Making Imaginary Roots Real

"How can we make sense of what we learned today?" This is a question I commonly pose to my algebra students in an effort to have them think about the connections between the new concept they are learning and concepts they have previously learned. For students who have a strong, expansive understanding of previously learned topics, determining how to "make sense" of new concepts may be relatively straightforward. However, for students who have limited understanding of foundational concepts, "making sense" of new material may not be as simple.

NCTM defines sense making as "developing understanding of a situation, context, or concept by connecting it with existing knowledge" (Martin et al. 2009, p. 4). Providing opportunities for students to engage in sense making allows them to extend their understanding of mathematical concepts by analyzing the relationship between different mathematical ideas; they are then able to use mathematics flexibly as a tool to reason through and solve varied problems. Sense making is instrumental in determining the meaning of problems and identifying appropriate entry points (CCSSI 2010); as such, it is a critical aspect of students' mathematical understanding that should be inextricably interwoven into all high school programs (Martin et al. 2009).

As students' sense-making skills are cultivated, each class of students navigates its own unique path to attempt individually and collectively to "make sense" of the mathematics that they are

Students in an entry-level algebra class attempt to make sense of the relationship between imaginary zeros and their nonimaginary counterparts.

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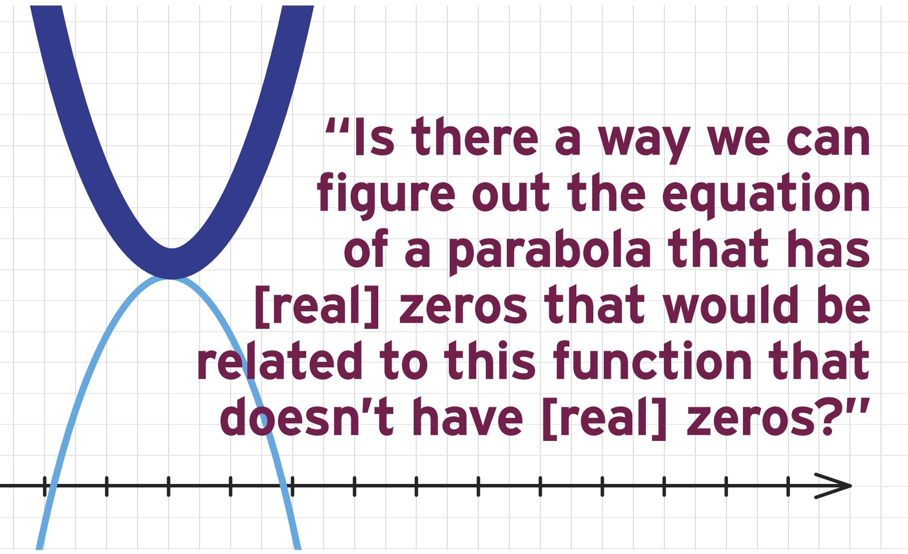
learning. As teachers support students along their path of sense making, it is critical to consider that the extent to which students are able to make sense is related to the extent to which they can connect their learning with their *existing knowledge*. Therefore, as students grapple with making sense of new knowledge, it is important that they fortify their foundational knowledge that will serve as their *existing knowledge* on which they can build new understanding. In this article, I share the journey of some of my algebra students as they sought to "make sense" of imaginary zeros through their existing knowledge of real zeros.

CLASSROOM CONTEXT

The students who comprised my first-year algebra classes were heterogeneous with respect to ethnicity, gender, and income level. With an average class size of 32, these classes also included students with varying levels of ability and interest; some students eagerly engaged with challenging mathematics problems; others continuously had difficulty with foundational algebraic concepts. At the time in the academic year when the students encountered quadratic functions without real zeros, most students were able to calculate the zeros of a quadratic function using multiple methods, including the quadratic formula. They were able to estimate the nonintegral zeros of a function by observing its graph. They were also able to determine how a function is affected graphically when it is translated by adding a constant to the x- or y-variable. They were not, however, familiar with imaginary numbers because in the curriculum we follow, complex numbers are introduced in second-year algebra.



"Is there a way we can figure out the equation of a parabola that has [real] zeros that would be related to this function that doesn't have [real] zeros?"

EXTENDING A CLASSROOM TASK

During one of the tasks in class, I provided students with various quadratic functions in the form $f(x) = ax^2 + bx + c$, each of which had real zeros. As they worked with their peers in small groups, students identified some of the characteristics of each quadratic function and its graph (e.g., axis of symmetry, vertex, and zeros), graphed each, explored how each characteristic above related to the graph of the respective function, identified the similarities and differences between the given functions, and discussed how translating a function by adding a constant to the x- or y-variable affected the graph and equation of each function. During their discussions, I joined each group for a few minutes to assess their understanding of the mathematics and to push their thinking further. Either on their own or with my support, they were able to identify the axes of symmetry, vertices, and zeros of the functions; most students were also able to discern how changes in the a, b, or c value of each quadratic affected its graph, roots, axis of symmetry, and vertex.

