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Young Children's Intuitive Models of Multiplication and Division

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In this study, an intuitive model was defined as an internal mental structure corresponding to a class of calculation strategies. A sample of female students was observed 4 times during Grades 2 and 3 as they solved the same set of 24 word problems. From the correct responses, 12 distinct calculation strategies were identified and grouped into categories from which the children's intuitive models of multiplication and division were inferred. It was found that the students used 3 main intuitive models: direct counting, repeated addition, and multiplicative operation. A fourth model, repeated subtraction, only occurred in division problems. All the intuitive models were used with all semantic structures, their frequency varying as a complex interaction of age, size of numbers, language, and semantic structure. The results are interpreted as showing that children acquire an expanding repertoire of intuitive models and that the model they employ to solve any particular problem reflects the mathematical structure they impose on it.

Several recent studies have shown that students can solve a variety of multiplicative problems long before formal instruction on the operations of multiplication and division. For example, Kouba (1989) found that 30% of Grade 1 and 70% of Grade 2 students could solve simple equivalent group problems and Mulligan (1992) found a steady increase in success rate on similar problems from over 50% at the beginning of Grade 2 to nearly 95% at the end of Grade 3. More recently Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) found that even kindergarten students could learn to solve multiplicative problems.

Students use a range of solution strategies to solve multiplication and division word problems, and from this it has been inferred that they acquire various intuitive models of multiplication and division (Fischbein, Deri, Nello, & Merino, 1985; Kouba, 1989; Greer, 1992). The interest in intuitive models lies in the proposition that they are formed early on in elementary contexts and can strongly influence students' understanding of more complex multiplicative situations in secondary school and adulthood, often negatively (Fischbein et al. 1985; Graeber, Tirosh, & Glover 1989; Simon, 1993). However, it is not clear what intuitive models young children form, how they are related to the semantic structure of the problems to be solved, and how the models develop over time. The present paper attempts to throw light on these questions using data from a longitudinal study of students in Grades 2 and 3.

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We will use the term *multiplicative* to describe situations that lead to either multiplication or division, and we will restrict our attention to one-step word problems.

BACKGROUND

Semantic Structure

Multiplication situations can be classified according to the nature of the quantities involved and the relation between them (Nesher, 1988; Vergnaud, 1988). Greer (1992) lists four categories that primarily apply to problems involving the multiplication of whole numbers.

- equivalent groups (e.g., 2 tables, each with 4 children)
- multiplicative comparison (e.g., 3 times as many boys as girls)
- rectangular arrays (e.g., 3 rows of 4 children)
- Cartesian product (e.g., the number of possible boy-girl pairs)

Greer also lists six other categories that readily admit fractions and decimals.

Every multiplication situation can lead to various division problems. Equivalent groups division problems have classically been categorized as partition (sharing) and quotition (measurement) situations (Fischbein et al., 1985; Kouba, 1989). For example, "8 children, 4 tables; how many at each table?" is a partition problem, and "8 children, 4 at each table; how many tables?" is a quotition problem.

Following Nesher (1988), we will refer to the category to which a multiplicative problem can be assigned as its *semantic structure*. It has been found that mathematically equivalent problems of different semantic structure evoke different solution strategies and vary widely in difficulty (Bell, Fischbein, & Greer, 1984; Bell, Greer, Grimison, & Mangan, 1989; Brown, 1981; De Corte, Verschaffel, & Van Coillie, 1988; Fischbein et al., 1985; Nesher, 1988; Vergnaud, 1988).

Classification of semantic structure is clearly somewhat arbitrary in that the categories can be extended, collapsed, or refined depending on the purpose of the investigation. For example, Schmidt and Weiser (1995) recently presented a fourfold classification of semantic structures with several subclasses. The crucial point is that the semantic structure of a problem is determined by a researcher prior to its presentation to students and does not necessarily indicate how students will solve the problem.

Solution Strategies

Several studies of the actual strategies students correctly use to solve whole-number multiplicative problems have yielded fairly consistent results (Anghileri, 1989; Boero, Ferrari, & Ferrero, 1989; Carpenter et al., 1993; Kouba, 1989; Mulligan, 1992; Mulligan & Mitchelmore, 1996; Murray, Olivier, & Human, 1992; Steffe, 1994). Strategies have been classified in two ways. One way of classifying solution strategies is by the calculation strategies students employ (Anghileri, 1989)—called *degree of abstractness* by Kouba (1989) and Mulligan (1992). Combining descriptions from various studies leads to the five categories summarized in Table 1. Variations of all strategies occur in both multiplication and division problems. In particular, we interpret both *unitary counting* in multiplication problems and *sharing* or *dealing* in division problems as direct counting. We will return later to the questions of what unitary counting and sharing involve and whether it is justified to put them in one category.

Table 1
Children's Calculation Strategies for One-Step, Whole-Number, Multiplicative Word Problems

Strategy	Definition
1. Direct counting	Physical materials are used to model the problem and the objects are simply counted without any obvious reference to the multiplicative structure.
2. Rhythmic counting	Counting follows the structure of the problem (e.g., "1, *2*; 3, *4*; 5, *6*" or "*6*; 5, *4*; 3, *2*.") Simultaneously with counting, a second count is kept of the number of groups.
3. Skip counting	Counting is done in multiples (e.g., "2, 4, 6" or "6, 4, 2"), making it easier to keep count of the number of groups.
4. Additive calculation	Counting is replaced by calculations such as "2 + 2 = 4, 4 + 2 = 6" or "6 – 2 = 4, 4 – 2 = 2."
5. Multiplicative calculation	Calculations take the form of known facts (e.g., "3 times 2 is 6" or derivations from a known fact (e.g., "3 × 2 = 2 × 2 + 2.")

