

9 | DEVELOPING ALGEBRAIC STRATEGIES WITH MINILESSONS

In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual.

—Galileo Galilei

Minilessons are presented at the start of math workshop and last for ten or fifteen minutes. In contrast to investigations like the ones described in previous chapters, which characterize the heart of the math workshop, minilessons are more guided and more explicit. They include computation problems that when placed together are likely to generate discussion on certain strategies or big ideas that are landmarks on the landscape of learning. We call these groups of problems *strings* because they are a tightly structured series (a string) of problems that are related in a way that supports the development of numeracy and algebra.

Minilessons are usually presented to the whole class, although many teachers use them with small groups of students as well as a way to differentiate. During whole-group sessions at the start of math workshop, young children often sit on a rug. Older students can sit on benches placed in a **U**. Clustering students together like this near a chalkboard or whiteboard facilitates pair talk and allows you to post the problems and the strategies used to solve them.

The problems are presented one at a time and learners determine an answer and share the strategy they used. The emphasis is on the development of mental math strategies. Learners don't have to solve the problems *in* their head, but it is important for them to do the problem *with* their head! In other words, they are encouraged to examine the numbers in the problem and think about clever, efficient ways to solve it. The relationships between the problems in the string support them in doing this. The strategies that students offer are represented on an open number line.

TEACHING AND LEARNING IN THE CLASSROOM

Carlos, a fifth-grade teacher in California, is presenting a minilesson on variation using the following math string:

Here is an unknown amount on a number line. I call it j .
If this is one jump, what does $3j$ look like?
How about one jump and seven steps?
Now, what do three jumps and one step backward look like?
What if $j + 7 = 3j - 1$?
What if $j + 11 = 3j - 1$?
What if $j + 11 = 3j - 5$?

He reminds the students about representing distances using an open number line and begins the string by drawing a small jump on the line, telling the students it is j . Then class members represent the second, third, and fourth jump descriptions. (Heidi and Alyssa's representations are shown in Figure 9.1.)

Next Carlos asks whether the drawings include an accurate representation of the equation $j + 7 = 3j - 1$. Alyssa says, "Ours isn't going to work. We're going to have to draw it all over again."

Carlos grins. "Did you draw it wrong? I thought we agreed you were correct."

"You tricked me," Heidi declares. "The equation makes what I was doing look wrong. But we didn't know then that j plus seven had to equal three j s minus one."

"The problem isn't with Heidi's drawing," Alyssa adds, "we're just going to have to draw it over again."

The string has succeeded in generating a spirited conversation on variation. As pointed out in Chapter 6, variables describe relationships. All the drawings thus far are correct given the information known at the time. As more information is given—such as 3 jumps and a step back is equivalent to a jump and 7 steps—the value for j is forced. Understanding variation is a big idea on the landscape of learning. To support his learners in constructing this idea, Carlos encourages further reflection.

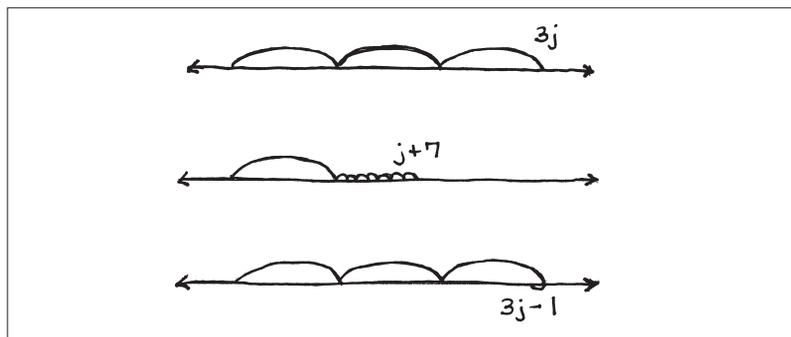


FIGURE 9.1
Heidi and Alyssa's
Representations

“Does everyone understand what Alyssa and Heidi are worried about?” Carlos asks. “Talk with your neighbor for a few minutes to make sure you understand the issue.”

After a few minutes Carlos asks Juan to explain. “Well, the point is,” Juan says, “that Heidi didn’t know how big the jumps would end up. So they have to fix it. That’s what Alyssa means. What Heidi did is okay, but the step length has to be related to the jump.”

“Well then, what could the length of the jump be?” Carlos asks. “Heidi, do you want to add something?”