After a brief whole-class discussion, I presented students with the following quadratic equation: $f(x) = x^2 - 6x + 13$. As part of the task, students were to graph the function and determine its significant characteristics as before. All the students determined, on the basis of the graph, that the function did not have any (real) zeros. Students' methods for justifying their answers, however, were varied; many attempted to use the quadratic formula to support their answer. In the past, when they used the quadratic formula to determine the zeros of similar problems, they were perplexed by the presence of negative radicands in their calculations. I explained that they would learn more about these types of zeros in second-year algebra. For this problem, when the students concluded that

$$x = \frac{6 \pm \sqrt{-16}}{2},$$

they explained that since there was no (real) value equivalent to $\sqrt{-16}$, this meant that no (real) x-value would make $f(x) = 0$ and, therefore, this supported their conclusion that the function did not have any (real) zeros. In general, the class agreed that this was an acceptable explanation. One student, however, was uncomfortable with the finality of that explanation. Fiona (all student names are pseudonyms) commented that although she understood there were no (real) zeros, she wanted to "make sense" of the value obtained for x in relation to the specific function and its graph. In other words, she wanted to understand how the roots of $f(x) = x^2 - 6x + 13 = 0$ were specific to this quadratic function and different from a quadratic whose roots were, say,

$$-5 \pm \sqrt{6}.$$

Although she understood how to calculate the numerical answer, it seemed arbitrary and did not connect to her previously developed knowledge of zeros. Several students agreed. We did not have time during that lesson to address Fiona's question, so I worked with her and six other interested students the following day as they attempted to make sense of the imaginary zeros for this function.

RELATING IMAGINARY ZEROS TO REAL ZEROS

Before working with the smaller group of seven students the following day, I was concerned that their "existing knowledge" was not strong enough for them to make sense of complex zeros with nonzero imaginary values. Students had no prior notion of imaginary numbers nor had they learned how to perform calculations with them. Additionally, because they were in an entry-level algebra class, they rarely encountered quadratic functions without real zeros. I fully supported their quest to view an imaginary root as a value that was not arbitrary, as this could bring them further along a path to making sense of complex zeros.

Referring to the original function, $f(x) = x^2 - 6x + 13$, I told students they were going to determine whether there was a relationship between $f(x)$ and another function with which they were more familiar—a function with real zeros. I asked students about the types of functions they were familiar with graphing. They stated and sketched examples of functions with two zeros and one zero. They also distinguished between a function with a positive a-value graphed as a parabola that "looks like a 'U'" and a function with a negative a-value graphed as a parabola that "looks like an upside-down 'U.'" When I asked, "Why is the sign of the a-value significant?" they responded that the a-value determines the direction of the parabola; it is "flipped" by the additive inverse of the a-value.

As students individually drew a graph of f, I also drew it on the board. I asked, "What is different about this function compared with those we typically see in class?" They stated that the parabola does not intersect the x-axis and, therefore, it "has no [real] roots."

I then asked, "Is there a way we can figure out the equation of a parabola that has [real] zeros that would be related to this function that doesn't have [real] zeros?" As students began their discussion, they determined that the related parabola would have to intersect the x-axis to have zeros. And, for the parabola to intersect the x-axis, they would have to "flip" it. I asked, "How could we 'flip' it?"

Fiona stated that the "a-value would have to be negative."

As students continued their discussion, five of them decided to find the additive inverse of the a, b, and c values of $f(x)$ because "we need a negative value for a." Then they graphed their new equation, $g(x) = -x^2 + 6x - 13$, on the same graph as the original equation. They realized that although the parabola was "flipped," this did not resolve their initial issue of ensuring it would intersect the x-axis. I encouraged them to continue: "So, that didn't work out exactly as we'd hoped, but you're on the right track. Is there something else we could do to determine a related function that would be represented by a 'flipped' parabola that has real zeros?"

After a few moments, Omari said that the original parabola and the "flipped" parabola could still be related if they shared the same vertex. I asked the rest of the students if they agreed with him; they did. Since they agreed with Omari, I asked them to determine the equation of the parabola of the function that satisfied these conditions.

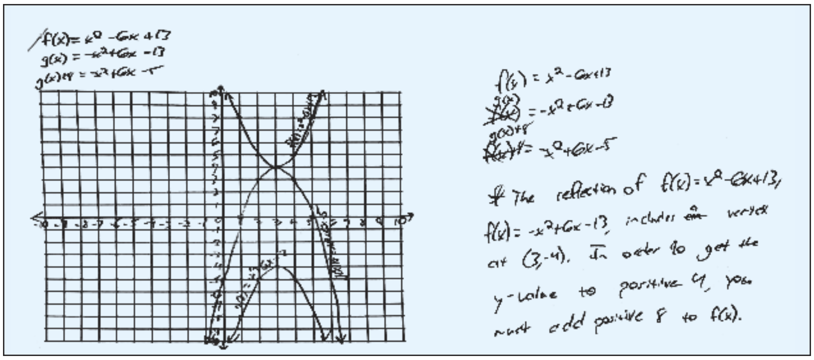


Fig. 1 Omari explained how he determined the equation for $g(x) + 8$.