The second classification is called use of physical objects by Kouba (1989) and modeling strategies by Mulligan (1992) and Carpenter et al. (1993). Students may model the problem situation using such objects as tokens or fingers; they may model it by drawing icons or tallies; or they may not model the situation externally at all. All these possibilities have been reported as occurring in conjunction with all five strategies in Table 1.

INTUITIVE MODELS

What Is an Intuitive Model?

The notion of intuitive models (also called implicit, tacit, or informal models) appears to have originated with Fischbein et al. (1985). They hypothesize that "each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model" (p. 4), which mediates the search for the operation needed to solve a problem. They also state, "When trying to discover the nature of the intuitive model that a person tacitly associates with a certain operation, one has to consider some practical behaviour that would be the enactive, effectively performable counterpart of the operation" (p. 5).

This definition seems to imply that an intuitive model is an internalization of the physical operation involved in the corresponding problem situation, and therefore, that a direct correspondence exists between intuitive model and semantic structure. Indeed, Fischbein et al. (1985) describe three intuitive models that directly correspond to three semantic structures. For multiplication, they hypothesize a repeated-addition model "in which a number of collections of the same size are put together" (p. 6)—reflecting the equivalent-groups semantic structure. For division, they describe intuitive models corresponding to partition and quotition semantic structures and claim that "the structure of the problem determines which model is activated" (p. 7). On the hypothesis that these are the dominant (but tacit) models among students in Grades 5, 7, and 9, they were able to predict with some success how performance on rational number multiplicative problems depends on the type of numbers involved. Fischbein et al. also conjectured that the hypothesized models persisted from intuitive models of multiplication and division, which had been formed much earlier in whole-number contexts.

Kouba (1989) was stimulated by the Fischbein et al. (1985) study to investigate these earlier intuitive models directly, by observing how young children solved whole-number problems. Problems were selected to fit the three semantic structures already described. For our purposes, her most pertinent finding was that partition and quotition problems did not generate different calculation strategies. She reported, without citing specific data, that both types of division were solved by using either repeated subtraction or by repeated building up. She concluded:

> There also are similarities in the intuitive models that students appear to have for measurement division and partitive division.… [A]t an intuitive level, many students may perceive these different types of division as being more related than is evident in the Fischbein et al. (1985) descriptions of the intuitive models.… The link probably has more to do with the children's use of counting as a means of doing all operations. (p. 157)

This finding implies that there may in fact be no direct correspondence between the semantic structure of a problem and the method that a child uses to solve it.

Kouba's (1989) study suggests that it would be valuable to examine young children's solution strategies in more detail and, in particular, to look for categories of similar calculation strategies used across a range of semantic structures. Each category of calculation strategy could then be seen as evidence for an internal mental structure that children impose on multiplicative situations and that reflects particular aspects of the mathematical structure. Kouba is clearly thinking of such internal mental structures when she uses the term *intuitive model*, and we will do the same.

It seems preferable to define an intuitive model in terms of calculation strategies rather than modeling strategies. For although modeling strategies are important in practice—it is usually more efficient to use visualization or arithmetical symbols than concrete models—it could be argued that they reflect children's familiarity with a particular calculation strategy rather than fundamental differences in the way children structure the problem. In other words, the same intuitive model could be used with a variety of modeling strategies. On this basis, what intuitive models can we expect to find?

Intuitive Models of Whole Number Multiplication and Division

Anghileri's (1989) results, obtained over six semantic structures, suggest only three intuitive models for whole-number multiplication: those corresponding to unitary counting, repeated addition, and multiplicative calculation. It seems reasonable to combine rhythmic counting, skip counting, and additive calculation strategies into the one repeated addition model if we observe that all these strategies are based on the same principle of double counting. As Anghileri argued, the increase in sophistication from rhythmic counting to additive calculation is more a result of a deepening understanding of addition than any basic change in calculation strategy. It is important to emphasize that we are using the term *repeated addition* here to refer to the structure of a particular class of calculation strategies and not, as many authors do, to the semantic structure of equivalent groups.

For whole-number division, Kouba's (1989) results suggest at least four intuitive models: sharing, repeated taking away, building up, and multiplicative calculation. Both repeated taking away and building up appear to be based on the same principles as repeated addition. Other classes of calculation strategies might appear if a wider range of semantic structures were included.

The above interpretation naturally raises the following questions:

1. Can the proposed intuitive models be identified in new data? In particular, is it justified to put unitary counting and sharing in one model?
2. Do any new intuitive models appear when a broader range of semantic structures is included?
3. Does the semantic structure of the problem influence children's intuitive models?
4. How do children's intuitive models change over time, especially as a result of instruction?

The longitudinal study to be described below was designed to provide some answers to these questions.

METHOD

Design

Clinical interviews were conducted in March/April and November/December (i.e., early and late in the Australian school year) in two successive years when the students were in Grades 2 and 3. At the time of the first interview, students had received teacher instruction in basic addition and subtraction but not in multiplication and division. Between the third and fourth interviews, all students were given instruction in basic multiplication facts with the 2–10 times tables but not in related division facts. The mathematics teaching practice was monitored over the entire study. All participating teachers followed the official K–6 mathematics syllabus (New South Wales Department of Education, 1989), which emphasizes the acquisition of basic facts and computational skills. No teacher reported giving instruction in solving word problems.