Heidi explains, “We didn’t know this information before. I think the way we drew $j + 7$ is really just a way of showing there is one jump and seven steps and it’s just a way to think about it. So what we did is still fine. It’s just that when we know that j plus seven has to equal $3j$ minus one, then, well, like Alyssa says, we have to redraw it.”

Carlos says, “Do all of you agree? Heidi is saying that this picture represents the total distance traveled in one jump and seven steps and that the amount traveled depends on the size of the jump in relation to the steps. So we can think of a whole bunch of possibilities and this picture represents just one of these possibilities.” Most students nod in agreement. Carlos continues, “And the same for these three jumps minus one. It’s a picture that helps us think about the possibilities, but to solve a particular question we may have to work a little more to find the jump length.”

“Yeah, it’s like an all-in-one picture,” Juan blurts out. *Variation is now the focus.*

WHAT IS REVEALED

Carlos is using this string to encourage students to represent related algebraic expressions and to treat variables with variation. He represents j , and the students quickly agree that $3j$ is just 3 times as big, and that $j + 7$ would just be 7 small steps more. The next representation is also easy: three equal jumps. The fourth requires just one step back. The third and fourth representations do prompt students to comment that they don’t know how big the jump is in relation to the steps, which is true. Nevertheless, asking students to draw what the question indicates produces a variety of meaningful representations. Even though these representations cannot be used to determine the value of a variable, the students are treating the expressions as objects. The equation introduces an equivalence that causes students to have to redraw. As the string continues, new equivalents are introduced, causing further adjustments.

Too often students believe there is only one way to complete a mathematics problem. Flexibility in thinking is crucial for making sense of the big idea of variation. When more information is provided—in this case, $3j - 1$ is equal to $j + 7$ —an exact relationship between jump lengths and step

lengths can be nailed down. Without this added information, there are many possible representations for the expressions $3j - 1$ and $j + 7$.

In previous years, Carlos asked his students to find mystery numbers: “Find for me the mystery number that satisfies the rule that $m + 7$ is the same as $3m - 1$.” His students would set to work with calculations, trying many numbers, 1, 7, 3, 10, 5, and in time they found that 4 was the mystery number. For them, the question was based on simple arithmetic, and they knew that with a little patience they could solve the mystery. (Often his students would base a new guess on information from the previous guess, so the process was not totally random—more a guess-and-revise strategy.) Once a mystery was solved, they called out their answers and were ready for a new mystery. Students were engaged, and it was fun, but something was missing. The students were not using the relationships encoded in the mystery—they weren’t structuring.

The jump contexts, number lines, and strings Carlos uses now prompt his students to think, evaluate, and reevaluate. In another four years when they are in high school, Carlos’ students will study functions; the work he is doing now is an important part of their preparation. When Juan says, “It’s like an all-in-one picture,” he is explaining that by representing $j + 7$ and $3j - 1$ as Heidi did in Figure 9.1, a relationship is at play; for a possible jump length, there is a resulting total length indicated by the representation.¹ As the students explore this string, they use these relationships, not guess-and-check arithmetic, to think about $j + 7 = 3j - 1$.

BACK TO THE CLASSROOM

Carlos asks his students to work with a partner and draw a picture that represents $3j - 1 = j + 7$. After a few minutes he asks Sam and Ramiro to share their thinking.

Ramiro begins. “You see, we have to fit the eight steps into two jumps so we think it is four.” Carlos asks for more details and Ramiro rephrases. “See, the three jumps and the backward step. Those two jumps and the backward step have to be the same as seven steps, so we have to fit seven steps, I mean, eight steps, all into two jumps. That means there are four steps in a jump.”

Carlos creates the representation in Figure 9.2 and asks, “Do you want to add something Sam?”

“Yeah, it’s like we used a storage box. We put the first jump both times in the storage box and just looked at the other jumps and steps.”

Carlos continues to probe. “This new diagram is different from what we had when Heidi had me draw three jumps and one step backward. Is

¹In terms of function language, the possible jump lengths are the input values (or domain) of the function, and the total length represents the output values (or range) of the function.

Heidi's diagram right? Give me a thumbs-up if you think her diagram is still okay." Some thumbs go up, but not too many. "Maria, you've got your thumb down, why?"

"Because it won't work. The eight steps don't fit."

"Heidi, you don't seem to agree."

