Promoting a Conceptual Understanding of Mathematics

This article provides an overview of the eight effective mathematics teaching practices first described in NCTM's *Principles to Actions: Ensuring Mathematical Success for All.*

Margaret Smith, Victoria Bill, and Mary Lynn Raith

The vision of mathematics learning advocated by NCTM for twenty-five years (1989, 1991, 2000, 2014) positions students as active learners, constructing their knowledge of mathematics through exploration, discussion, and reflection. World-class standards put into place by states and provinces over the last decade support this vision by emphasizing reasoning, problem solving, and perseverance. The challenge that teachers face is how to make this vision, and the standards that embody it, a reality in their classrooms.

Principles to Action: Ensuring Mathematical Success for All (NCTM 2014) provides guidance for meeting this challenge by articulating a set of eight teaching practices that provide a framework for strengthening the teaching and learning of mathematics (see **fig. 1**). These eight effective teaching practices describe the intentional and purposeful actions that teachers must take to support the engagement and learning of each and every student.

So what do these practices actually look like when thoughtfully enacted by a teacher during instruction? How can teachers begin to develop these skills? In the remainder of this article, we address these two questions.

ENACTING THE EIGHT EFFECTIVE TEACHING PRACTICES

The discussion of the eight effective teaching practices that follows is based on Exploring Fraction Division: The Case of Mr. Kevin Richard. In the lesson, Mr. Richard used Max's Dog Food task (see **fig. 2**) to help his students explore fraction division. The case is online with this article at **https://www.nctm.org/mtms**. We encourage you to stop and read it before proceeding. As we discuss each of the practices, we will identify specific line numbers from the case that exemplify the point we are trying to make.

Establish Mathematics Goals to Focus Learning

Richard had a clear goal for student learning. He wanted his students to understand that "1. Dividing one number a by another number b means determining how many times b is contained in a; and 2. when dividing by a fraction the remainder is expressed as a fraction of the divisor" (lines 2–4). Although he ultimately wanted students to develop procedures for dividing fractions, in the lesson featured in the case, his goal was for students to understand what happens when you divide a mixed number by a fraction and how to interpret the answer. This goal is grade-level appropriate and connects to rigorous standards such as those found in the Common Core: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions and use visual models to interpret the problem (CCSSI 2010).

Richard used the goal to focus his decision making during the lesson. For example, he selected students to present two different representations (an area model and a number line model, shown in **figs. 3** and **4**), both of which highlighted the fact that 12 1/2 was being grouped into sets of three-fourths (his first goal). He made clear that the fractional 1/2 pound remaining needed to be interpreted in terms of a portion of the 3/4 serving (goal 2). Toward the end of class, he confirmed that the task was in fact a division problem (lines 103–4) and gave students

Fig. 1 *Principles to Actions* contains these eight effective mathematics teaching practices (NCTM 2014, p.10).

- Establish mathematics goals to focus learning
- Implement tasks that promote reasoning and problem solving
- Pose purposeful questions
- Use and connect mathematical representations
- Facilitate meaningful mathematical discourse
- Elicit and use evidence of student thinking
- Support productive struggle in learning mathematics
- Build procedural fluency from conceptual understanding

Fig. 2 Max's Dog Food task explored division of fractions.

Dog food is sold in a 12 1/2 pound bag. My dog, Max, eats a 3/4 pound serving every day. How many servings of dog food are in the bag? Draw a picture, construct a number line, or make a table to explain your solution.

Source: Institute for Learning at the University of Pittsburgh (2016)

both an exit ticket and homework (lines 108–111) that would give him insight on what individual students understood about fraction division.

It is noteworthy that Richard's goals focused on what he wanted students to learn as a result of engaging in the lesson, not on what students would do. Although students' learning and doing are both important, clarifying what students will learn provides guidance in determining what students actually understand. Simply getting an answer of 16 2/3 servings would tell Richard that students could do the problem, but it would provide no insight regarding what they understood about fraction division.

Implement Tasks That Promote Reasoning and Problem Solving

Max's Dog Food task is what has been referred to as a high-level or cognitively challenging mathematical task (Stein and Smith 1998). If students have not previously learned an algorithm for solving the task (as was true of Richard's students), they have to make sense of the situation and determine a course of action. The task can be approached in many different ways, using repeated addition, repeated subtraction, a ratio table, a tape diagram or area model, or a number line model. Such tasks promote equity by ensuring that each and every student can enter the task and in so doing demonstrate to the teacher what he or she understands about the situation.