Sample

The interview sample initially comprised 70 girls ranging from 6.5 to 7.5 years of age at the time of the first interview; of these, 60 remained at the final interview. The students were randomly selected from eight schools in Sydney. The interview sample was limited by gender to avoid the effect of possible gender differences. Two students with very poor reading ability were excluded.

The Problems

The problems used are shown in Table 2. All problems were set in familiar contexts and all involved only whole-number data and answers.

Table 2
Multiplication and Division Word Problems

Multiplication	Division
Equivalent groups	*Partition*
1. There are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether? (4, 7)	7. There are 8 children and 2 tables in the classroom. How many children are seated at each table? (28, 4)
2. Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether? (3, 7)	8. Six drinks were shared equally among 3 children. How many drinks did they each have? (14, 7)
Rate	*Rate*
3. If you need 5 cents to buy 1 sticker, how much money do you need to buy 2 stickers? (5, 7)	9. Peter bought 4 lollipops with 20 cents. If each lollipop cost the same, how much did 1 lolly cost? (8, 40)
Comparison	*Comparison*
4. John has 3 books, and Sue has 4 times as many. How many books does Sue have? (6, 5)	10. Simone has 9 books. This is 3 times as many as Lisa. How many books does Lisa have? (40, 8)
Array	*Quotition*
5. There are 4 lines of children with 3 children in each line. How many children are there altogether? (3, 8)	11. There are 16 children, and 2 children are seated at each table. How many tables are there? (36, 4)
Cartesian product	
6. You can buy chicken chips or plain chips in small, medium, or large packets. How many different choices can you make? (8, 2)	12. Twelve toys are shared equally among the children. If they each had 3 toys, how many children were there? (24, 6)

Note. The text gives the small-number problems. The large-number problems were worded identically except for the substitution of the numbers given in parentheses.

The multiplication problems were chosen to represent 5 of the 10 semantic structures identified by Greer (1992). The other 5 structures were excluded on the basis that they involved measurement concepts that would be unfamiliar to students in Grade 2. Because the equivalent groups structure is the one most commonly used in textbook word problems, two such problems were included.

The division problems were constructed as inverse problems in the equivalent groups, multiplicative comparison and rate structures; inverses in the array and Cartesian product structures were judged to be inappropriate for the age level tested. Two problems were chosen for both the partitive and quotitive division structures.

Two categories of number size were selected to compare the influence of small and large numbers: The product was either between 4 and 20 (small numbers) or between 20 and 40 (large numbers). These number domains were used because the Grade 1 curriculum focused on whole-number combinations between 0 and 20 and extended to 100 by Grade 3.

Interview Procedures

All interviews were conducted by the first author in a small side room. Problems were written on cards and read to the child by the interviewer. The problems were re-read to the child as often as necessary to assist her in remembering the details. Forty small cubes were available on the table, but no paper or pencil. The interviewer explained to the child that she could use the cubes to assist in solving the problem if she wished. In this way students were forced to use either concrete modeling, mental calculation strategies, or both.

The small-number multiplication problems were administered first followed by the small-number division problems, in the order shown in Table 2. The large-number problems were then posed. However, a large-number problem was asked only if the child had been successful on the corresponding small-number problem. Students were praised for their attempts, but no feedback was given as to the correctness of their responses. The interviews lasted from 15 to 55 minutes with an average time of 30 minutes.

When modeling with cubes was not used, or a response was unclear, additional neutral questions were asked by the interviewer, for example, “Can you tell me how you solved the problem?” and “Did you see anything in your mind when you were solving the problem? Can you describe what this looks like?” When a child gave an explanation that appeared to conflict with anything the interviewer had observed, she was asked to describe what she was thinking at that time.

Data Coding

The interviewer recorded children’s responses as they solved each problem, drawing diagrams of children’s modeling and noting gestures and finger movements. She also coded each response as correct, incorrect, uncodable, or nonattempt. Where the child began using a correct strategy but gave an incorrect answer, the response was coded as incorrect. Each interview was also audiotaped. A research assistant checked the interviewer’s notes and audiotapes and clarified uncertainties in consultation with the interviewer.

RESULTS

For two reasons, the data analysis is confined to those responses coded as correct. First, the vast majority of incorrect responses resulted from superficial strategies (mainly, adding the two given numbers) or from incorrect models of the problem situation. No intuitive model of multiplication or division could be inferred from such responses. Second, children who used a correct strategy but made an error in the calculation usually corrected themselves spontaneously in the process of explaining their procedure.

Identification of Intuitive Models

Procedure

A two-stage procedure was used to identify children's intuitive models. First, children's responses were examined to find if their calculation strategies could be reliably identified. Second, the calculation strategies were grouped to infer underlying intuitive models.

The first author initially coded children's calculation strategies separately according to whether they used concrete modeling or no modeling. Where a child used two or more calculation strategies to solve the same problem, the most dominant strategy was coded. Several strategies were used both with and without concrete modeling, and some were used in both multiplication and division. Eventually, 12 different calculation strategies were identified and defined. They were essentially the same as those reported in a previous study (Mulligan, 1992).

Table 3
Intuitive Models for Multiplication and Division

Intuitive model	Calculation strategies
	Multiplication
1. Direct counting	Unitary counting
2. Repeated addition	Rhythmic counting forward Skip counting forward Repeated adding Additive doubling[a]
3. Multiplicative operation	Known multiplicative fact Derived multiplicative fact
	Division
1. Direct counting	One-to-many correspondence Unitary counting Sharing Trial-and-error grouping
2. Repeated subtraction	Rhythmic counting backward Skip counting backward Repeated subtracting Additive halving[b]
3. Repeated addition	Rhythmic counting forward Skip counting forward Repeated adding Additive doubling[a]
4. Multiplicative operation	Known multiplicative fact Derived multiplicative fact

[a]For example, "3 and 3 is 6, 6 and 6 makes 12."
[b]For example, "Cut 8 into two halves makes 4 and 4."