"No, mine is still right. Like Juan said, mine just shows how it goes and we didn't know the amount of the jump. So it's still good. But the four steps in a jump work in Ramiro's picture because he had another equation."

"Not everyone seems convinced. Let's come back to this in a moment after we try one more problem in this string." *Variation is a difficult idea and Carlos decides to continue to examine this idea.* "Take a minute and give me a thumbs-up when you know what would happen if $j + 11 = 3j - 1$."

After a few minutes, Carlos calls on Maria. "We had to change it again. Now we had to put twelve steps into two jumps instead of eight. Now the jumps are bigger."

Rosie adds, "They are six now, because six and six is twelve."

"I agree," Heidi says tentatively, "but it's the same strategy as in my picture and that picture is still right too."

"And they are still using the storage box for the first jump like Sam said," Keisha points out. "See, the first jump didn't matter, it's the eleven plus one that matters for two jumps."

Carlos lets his students negotiate this terrain for a bit. Once they seem comfortable with Maria's answer, he asks them to consider $j + 11 = 3j - 5$. As the string continues, the class has more opportunities to consider the idea of variation, and they have a chance to talk about generalizing the approaches used for this type of problem.

WHAT IS REVEALED

In this brief minilesson, Carlos chooses a series of related problems and asks his students to solve them. Together they discuss and compare different strategies and ideas and explore relationships between problems. The relationships between the problems are the critical element of the string. As he works through it, Carlos uses the double open number line

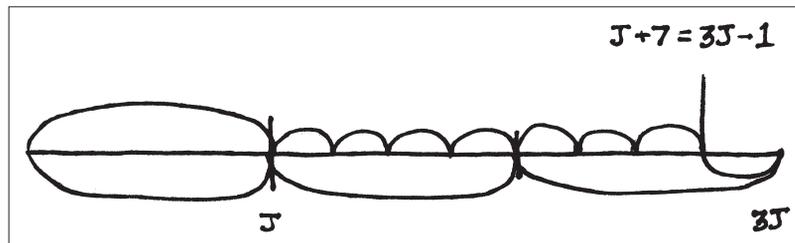


FIGURE 9.2
 Representation of
 Ramiro's Strategy

as a representational tool. This representation enables children to examine the equivalence of the expressions.

Carlos succeeds in bringing to the fore some important big ideas—that variables describe relationships and are not merely unknown quantities (variation) and that equivalent amounts can be separated off in solving a problem. The class has also had an opportunity to generalize about a strategy that is useful in considering similar problems of this type. The conversation early on with Heidi and the validation of her original representation shows students they may often have to redraw to account for variation.

Good minilessons always focus on problems that are likely to develop certain strategies or big ideas that are landmarks on the landscape of learning. Designing such strings and other minilessons to develop algebraic ideas and strategies requires a deep understanding of their development—the choice of numbers, representational models, and contexts used are not random.

JOURNEYING THE LANDSCAPE

Choosing the Numbers

Equivalence is a big idea on the algebra landscape. Underlying this idea is another big idea—that expressions can be treated as objects (rather than simply as procedures) and placed in relation to one another. These ideas are precursors to the idea that variables describe relationships and are not merely unknown quantities (variation). All of these ideas enable the important algebraic strategy of separating off equivalent amounts. Knowing that these ideas and strategies are important, Carlos has carefully crafted a string of problems to support their development along the landscape of learning.

Here's a similar string crafted around the same big ideas:

Here is an unknown amount on a number line. I call it j . Where is 2 times j ?

Where is 2 times these 2 j s?

Where is $4j + 6$?

Where is $4j - 6$?

Suppose I tell you that $4j - 6$ is the same as $2j$. What now?

How about $2j - 3$? Where is it? Why?

What if I told you that $4j - 6$ is the same as $3j$. What now?

Now where is $2j - 3$? Why?

In the first problem, j is represented and $2j$ would just be twice it. The next problem will probably also be easy for the students: There are four equal jumps. The third problem introduces $+6$ and the fourth -6 and here students may begin to comment that they don't know how big the jump is in relation to the steps. This is true, and such comments should be encour-

aged, but it is important to ask students to draw what the question indicates and share a variety of representations. Their diverse representations will promote discussion of this unknown quantity, since the size of the jump could vary—and now variation is up for discussion. The next problem introduces an equivalent expression that may be a surprise and most likely will cause students to have to redraw. As the string continues, new equivalents are introduced, causing further adjustments.