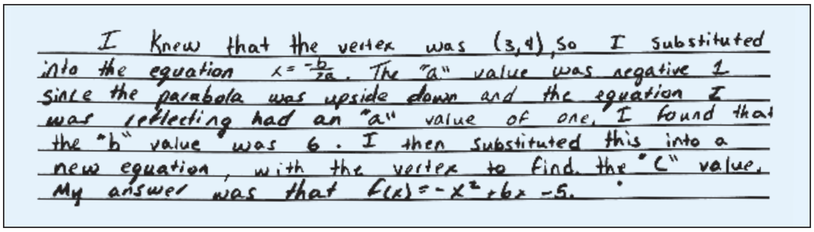


Fig. 2 Evelyn determined the equation of the function using a different method.

Some students decided to reflect the parabola over the x-axis and then translate the reflected function to obtain a parabola with their desired conditions. Omari used an approach that was similar to some of his classmates' (see **fig. 1**). In Omari's explanation, although he referred to the equation of the reflected image as $f(x)$, which he indicated as $g(x)$ in the rest of his work, he explained how he arrived at the equation for the function that shares the same vertex with f but has real zeros.

Two students determined the equation of the related function using a different method. Evelyn decided not to find the parabola of the image reflected over the x-axis. Instead, she used the equation of the axis of symmetry to determine the b-value of the new equation. Since she already deduced that the a-value would be −1, she used the x- and y-values from the vertex of the original equation to determine the c-value of the new equation that she referred to as $f(x)$ (see **fig. 2**).

Although students determined the equation of the new function using different methods, from this point forward, I will refer to the equations using Omari's notation: $f(x) = x^2 - 6x + 13$ as the original function, $g(x) = -x^2 + 6x - 13$ as the function reflected over the x-axis, and $g(x) + 8 = -x^2 + 6x - 5$ as the translation of the reflected parabola that shares the same vertex with $f(x)$ (see **fig. 3**).

After students shared how they had arrived at the equation for $g(x) + 8$, I told them to determine its zeros so they could decide whether they were related to those of $f(x)$. The students used the quadratic formula to determine $\{1,5\}$ as the zeros. During their discussion, several students stated they did not see a relationship between the zeros of $f(x)$ and $g(x) + 8$; they began to discuss whether a different related parabola would be better suited for this exploration. I told them that they could not readily see the relationship because they had not yet formally learned how to work with negative radicands. Two students, however, were able to identify that the nonreal zeros of $f(x)$ were related to the real zeros of $g(x) + 8$. Before the two students shared the relationship they identified, I showed the entire group how Alexis calculated the zeros of $f(x)$ next to her work for the zeros of $g(x) + 8$ so that other students would have an opportunity to determine whether a relationship existed. Although Alexis incorrectly referred to the related function as $f(x) + 8$, her work was structured in a manner that allowed her peers to identify the relationship in the sixth line of her work for each function (see **fig. 4**).

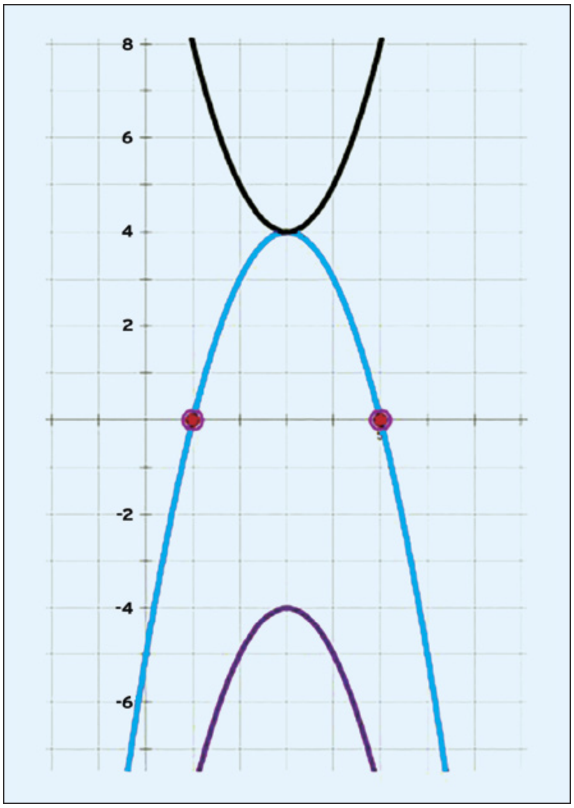


Fig. 3 The black parabola represents f(x), the maroon parabola represents $g(x)$, and the blue parabola represents $g(x) + 8$ with its zeros.