A research assistant then independently used the strategy definitions to code every fifth interview (48 transcripts). A 92% agreement rate was found and regarded as satisfactory.

Finally, the 12 calculation strategies were examined to identify overarching principles. They could be grouped into the proposed intuitive models; no further models appeared. The resulting intuitive models, and the strategies that correspond to them, are shown in Table 3.

We now give a more precise description of each model, as identified in our data, concentrating on features that have not been widely described to date.

Intuitive Models for Multiplication

Direct counting. The classic strategy of modeling the problem (using either cubes or visualization) and then counting the cubes one by one was frequently observed. Some other strategies also failed to take advantage of the equal sizes of the groups. For example, in response to Problem 1, Michelle put out three blue and five red cubes in two groups of four, said there were three boys and five girls, and calculated "3 + 5 = 8."

We infer from these responses an intuitive model that represents the problem situation correctly but does not impose on it an appropriate mathematical structure. The problem is essentially solved by the concrete materials themselves, together with an independent counting procedure. As Steffe (1988) puts it, a child using direct counting has not yet made the leap of regarding "three ones" as "one three."

Repeated addition. As found in previous studies, many students successfully solved multiplication problems by rhythmic counting, skip counting, or additive calculation. In effect, these methods all create an appropriate sequence of multiples. Repeated addition is an advance on direct counting because it takes advantage of the equal-sized groups present in the problem situation.

Multiplicative operation. This model was inferred when students gave correct responses without appearing to form the entire sequence of multiples. A typical response was, "I made one group of three and timesed it." The few students who used concrete modeling with this intuitive model only made one group of objects. More developed techniques were evident in responses that included recalled multiplication facts; for example, "I knew it was multiplication straight away.… You just do the multiplication.… Three fours are twelve." We justify including the use of derived facts with this model because, even though addition is used, the basic aim is to calculate a product without creating the entire sequence of multiples.

We call this an *operation model* because multiplication appears explicitly as a binary operation for the first time in this model. Both Fischbein et al. (1985) and Kouba (1989) comment on the importance of seeing multiplication as an operation, and Anghileri (1989) found that use of multiplication facts was much more common among students classified as above-average ability than among the others in her sample.

In the operation model, the final term is extracted from the implicit sequence of multiples and treated as a single entity. In this respect, the operation model is based on the repeated addition model but is distinctly different from it. Julianne typified the difference when she remarked at Interview 4, "I just thought of using the numbers and the multiplying sign in my head, I didn't need to count."

Intuitive Models for Division

Direct counting. Several strategies relied on concrete modeling followed by an independent counting process. The first step in all these strategies was to count out a number of cubes equal to the dividend. For partition problems, one strategy was simply to make a tentative grouping of the cubes and then move them from one group to the other until the numbers were equal. Another strategy was to deal out the cubes successively in ones or twos to the specified number of groups until they were exhausted. For quotition problems, a typical strategy was to successively separate out groups of the specified size until the cubes were exhausted and then to count the number of groups.

All these strategies seem to represent little more than a correct modeling of the problem situation together with accurate counting, as in the direct-counting model for multiplication. It is true that they achieve the aim of creating equal-sized groups, but the subsequent calculation procedure does not reflect this structure. Direct counting for multiplication and direct counting for division therefore seem to indicate essentially the same intuitive model.

Repeated subtraction. The strategies in this model all start with the dividend and use a systematic calculation procedure in which the number in each group is repeatedly taken away. For the quotition problems, the method is direct. For example, in response to Problem 11, Amy counted out 16 cubes and then took away groups of 2 cubes, saying "16, 14, 12, 10, 8, 6, 4, 2, nothing left … that's 8 tables." The important distinction between Amy's model and a direct counting model is that Amy simultaneously counted both the number of cubes left and the number of groups already formed. For nonquotition problems, a number has to be guessed and repeatedly taken away the specified number of times to check if the result is zero. Students seemed to be extraordinarily good at determining such numbers, without revealing any direct process for doing so.

All calculation strategies in this model, including additive halving, create a decreasing sequence of multiples starting with the dividend.

Repeated addition. This model is similar to the repeated subtraction model except that, instead of starting with the dividend, the child builds up to the dividend. This model is also direct for quotition problems and indirect for other problems. For example, in response to Problem 11, Susan counted aloud the sequence of multiples 2, 4, 6, …, and after each count identified the number of tables (shown in parentheses): "2 (1), 4 (2), 6 (3), 8 (4), 10 (5), 12 (6), 14 (7), 16 (8)—8 tables." Kouba (1989) called these strategies *building up*.

All strategies in this model create an increasing sequence of multiples. It is therefore basically the same model as the repeated-addition model for multiplication. We also argue that it is a more advanced model than repeated subtraction because it allows the same model to be used for both division and multiplication problems. Such unification would be expected to reduce cognitive load and lead to greater efficiency and a more rapid adoption of a multiplicative operation model.

Multiplicative operation. Strategies in this model used multiplication as an operation. In some cases, the solution was guessed and checked by multiplication. In others, the student appeared to search for a multiple of the divisor that was equal to the dividend—but without generating the entire sequence of multiples. However, all these strategies used known or derived multiplication facts. Only a few students demonstrated an explicit awareness of division as an operation, mostly in a halving strategy.