JOURNEYING THE LANDSCAPE

The double number line that Carlos has chosen as a representational model prompts students to examine how the expressions are related and to use equivalence. In each of the strings there are equations in which the numbers and the representation of them on double number lines potentially lead students to think about removing equivalent expressions.

In algebra classes, students traditionally are taught to “cancel out” or “add equal amounts to both sides of an equation.” But too often they are taught these rules before they have made sense of what an expression such as $j + 7$ actually means. Carlos is very careful to validate Alyssa and Heidi’s diagrams for $j + 7$ and $3j - 1$, because both are correct and complete. Up to this point Carlos has been working to ensure that his students understand that an algebraic expression can be treated as an object (not only as a procedure)—that the multiple representations on the number line as students suggest small steps or big steps provide possible mental images of this object.

With the added condition that $j + 7 = 3j - 1$, the diagrams have to be redrawn because the jump length is now specified by the relationship of these two expressions. In the new drawing (Figure 9.2), the representations of $j + 7$ and $3j - 1$ still have the same structure as Heidi’s drawings did in Figure 9.1, but now they are aligned to end at the same point. The act of redrawing is an action in which *variation* is implicit. There is a continuum of possible diagrams for $j + 7$ that are correct, in which the length of j can stretch or shrink, and this idea is used to create a diagram in which the ends line up. It is at this point that it becomes apparent that the initial jump in each sequence can be separated off. This is what Ramiro accomplishes when he says “eight steps all into two jumps.” By putting the first jump aside (like in a “storage box”) Ramiro has reformulated solving $j + 7 = 3j - 1$ as equivalent equations: $7 = 2j - 1$, or $8 = 2j$. He has used equivalence to remove an instance of the variable j .

Choosing the Model

The double number line is a powerful tool for examining equivalence and developing algebraic strategies in the early grades, too. Patricia is using the

following string with her second graders to help them treat numeric expressions as objects:

$$\begin{aligned}10 &= 5 + 5 \\10 + 10 &= 5 + 5 + 5 + 5 \\5 + 20 &= 10 + 10 \\5 + 20 + 4 &= 4 + 10 + 15 \\13 + 8 + 6 &= 5 + 9 + 13\end{aligned}$$

Each time she writes a statement, she asks the children to determine whether it is true or false. If the statement is false (the third equation in the string, for example), it is made true by replacing the equals sign with an inequality sign.

The string begins with an easy equation that supports the second, third, and fourth. The hope is that as the children work through the string, someone will suggest that a determination can be made without adding up all the numbers—that the numeric expressions can be treated as equivalent objects. For example, in the last problem the students who see $8 + 6$ as an object equivalent to $5 + 9$ won't have to add the numbers to see whether the statement is true.

The class is discussing the fourth problem in the string. Patricia writes the statement $5 + 20 + 4 = 4 + 10 + 15$, and asks her students to give a quiet thumbs-up when they are ready to say whether the statement is true or false. (Her students have learned not to disturb their classmates by calling out answers. Also, thumbs held up in front of the chest allow class members who might be distracted by waving arms crucial time in which to think.)

When most thumbs are up, Patricia calls on Ian. "I say it's true," he declares with conviction.

"Okay, it looks like others agree, so tell us why you think that."

"Well, the fours don't matter because they are on both sides. And the five plus twenty is a twenty-five and the ten plus fifteen is another twenty-five. So it's true."

For many children, the only way to solve this problem is to use an arithmetic strategy: Add up both sides and compare answers to see if they are the same. For example, here they would produce $29 = 29$. But Ian's first sentence shows this isn't his strategy—he is using a more algebraic strategy by noticing the equivalence of the 4s on each side of the equation and ignoring them. He is also implicitly using the commutative property of addition ($25 + 4 = 4 + 25$)—it doesn't matter the order in which you add things.

Patricia pushes Ian to say more.

"Because you are adding them up and four more is the same as four more," Ian clarifies.

"How many of you can explain what Ian is thinking?"

Mia gives it a go. "I think what Ian is saying is that five plus twenty is twenty-five and that ten plus fifteen is twenty-five, and these are the same, so when you go four more you get the same thing and so it's the same on both sides."

Patricia draws the representation shown in Figure 9.3 and asks Ian and Mia if this represents their thinking. Mia Chiara nods in agreement, but Ian is not convinced.