Critical to the success of his lesson was the fact that the task Richard selected aligned with his goals for the lessons. Although the task had the potential to help him accomplish his goals, the way in which he enacted the lesson led to its ultimate success. He never suggested that students follow a particular pathway, and he asked students questions throughout the lesson to help them make sense of the situation. When students found an answer of 16 1/2, he did not tell them that they were wrong. Instead, he invited students to give explanations that provided the class with several opportunities to consider why 16 2/3 made sense. For example, referring to **figure 3**, the following explanation occurred (lines 39–42; 44–46):

Phoebe: Well every time we had 3 of the fourths we knew this would be one serving. So we started with the first group of 3 one-fourths and labeled them with 1s to show they were all part of the first serving. Then we just kept going. We found we had 16 groups of 3/4.

Teacher: So what did you do with the 1/2 pound?

Phoebe: So we made the 1/2 box a whole box and shaded in the 1/2 pound. We knew that a serving was 3/4 of a pound, and that would be 3 pieces of the box. So we had 2 of the 3 pieces needed for a serving. The answer is 16 2/3 servings.

Subsequently, another student, Kate, explained why the answer was 16 2/3 and not 16 1/2 using the area model in **figure 3** (lines 54–59). Tabitha provided a similar argument using the number line model in **figure 4** (lines 62–68).

Pose Purposeful Questions

Throughout the lesson, Richard asked his students many questions, most of which were open-ended. These questions helped students explain, clarify, and elaborate on their thinking, or pressed students to consider mathematical ideas more explicitly. For example, during the whole-group discussion, Richard asked the members of group 1 to explain what they did (line 31) and then asked group members to be more specific about how they determined their answer (lines 37–38). At several points during the

Max's Dog Food task promotes equity by ensuring that each student can enter the task and demonstrate understanding.

lesson, the teacher pressed students to look for connections between different approaches (lines 71–72; line 93), which served to highlight how the different representations were used to model division and particularly how the representations showed a fraction divided by a fraction.

Asking questions gives teachers information about students' thinking that can help the teacher determine the next course of action in a lesson. When Richard asked whether students agreed or disagreed with Sarah and Chris's assertion that this was a division problem (lines 84–85), he learned that Reilly did not agree. Reilly indicated that his group did not divide, they added (line 87). The teacher then asked Reilly to "tell us what you did" (line 88). By pursuing Reilly's position, Richard was able to understand his point of view and ultimately help the entire class see how repeated addition was used to determine the number of 3/4 in 12 1/2—the meaning of a division situation.

Fig. 3 An area model for the division of fractions was one representation given.

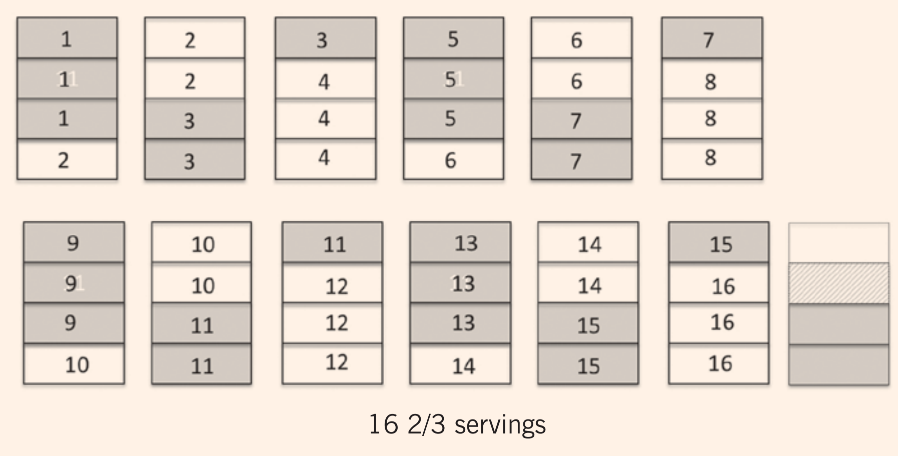
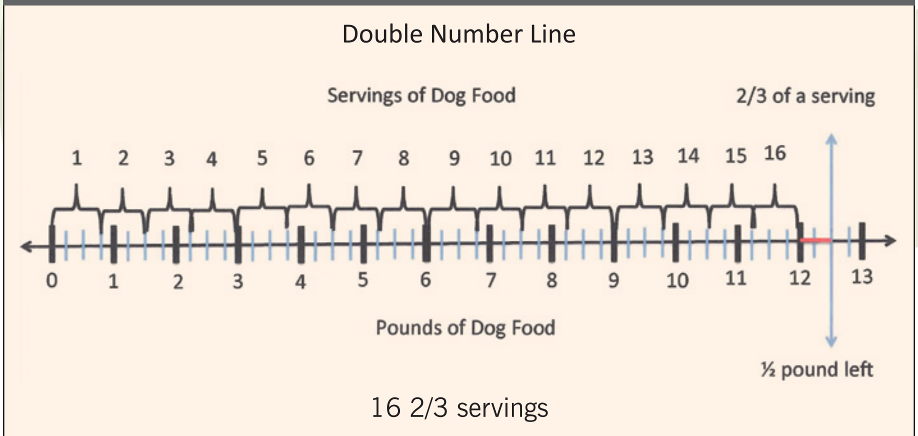