The multiplicative operation model of division is essentially the same model as the multiplicative operation model for multiplication. It appears to be related more closely to the repeated-addition model of division than to the repeated-subtraction model.

Variation in Intuitive Models

We turn now to an analysis of how children's intuitive models were related to the semantic structure of the problem and how they changed over time. The data show many complex interactions, and we can only summarize general trends here. To simplify the discussion, data on the duplicated problems (equivalent-groups multiplication, partition and quotition division) will be combined. Tables giving the percentages of the sample using each intuitive model correctly for each problem at each interview are available from the authors.

Multiplication

Figure 1 shows the overall distribution of the intuitive models employed in correct responses to the multiplication problems at each interview stage. The percentage of correct responses increased steadily from 31% to 68%, mainly because of a 12% increase in the successful use of repeated addition between Interview 1 and Interview 2 and a 24% increase in the successful use of the multiplicative operation model between Interview 2 and Interview 4. The percentage of correct responses using the direct-counting model remained steady at about 10%.

There was a consistent difference in performance between problems at all interviews and for both small and large numbers. The equivalent-groups, rate, and array problems were approximately equal in difficulty, the success rate increasing from an average of 45% at Interview 1 to 86% at Interview 4. The comparison problem was intermediate in difficulty, increasing from 6% to 52% over the period of the study. The Cartesian product problem was very difficult, averaging 1% correct over Interviews 1–3 but jumping to 14% at Interview 4.

Correct use of the direct-counting model was only frequent for the array problem (22% overall) and the equivalent-groups problems with large numbers (20%). Correct usage rarely exceeded 10% for the other problems, and it was never the most successful strategy.

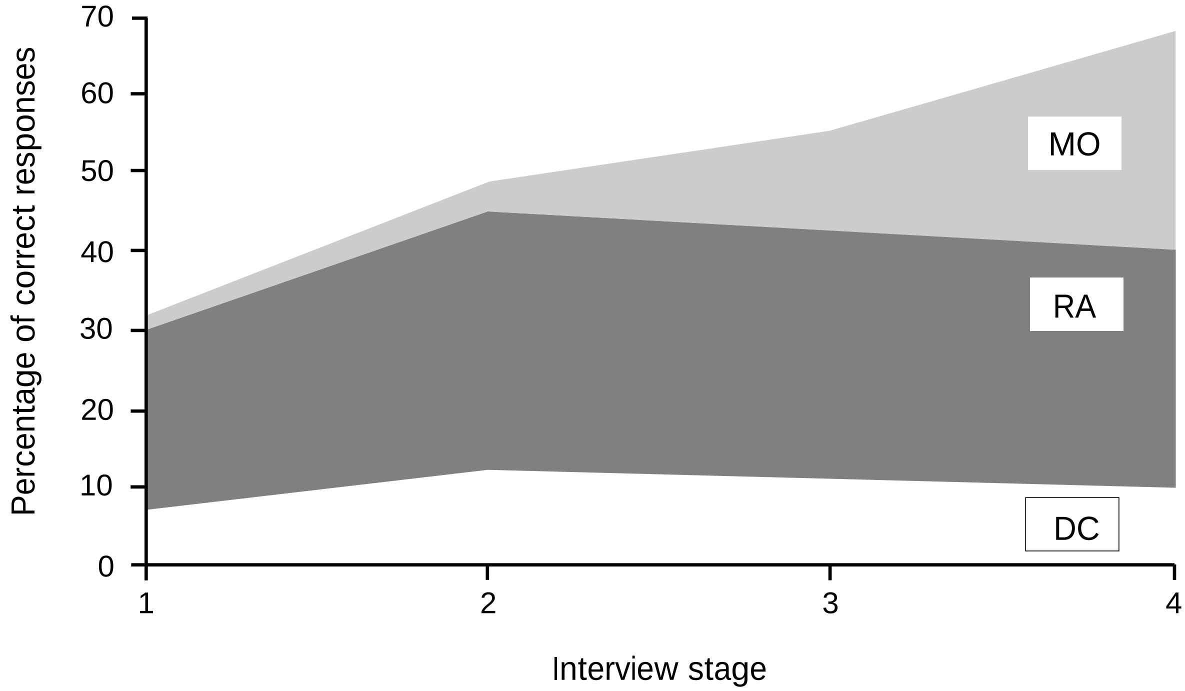

Figure 1. Percentage of sample giving correct responses to multiplication problems at each interview stage. Responses are classified by intuitive model: direct counting (DC), repeated addition (RA), or multiplicative operation (MO). Data pooled from 12 problems.

The repeated-addition model was the most frequent correctly used model on almost all occasions for all semantic structures except comparison. Figure 2 compares the various problem types. The relatively high success rate for rate problems is likely the result of the particular numbers involved. Both problems involved multiples of 5, which seem to be second only to multiples of 2 in terms of their familiarity to young children. However, in the small-number problem, doubling 5 was most often solved using a "5 and 5 makes 10" argument. In the large-number problem, the odd multiple of 5 seemed to be relatively unfamiliar and students simply counted in fives. The overall tendency shown in Figure 1 for the successful use of the repeated-addition model to increase and then decrease again was only present for the equivalent-groups, array, and comparison problems.

Figure 3 shows the variation in the successful use of the multiplicative operation model. Success was rare in the first two interviews, but it began to increase at Interview 3—just prior to formal instruction in multiplication—and had become fairly common by Interview 4 on all structures except the Cartesian product. It is notable that in Interviews 3 and 4, the comparison and equivalent-groups problems—although differing in overall difficulty—elicited almost equal numbers of correct responses using the multiplicative operation model. In fact, the multiplicative operation model was the most frequent correctly used model for the comparison problem. The cause seemed to be the linguistic cue "times"; for example, Lisa responded to the small-number problem by saying, "times as many … that's multiply … three fours."