“Ian, you’re not convinced?” Patricia says. “But you were sure a moment ago.”

“No, I’m convinced the equation is true, but your picture isn’t what I’m talking about.”

“What’s not right about the picture?”

“Well, I said the fours don’t matter, but you put them both on the same side and that’s not where they are.” Patricia had placed the fours one above the other on the number line, hoping to show the equivalence Ian had mentioned. Surprisingly, although Ian has used equivalence in his determination that the statement is true, when representing addition on an open number line the order matters to him. Patricia draws another double number line on which the jumps appear in the order presented in the equation (see Figure 9.4) and asks Ian if this is what he means.

“Yes, but you have to put the twenty-fives in.”

Patricia adds the 25s (see Figure 9.5) and Ian nods.

Sensing that the big idea of the commutative property needs further discussion, Patricia asks the children to talk with their partner about these two diagrams and about how Ian and Mia are thinking about the problem.

Rosie offers some thoughts, “I think they both see the twenty-fives, but Ian wants to add the fours first and second while Mia wants to add the fours at the end both times. But it doesn’t matter, really.”

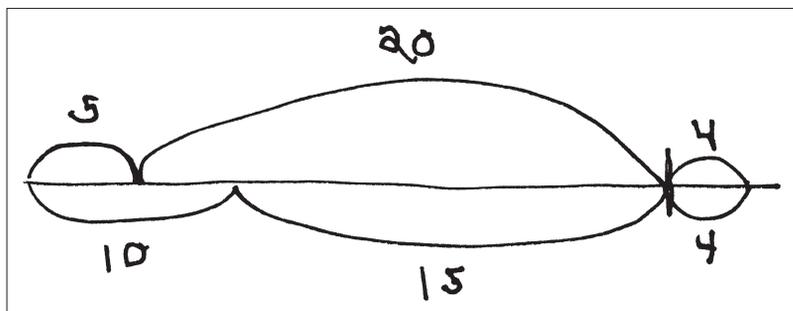


FIGURE 9.3
First Representation
of Ian and Mia Chiara’s
Strategy

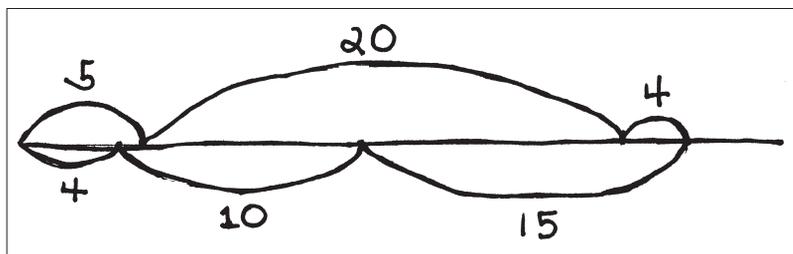


FIGURE 9.4
Second Representation of
Ian and Mia Chiara’s
Strategy

“Yeah,” adds Camille, “because you are just adding them up and you can do it both ways.”

“That’s what I meant when I said the fours don’t matter because we are adding it all up,” Ian points out. “They don’t matter. Both are four and twenty-five.” *The landmarks of commutativity and equivalence have now been reached.*

CHOOSING A CONTEXT

Teaching and Learning in Another Classroom

Maia’s second graders are used to playing a version of twenty questions in which the goal is to determine what coins (nickels, dimes, pennies, and quarters totaling fifty cents) she has in her hand. Today she is playing a variation on this game in her minilessson. She has set down two small bags with coins in them; the children are to determine which bag has the most money or whether they contain equal amounts. The first bag has one quarter, three dimes, four nickels, and one foreign coin; the second bag has one quarter, two dimes, six nickels, and one foreign coin. Both foreign coins are the same but of unknown value. The children are midway through the game.

“So far you have figured out that there is one quarter in each bag and two dimes in this bag on the left. Let me write that down. What sign should I use so far?” asks Maia.

“Greater than” several children call out.

“Why, Sam?”

“Well, the quarters are the same, but the bag on the left also has two dimes.” Sam has used equivalence to separate off equal amounts. He doesn’t need to add the known values in the bags, because the quarters are equivalent.

