40 and 35 add up to 75. Often, these students may write only the answer, and some teachers think that they have cheated. Students’ written work for the procedure may also look different, as in the following:

$$\begin{aligned} x + 35 &= 75 \\ 40 + 35 &= 75 \\ 75 &= 75 \end{aligned}$$

Students may not write $x = 40$ explicitly. These students may be able to do one-step equations, but they may balk at two-step equations because their “internal” algorithm does not work with those equations.

Multiplication of binomials

Mexican textbooks use arrows to show multiplication of binomials and polynomials (Serralde et al. 1993) (see **fig. 6**). To multiply two binomials, the diagram indicates that the first term in the first set of parentheses is multiplied by each of the terms in the second set of parentheses; then the second term in the first set of parentheses is multiplied by each of the terms in the second set of parentheses. This process results in $9x^2 + 6x + 18x + 12$. Notice that the emphasis in this method is in systematically multiplying successive terms of the first algebraic expression by all the terms of the second expression. The algorithm can be readily extended when the algebraic expressions have more than two terms. In contrast, a shift of emphasis is neces-

sary to generalize the popular FOIL method (first, outer, inner, last) used in the United States.

Summary of Algorithmic Differences

THESE ALGORITHMS AND NOTATIONAL DIFFERENCES are not all the mathematical distinctions that immigrant students face in the United States. The differences outlined here, however, establish the idea that students from other countries may use different methods and may become confused when even basic ideas are presented through different representations.

Considerations for Success

WE HAVE IDENTIFIED MATHEMATICAL SITUATIONS that immigrant students from Latin America may find difficult. What can the classroom teacher do to help such students be successful in a school in the United States? The following paragraphs offer some recommendations.

First, validate students' previous experiences both linguistically and mathematically. Emphasize the richness and diversity of students' knowledge and their experiences with other systems of learning and expression. Most students have had schooling in their countries of origin, and those experiences differ from the school experience in a new country. Refer to those experiences in a positive light, and eliminate the notion that the methods used in schools in the United States are inherently better. Take time to explain the differences that students encounter so as to create a comfortable environment in which students know that they can express themselves without fear. When students know that different approaches will be accepted, they are more relaxed and confident.

Second, find common beginning points for students to start their experiences in the United States. Two successful beginning points are graphing calculators and writing. Most immigrant students did not use calculators in their native countries. In mainstream classrooms, these students can be teamed with others to give immigrant students an opportunity to learn about graphing calculators and to hone English skills. In a sheltered English, ESL, or bilingual classroom, an introduction to graphing calculators becomes direct instruction, as few of the students will have had any experience with this tool. Using written explanations is also a good beginning point; students are not used to writing explanations in mathematics. All students benefit because they develop the ability to analyze and express mathematics in a written format, and immigrant students increase their English skills. Classroom experience shows that students balk in

the beginning, but they quickly develop the ability to listen critically to teacher explanations and raise questions when they feel explanations are unclear.

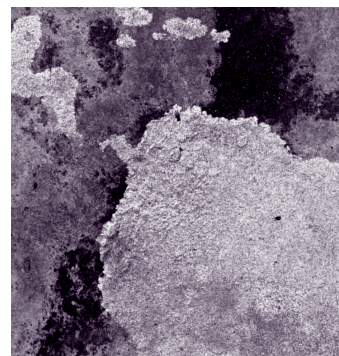
Of course, in addition to differences in representations, differences can also be found in the sequence of presentation of mathematical topics in other countries. The teacher can provide enrichment and challenging activities for a topic that may be known to immigrant students but is new to those from the United States.

Finally, establish a sense of rapport in which both students and teachers are learners. For example, the teacher's mastery of Spanish may not be as good as that of the students. Students can help improve the teacher's Spanish as the teacher helps them master English. Allow class time for students to practice the variety of language used in mathematics, in both formal and informal settings and in both English and Spanish. Allow students to share algorithms learned in different countries to enrich the class as a whole.

The teacher plays a pivotal role in the success of immigrant students. The essential element is the teacher's decision to actively guide that success.

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Validate students' experiences and establish rapport