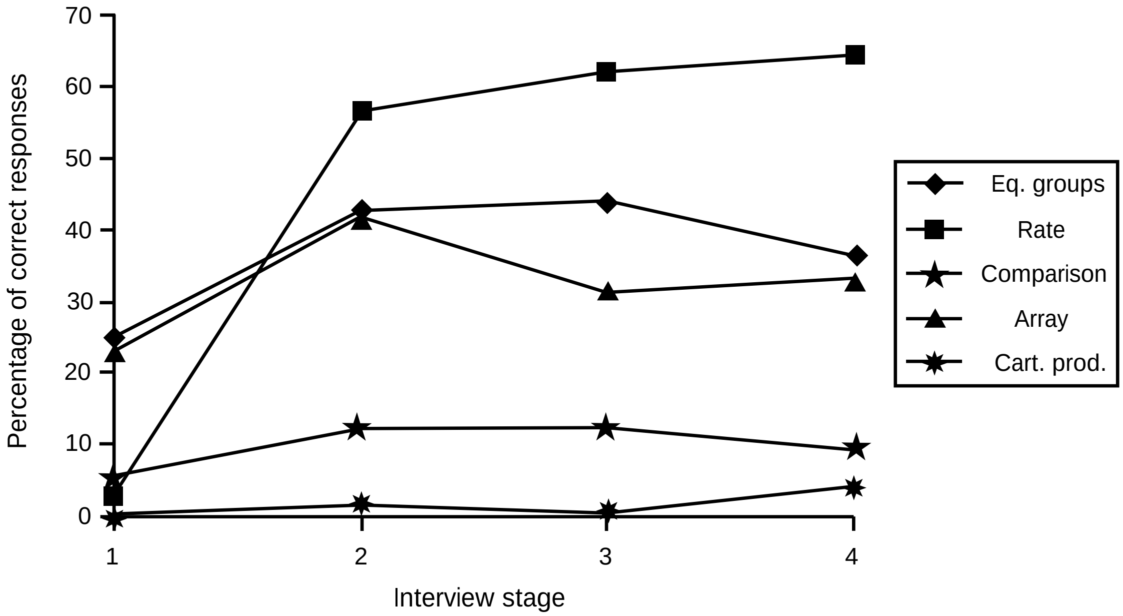

Figure 2. Percentage of sample using a repeated-addition model in correct responses to various multiplication problems at each interview stage.

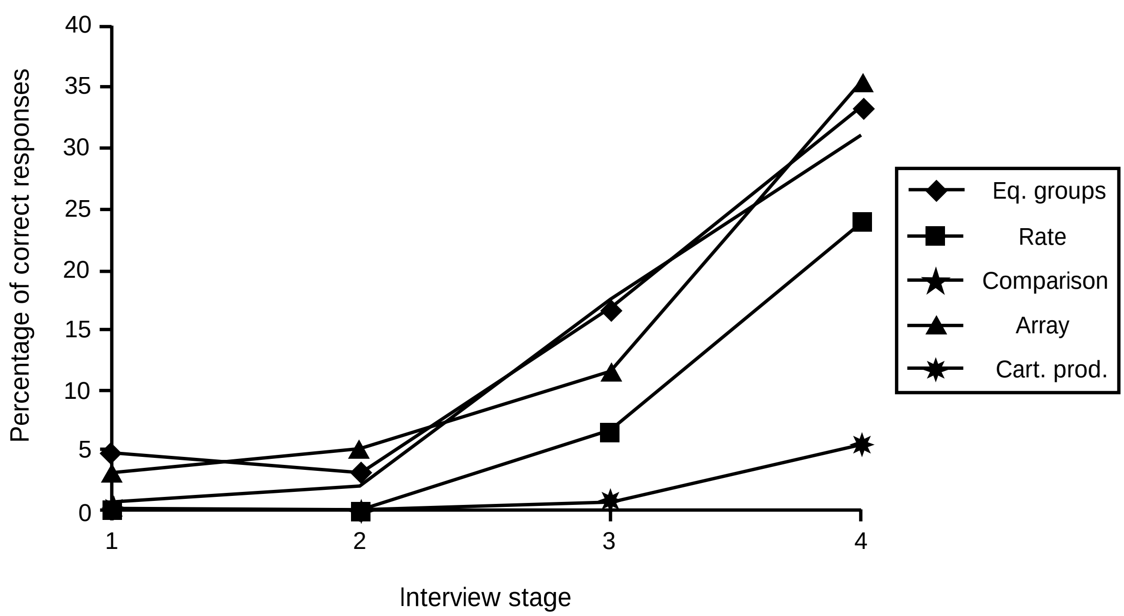

Figure 3. Percentage of sample using a multiplicative operation model in correct responses to various multiplication problems at each interview stage.

The size of the numbers in the problems had a fairly consistent effect. As might have been expected, overall success rate was lower on the large-number problems (43% correct) than the small-number problems (59%). Successful use of direct counting was more common for large numbers than small numbers (14% compared to 7%), whereas the reverse was true for both the repeated-addition model (20% compared to 38%) and multiplicative operation model (9% compared to 13%). Many students who had successfully used repeated addition for a small-number problem seemed to experience a "processing overload" when attempting to use the same strategy for the corresponding large-number problem; they then often solved the problem using direct counting. Students who had used a multiplicative operation for a small-number problem were often unable to retrieve the number fact required for the corresponding large-number problem and reverted to repeated addition.