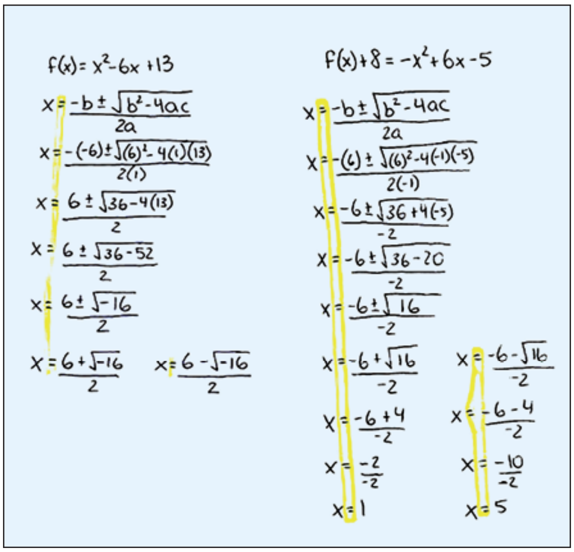


Fig. 4 Alexis's work allowed other students to see the relationship between $f(x)$ and $g(x) + 8$.

Students' unfamiliarity with imaginary numbers prevented them from being able to compare the zeros of $f(x)$, 3 ± 2, to those of $g(x) + 8$, $3 \pm 2i$. It would have been better had the computation of the real zero been delayed to allow students to see the imaginary and real zeros in parallel forms as, respectively,

$$x = 3 \pm \frac{\sqrt{-16}}{2} \text{ and } x = 3 \pm \frac{\sqrt{16}}{2}.$$

Alexis's work enabled them to see the relationship between the radicand in both functions by comparing the respective steps in the quadratic formula (see **fig. 5**). In their discussion about how they could make sense of what they learned, students determined that although they did not fully understand complex zeros at this stage, they were able to connect a nonreal zero to a related real zero, which allowed them to strengthen the foundation of their existing knowledge and to make sense of imaginary zeros in the future.

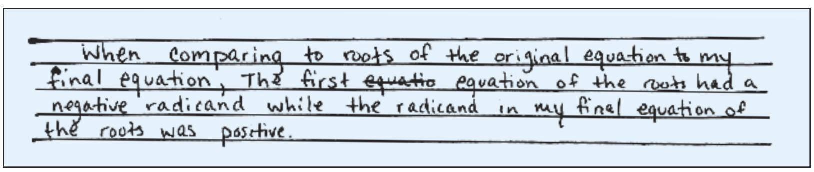


Fig. 5 Fiona noticed the difference between the radicands of $f(x)$ and $g(x) + 8$.

PREPARING FOR SENSE MAKING

Through this exploration, students were able to identify a relationship between the complex zeros of a function and the real zeros of another function reflected over its vertex. They were also able to determine that complex zeros are not arbitrary and that complex numbers will make sense to them when they study mathematics in subsequent

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classes. Supporting students' expansion of mathematical understanding on which they will build in higher-level mathematics classes provides them with the existing knowledge they will need to "make sense" of those concepts.

As the students continue to explore these ideas, they will be able to understand imaginary roots in a more meaningful and codified way. "As sense making develops, it increasingly incorporates more formal elements" (Martin et al. 2009, p. 4). When students learn more structured ways to discuss complex numbers with nonzero imaginary parts, they will be able to develop a stronger understanding of the relationship between imaginary roots or zeros expressed in $a \pm bi$ form and how they are related to the roots or zeros of the parabola reflected over the line $y = k$, where (h, k) is the vertex. They will also be able to determine the equation of the related function that has real roots based on what they learned and explored in first- year algebra. If the quadratic is represented in vertex form as $f(x) = a(x - h)^2 + k$, then the equation of the related function $g(x) + 2k = -(ax^2 + bx + c) + 2k$ or $g(x) + 2k = -f(x) + 2k$.

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CONCLUSION

Making sense of nonreal zeros can be challenging for students, particularly if they do not have the strong existing knowledge with which to connect these new ideas. I did not expect my students to make sense of nonreal zeros or imaginary numbers in an entry-level algebra class. However, now they have an additional experience on which they can build to make sense of these concepts in the future. This is how students can begin to understand that numbers that are "not real" still have "real" mathematical value.

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Let's Chat about Imaginary Roots

On Wednesday, September 26, at 9:00 p.m. ET,

we will discuss "Making Imaginary Roots Real," by Natasha T. K. Murray (pp. 28–33).

Join the discussion at #MTchat.