Fig. 4 The second example involved the number line model.



Use and Connect Representations

Richard selected a task that could be represented in different ways, knowing that his students had access to a range of appropriate representations they could draw on. This made it possible for his students to select a representation that made sense to them.

He pressed students to make explicit connections between the different representations that were shared. For example, after Tabitha explained her group's number line, Richard asked the class how the two methods that had been presented (the number line and area model) were the same and how they were different (lines 71–72). Later in the class when repeated addition was described, he asked a student to relate this approach to the number line model (line 93). As a result of these explicit connections, students were able to see how these three approaches were related to one another and to the operation of division.

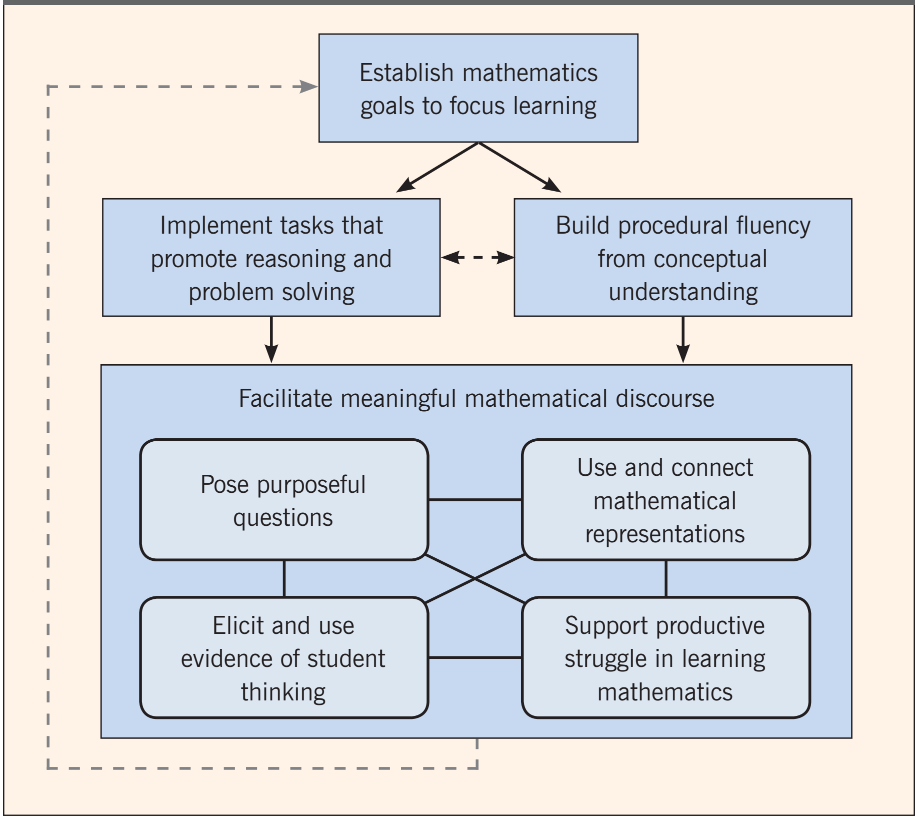
Making connections between different representations is critical in developing mathematical understanding. By allowing students to explore fraction division by first using representations that make sense to them, they come to understand mathematics more deeply.

Facilitate Meaningful Mathematical Discourse

The discussion that occurred in Richard's class was built almost exclusively on the thinking of his students. He invited two different groups to share the representations they had constructed because these representations served to highlight the key points he was trying to make in the lesson. He invited participation from other students in the class to seek clarity (lines 37–38; line 53) or consensus (lines 80; lines 84–85) and to make connections between approaches (lines 71–72). He later invited a third group to present its repeated-addition strategy and invited the class to explain the strategy and relate it to the number line. Through his efforts, Richard's students developed considerable investment in and ownership of the lesson.