Division

Figure 4 shows the overall distribution of the intuitive models employed in correct responses to the division problems at each interview stage. Overall performance increased from 33% at Interview 1 to 68% at Interview 4, despite the absence of formal teaching, which was virtually the same as for multiplication. As was the case for multiplication, correct use of direct counting was fairly constant (12% on average) and correct use of a multiplicative operation model increased substantially (by 16%) between Interviews 2 and 4. By contrast, correct use of the repeated addition and subtraction models increased steadily from 20% at Interview 1 to 38% at Interview 4.

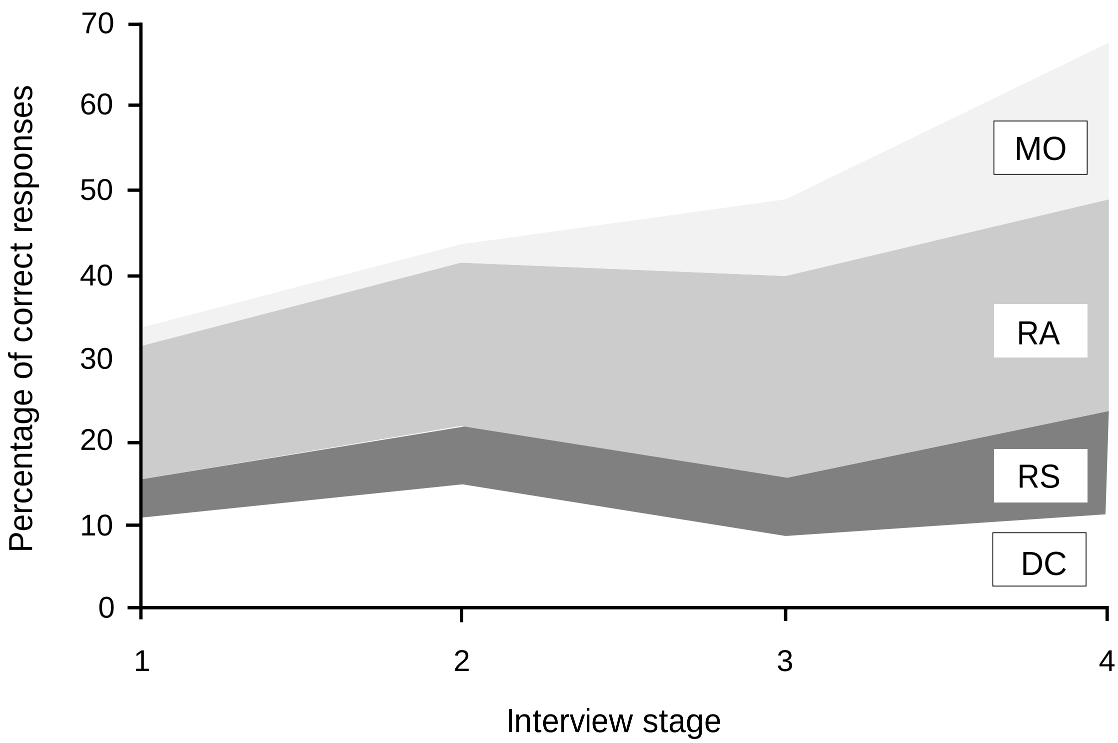

Figure 4. Percentage of sample giving correct responses to division problems at each interview stage. Responses are classified by intuitive model: direct counting (DC), repeated subtraction (RS), repeated addition (RA), or multiplicative operation (MO). Data pooled from 12 problems.

Performance on the partition, rate, and quotition problems was approximately the same, ranging from 37% at Interview 1 to 78% at Interview 4. The comparison problem was extremely difficult, averaging 2% correct over Interviews 1–3 but jumping to 14% at Interview 4.

Correct use of the direct counting model was mainly observed in the quotition problems (23% of responses) and in the large-number partition problems (13%). For the quotition problems, it was the most frequent correctly used model in Grade 2 and was still common at the end of Grade 3.

Correct use of the repeated-subtraction model was only consistently common on the small-number partition problem, Problem 7 (average 31%). This was the only problem easily solved by additive halving, which many of the students used. The only other cases where the repeated subtraction was successfully used in more than 10% of the responses were the other partition problems and the rate problems at Interview 4.

On all problems except comparison, repeated addition was common and almost always the most frequent correctly used model. Figure 5 shows the relevant data. Most notable is the difference between the increasing pattern for the quotition structure and the more stable pattern for the partition and rate structures. From Interview 2 to Interview 3, correct use of direct counting dropped from 34% to 16% and repeated addition or subtraction climbed from 17% to 33%. In Grade 2, the quotition problems were more difficult to understand than the partition and rate problems, children often interpreting "There are 2 children at each table" as "The children are at 2 tables." By Grade 3, more children were modeling the quotition situations correctly. The quotition structure, once understood, seemed to encourage double counting more than partition or rates.

Figure 6 shows the variation in successful use of the multiplicative operation model for division. As for multiplication, it was rarely used successfully in the first two interviews but began to appear at Interview 3. The greater success of the multiplicative operation model on the rate problems is explained by the specific numbers involved. Both problems involved multiples of 5. On the small-number problem, the multiplication fact $4 \times 5 = 20$ was frequently recalled, and on the large-number problem, 8×5 was often derived from it by doubling. It is instructive to compare this result with the two rate multiplication problems, which also involved multiples of 5 but where the most frequent correctly used model was repeated addition. The difference seems to lie entirely with the particular multiples required to solve the problems.