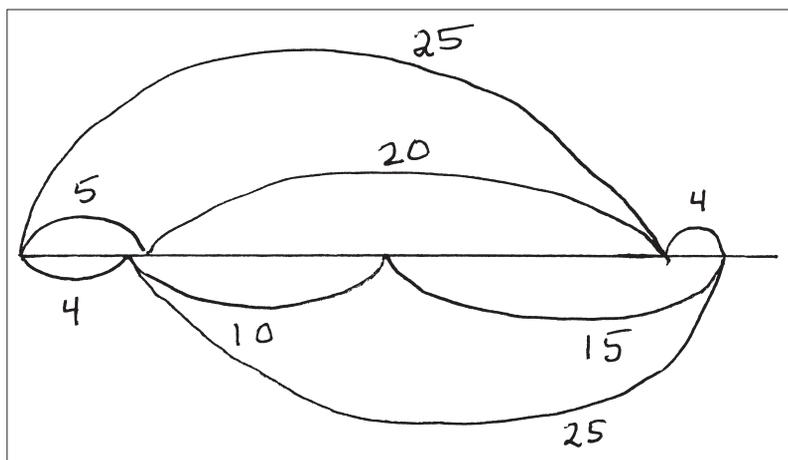


FIGURE 9.5
Adding in the 25s

Maia writes $1(25) + 2(10) > 1(25)$ on the board. “Okay. Do you have more questions for me?”

Juanita asks, “Are there more than two dimes in the other bag?”

“Yes,” replies Maia.

Kelly gets specific. “Are there three?”

“Do you mean exactly three?” Kelly nods. “Yes. So let’s write this down. Now you know the quarters and the dimes. Which sign do we need?”

“You have to turn the sign around,” replies Juanita.

Maia writes $1(25) + 2(10) < 1(25) + 3(10)$ on the board.

After a bit more discussion, the numbers of nickels are determined, pennies are ruled out, and the statement $1(25) + 2(10) + 6(5) = 1(25) + 3(10) + 4(5)$ is established. Explaining this Kelly says, “There is one more dime and two less nickels in the second bag, so they’re the same because one dime equals two nickels.” *Rather than adding all the values up to prove the values in the bags are equal—an arithmetic strategy—Kelly is using an algebraic strategy. She is mentally substituting a dime in the right-hand bag for two nickels.*

Now Maia teases the children with a smile. “You think you’re done now, don’t you? Actually there is one more coin in each bag and it is the same type of coin in each bag.”

“What is it?” Juanita asks what everyone is wondering.

“I don’t know what this type of coin is worth. They are foreign coins that I got a long time ago when I was traveling. I put one in each bag. Let’s call it c for coin because we don’t know what it is worth. What sign should we use?” As the children ponder her question she says, “Here is what we know so far,” and writes $1(25) + 2(10) + 6(5) + c ? 1(25) + 3(10) + 4(5) + c$.

“We can’t do it if you don’t tell us,” Sam says with exasperation. “How can we add it if we don’t know what it is?”

Sam and his classmates are still relying on arithmetic to verify equivalence, so the problem seems impossible. They do not yet have a strong sense of equivalence, and they also assume an expression represents a procedure. It is impossible to add something to something if you don’t know what the something is. Maia wants to make sure her students develop an understanding that equivalence can be understood without computation.

She asks the class, “Do you have to add it?”

“If it’s a nickel, it’s still equal,” replies Rosie.

“It works for a penny or a dime, too,” adds Juanita.

“Does it work for other numbers, too?” *Maia encourages students to consider several numbers as a way of developing the idea of a variable. But here, the variable is not an unknown number that has to be found. Instead it represents many possibilities (its value is in some range of amounts)—once again this is variation.*

Keshawn begins to grasp the big idea. “It works for any number, because it’s the same in both bags. You don’t have to know what it is.”

“Yeah, Keshawn, you’re right!” Isaac says in awe. “If it’s the same coin and it’s in both bags you don’t have to worry about it. It’s the same on both sides so it’s still an equal sign.”

Maia continues to use the context of the foreign coin to support the development of variation. “Could c be any amount? Are Keshawn and Isaac right? Would this statement be true no matter what c is?” She writes in an equals sign: $1(25) + 2(10) + 6(5) + c = 1(25) + 3(10) + 4(5) + c$.

Isaac replies with conviction, “Yep, c could be any number. As long as they are the same, it doesn’t matter what the c coin is worth. It could be any number.”