Although having students present, discuss, and relate different strategies are critical components of a meaningful discussion, it is paramount not

Fig. 5 This framework for mathematics teaching and learning highlights the relationships between and among the eight effective mathematics teaching practices (Boston et al 2017, p. 215; Huinker and Bill 2017, p. 245; and Smith, Steele, and Raith 2017, p. 194).



to lose sight of the lesson's goal. A discussion can easily turn into a set of show-and-tell presentations in which it is not clear how one approach relates to others or to the mathematical ideas that were targeted in the lesson. This did not happen in Richard's class. He was very specific about the approaches that he asked students to share during the discussion and used the identified approaches to bring ideas to the surface that he wanted students to grapple with, and ultimately learn, during the lesson. He also continued to press students to explain and make connections.

What made the discussion successful? Although it appears that Richard was able to make many in-the-moment decisions during the lesson, the success of the lesson was due in large measure to his advanced planning. Specifically, he engaged in the five practices for orchestrating productive discussions—anticipating, monitoring, selecting, sequencing, and connecting. (See Smith and Stein 2018 for more information on these practices.) These practices helped Richard consider what students would do, how he would respond, and how he could use the work of students to advance the mathematics learning of the class.

Elicit and Use Evidence of Student Thinking

Richard elicited evidence of students' thinking through his questions that pressed students to explain, clarify, discuss, compare, take a position, and make sense of division. Nearly every utterance from Richard was intended to make students' thinking visible. Once a student's thinking had been made public, he used what he learned to move the student (and the rest of the class) forward. For example, after Marcus, Phoebe, and Jasmine had explained why the answer was 16 2/3, Richard asked students if they had any questions for the group (line 51). Duncan indicated that he got 16 1/2 and did not see how the answer could be 16 2/3. Because of Duncan's confusion, the teacher asked another student to explain why it had to be 16 2/3. In so doing, the teacher used Duncan's confusion to elicit further explanations (first Kate's, and later Tabitha's) to help clarify how to interpret the remainder.

Asking questions that elicit student thinking is critical in determining what students understand. However, the student's response to the question must give the teacher information for his or her next move. If the teacher learns that a student is confused, then the teacher must determine what to do next to address the confusion. Simply telling the student that he or she is wrong or indicating the correct answer will not help a student move forward in understanding. The course of action that a teacher takes must give the student the opportunity to continue to make sense of the situation.

Support Productive Struggle in Learning Mathematics

Productive struggle begins by presenting students with a task that is within their reach, but not something they already know how to do, and then giving them support that will allow them to make progress on the task without taking over their thinking. Richard presented his students with a challenging task that they were able to make sense of, and ultimately solve, on the basis of their prior work with different representations and their knowledge of whole-number division. When students struggled

Supporting productive struggle requires determining what students understand, what is causing confusion, and then determining a question to ask that helps students reach beyond the impasse.

initially (lines 15–18), he asked questions to help determine what they understood about the problem situation and then made suggestions that would likely help them get a foothold on the problem.

During his monitoring of group work, Richard noted that some students had determined that the answer was either 16 1/2 servings or 16 2/3 servings. Rather than correct students' incorrect answers, he used the whole-class discussion to give them the opportunity to see different representations and hear different explanations, which were all intended to help them make sense of the situation and revise their initial thinking.

Supporting productive struggle requires determining what students understand, what is causing confusion, and then determining a question to ask that helps students beyond an impasse that they have reached from their own actions rather than being told what to do and how. Although telling students what to do to help them negotiate an impasse will help students get an answer, such actions have no long-term benefit since they are not based on developing understanding.

Build Procedural Fluency from Conceptual Understanding

The lesson featuring Max's Dog Food was not focused on building procedural fluency. This was clearly a lesson intended to help students developing an understanding of fraction division. However, Richard was laying a foundation on which procedural fluency could be built. For example, in both of the models students created (shown in **fig. 3** and **4**) students divided the 12 1/2 pounds into fourths, resulting in 50/4. They divided the 50/4 into groups of 3/4, an action that can be modeled by 50/4 ÷ 3/4. The quotient resulting from the use of the models (16 2/3) is the same as what results from dividing 50 by 3. Hence, the work done with the two representations could ultimately be used to develop the common denominator algorithm for dividing fractions.

PULLING IT ALL TOGETHER

The success of Richard's lesson was due not to his use of any one of the eight effective teaching practices but rather to the synergy that was created from integrating the practices in a coherent way. The model shown in **figure 5** depicts the relationship between and among the practices and how they work together to support ambitious instruction, as evidenced by what transpired in Richard's class.

As the model suggests, the first step is to define lesson goals to provide a clear direction for the lesson. The second step is to determine the focus of a particular lesson: Will it focus on developing conceptual understanding by engaging students in reasoning and problem solving, or will it focus on developing fluency based on previously developed conceptual understanding? The double-headed arrow that connects these two practices in the model highlights the symbiotic relationship between them.

The large rectangle in the model shows the interaction between posing questions, using and connecting representations, eliciting and using evidence of student thinking, and supporting productive struggle and how these four practices contribute to facilitating meaningful mathematics discourse.

For example, the questions that Richard asked elicited students' thinking and supported their productive struggle; the representations that students used supported their ability to make sense of the situation and to communicate their thinking in the public forum. Engaging in these practices during instruction and planning for their use before instruction can help improve the quality of teaching.

Let's Chat about Promoting a Conceptual Understanding

On Wednesday, September 19, 2018, at 9:00 p.m. ET, we will discuss "Promoting a Conceptual Understanding of Mathematics" (pp. 36–43), by Margaret Smith, Victoria Bill, and Mary Lynn Raith. Join us at #MTMSchat.

DEVELOPING ESSENTIAL TEACHING SKILLS

How can you get started making these teaching practices a central part of your instructional repertoire? Several resources are available that may help you in your journey to improved teaching:

- *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014) provides an introduction to the eight effective teaching practices. (http://www.nctm.org/Store/Products/Principles-to-Actions--Ensuring-Mathematical-Success-for-All/)
- The Professional Learning Toolkit includes professional development modules featuring narrative and video cases, each of which focuses on a subset of the eight effective teaching practices (http://www.nctm.org/PtAToolkit/)
- *Taking Action: Implementing Effective Mathematics Teaching Practices* presents grade-level activities designed to support learning each of the eight effective teaching practices (http://www.nctm.org/store/takingaction/)

These resources can be explored individually or with a group of colleagues who are sharing insights and reactions, working through activities together, trying things out in the classroom, and sharing experiences and next steps. You may want to co-plan lessons with colleagues using the eight effective teaching practices as a framework and engage in observations and analysis of teaching (live or in narrative or video form) and discuss the extent to which the eight practices appear to have been used by the teacher and what impact they had on teaching and learning.

Changing one's teaching is hard work. It takes sustained and meaningful effort, but over time you will improve in your ability to enact the eight effective teaching practices. The payoff will be improved student learning outcomes.

REFERENCES

Boston, Melissa D., Fredrick Dillon, Margaret S. Smith, and Stephen Miller. 2017. *Taking Action: Implementing Effective Mathematics Teaching Practices in Grades 9–12*. Reston, VA: National Council of Teachers of Mathematics.

Huinker, DeAnn and Victoria Bill. 2017. *Taking Action: Implementing Effective Mathematics Teaching Practices in Grades K–5*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 1989.

Curriculum and Evaluation Standards for School Mathematics. Reston, VA: NCTM.

———. 1991. *Professional Standards for Teaching Mathematics*. Reston, VA: NCTM.

———. 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

———. 2014. *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: NCTM.

Smith, Margaret S., and Mary Kay Stein. 2018. *5 Practices for Orchestrating Productive Mathematics Discussions*. 2nd ed. Reston, VA: National Council of Teachers of Mathematics.

Smith, Margaret S., Michael D. Steele, and Mary Lynn Raith. 2017. *Taking Action: Implementing Effective Mathematics Teaching Practices in Grades 6–8.* Reston, VA: National Council of Teachers of Mathematics.

Stein, Mary Kay, and Margaret S. Smith. 1998. "Mathematical Tasks as a Framework for Reflection: From Research to Practice." *Mathematics Teaching in the Middle School* 3 (January): 268–75.

ACKNOWLEDGMENT

This article is based on the Taking Action series written by the team of authors that includes Victoria Bill, Melissa Boston, Fredrick Dillon, DeAnn Huinker, Stephen Miller, Mary Lynn Raith, Margaret Smith, and Michael Steele.

more4U

A case is online at **https://www.nctm.org/mtms**. More4U material is a members-only benefit.

Margaret Smith, pegs@pitt.edu, is a professor emerita at the University of Pittsburgh. She develops research-based materials for use in the professional development of mathematics teachers. **Victoria Bill,** vbill@pitt.edu, is a Fellow with the Institute for Learning at the Learning Research and Development Center, University of Pittsburgh. **Mary Lynn Raith,** mlr5813@aol.com, was a supervisor and teacher in the Pittsburgh Public School District. She is currently creating and critiquing mathematical tasks and conducting state-level standards alignment analyses.