Maia has pushed her students to consider the big idea that equivalent amounts can be separated off, or substituted, even when variables are involved. By necessity her minilesson has also involved the big idea of variation. In addition, the children have had to view expressions as objects, not merely as describing a set of operations. This interwoven web of ideas is critical to the development of algebra.

Operating on Expressions

Once equivalence is well understood, children can be challenged to consider how one can operate on expressions. Carlos is using the following string to develop this idea with his fifth graders:

Here is one jump and two steps. What else could it look like?
 So how about two jumps and four steps, how would I represent that?
 What about $2(j + 2)$?
 What about $3(j + 2)$?
 What about $4(j + 2)$?
 What about $2(2j + 4)$?

Carlos begins his string by drawing the representation shown in Figure 9.6. “Okay, here is one jump and two steps. Is this how it always looks? Carrie, what do you think?”

“Well, we usually keep the steps the same in our drawings, but the jump could be shorter or bigger, we don’t know.”

“So it could be like this, or this, or this?” Carlos draws three different representations of $j + 2$ (see Figure 9.7). Most of the students nod in agreement. “So how about $2j + 4$? Carrie, do you have a suggestion on this one?”

“Well, we do what you just did, except we do two jumps and four little steps. Each time the jumps have to be the same because of the frog

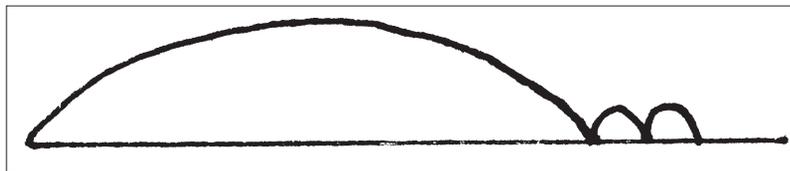


FIGURE 9.6
One Jump and
Two Steps

jumping rule.” Carlos invites her to draw two jumps and four steps below his first drawing of one jump and two steps on a double number line (see Figure 9.8). “What do you think? Jasmine, do you have something to add?”

“Wouldn’t it have been easier just to add a jump and two steps rather than starting all over again?” Jasmine offers shyly. Jasmine adds her idea to Carrie’s drawing (see Figure 9.9). Next Carlos asks the students to talk with their partner about what Jasmine and Carrie have drawn.

“I think it’s two j plus two,” Mario explains. “So no matter how you do it, it has to be the same.”

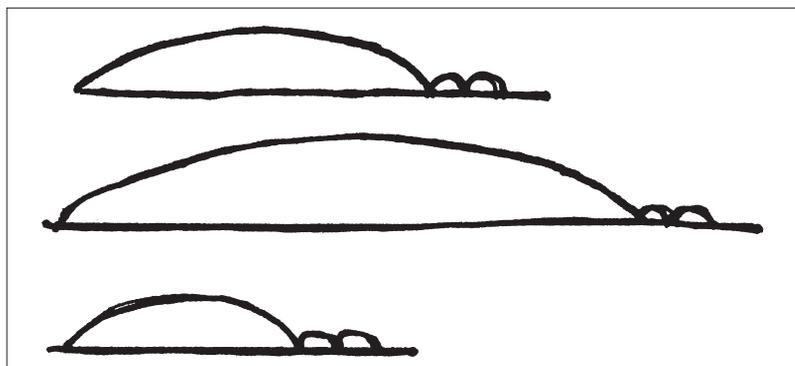


FIGURE 9.7
Other Versions of
One Jump and
Two Steps

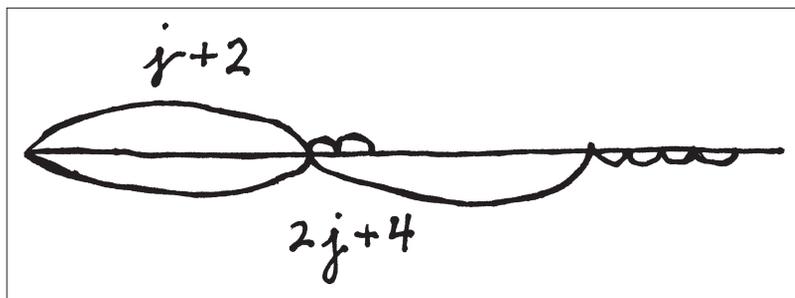


FIGURE 9.8
Two Jumps and
Four Steps

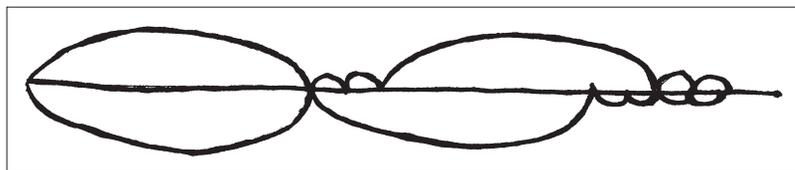


FIGURE 9.9
One Jump and
Two Steps Twice

Noticing that Louisa seems puzzled, Carlos attempts to bring her into the conversation. “Louisa, do you agree?”

“Well, I’m not sure. Doesn’t it have to be the same j each time? What if they were different frogs or something?”

Mario helps clarify. “No, it has to be one, the same frog for each jump. That’s what we always do.”

Carlos asks Mario to add to the diagram and show everyone where the $j + 2$ ’s are. Mario adds two arcs over the jumps and two steps on the top of the double number line and writes 1 and 2 above each arc (see Figure 9.10).

Once again Louisa looks puzzled. “Okay, I see them, but why did you write 1 and 2 above them? They aren’t one jump or one step.”

“Well, this is one jump plus two and this is the other jump plus two, so I just labeled them 1 and 2.” *Mario is treating the $j + 2$ as a single object and is operating with that expression as an object. This is a landmark in development analogous to the leap young children take when they begin to unitize a group of ten objects and count it as one ten. The ability to look at mathematical expressions and pull out chunks that can be viewed as single “entities” is an essential strategy throughout mathematics (collegiate and beyond!).*

Although Carlos has used the letter j in earlier discussions with his students, he has refrained from symbolizing to make sure that students are first making sense of the context and the representation. Now he senses the class is ready to interpret the symbolic expression, and he wants them to see correctly formulated algebraic equations that capture the focus of the minilesson.

Pleased with the discussion, Carlos asks the class, “Can we do this?” and writes $2(j + 2) = 2j + 4$ above Mario’s work. The class nods.

Mario interjects, “Yeah, that’s just writing what I’m saying another way.”

Because Carlos knows that many beginning algebra students will read $2(j + 2)$ as $2j + 2$, interpreting the symbols left to right, rather than seeing the $j + 2$ in the parentheses as an object, he continues with the string, having his students work with $3j + 6$ and $4j + 8$. Again they work with multiple chunks of $j + 2$ as they discuss how these jumping sequences are related. As the string progresses, equivalences like $3(j + 2) = 3j + 6$, $4(j + 2) = 4j + 8$, and $2(2j + 4) = 4j + 8$ are represented on double number lines and discussed.

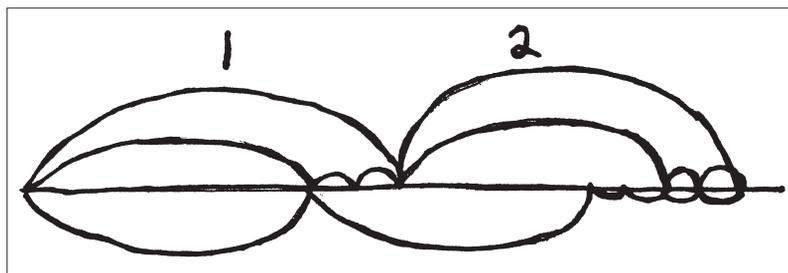


FIGURE 9.10
*Two Times One Jump
and Two Steps*

SUMMING UP

Good minilessons always focus on problems that are likely to develop certain strategies or big ideas that are landmarks on the landscape of learning. The big ideas of variation, equivalence, expressions as objects, that variables describe relationships and strategies such as using cancellation, commutativity, or equivalence can be further developed in minilessons. Designing strings or other minilessons to develop these ideas requires a deep understanding of the landscape; the choice of the questions and the models and contexts used are not random.

When Galileo said, “In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual,” he was describing his life-long struggle to have scientific ideas accepted in an era where religious authority dominated. But we cite it here for different reasons. When the humble reasoning of children is valued and nurtured in mathematics classrooms, doors open. When children are given the chance to structure number and operation in their own way, they see themselves as mathematicians and their understanding deepens. They can make sense of algebra not as a funny set of rules that mixes up letters and numbers handed down by the authority of thousands but as a language for describing the structure and relationships they uncover.