The size of the numbers in the division problems had an effect similar to that noted for multiplication. Overall success rate was lower on the large-number problems (38% correct) than the small-number problems (58% correct). Successful use of direct counting was slightly more common for large numbers than small numbers (12% compared to 11%). For the other models, correct solutions were less frequent for the large-number problems than the small-number problems: repeated subtraction (5% compared to 11%), repeated addition (16% compared to 27%), and multiplicative operation (5% compared to 9%). As for multiplication, large-number problems seemed to make demands on information retrieval or processing capacity that forced many students to revert to a more primitive and less demanding model.

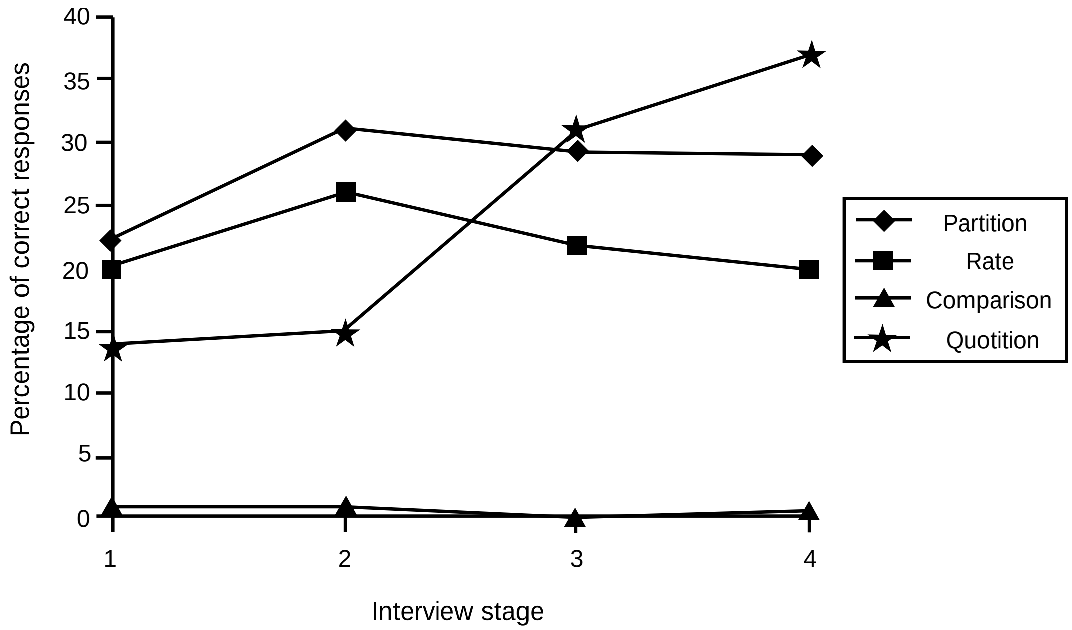

Figure 5. Percentage of sample using a repeated-addition model in correct responses to various division problems at each interview stage.

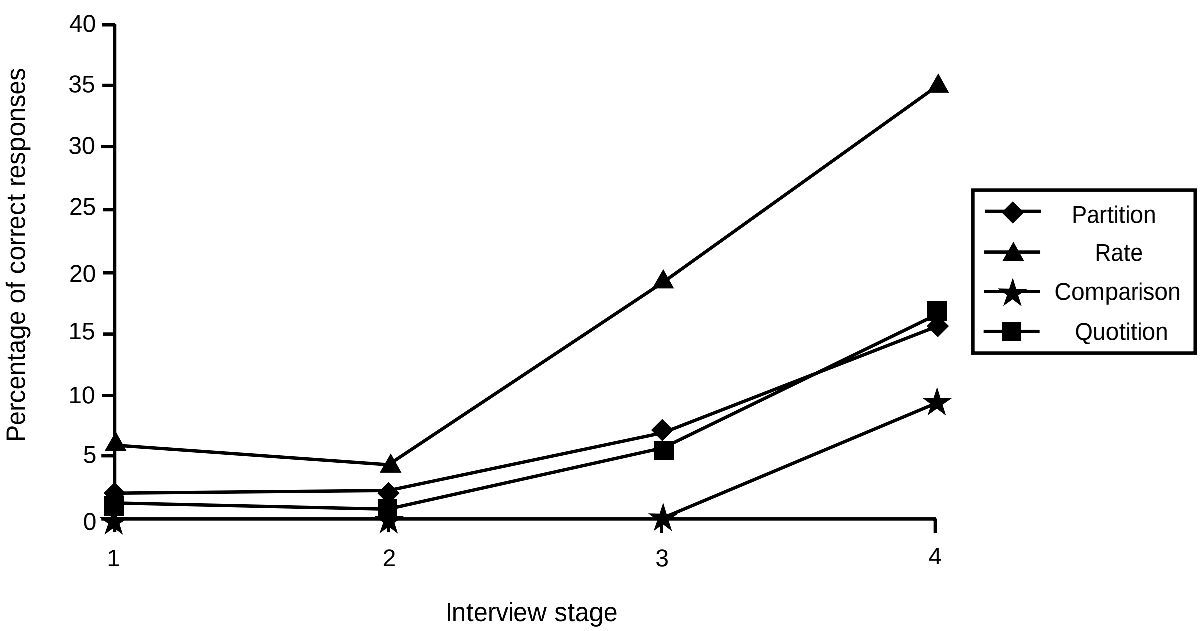

Figure 6. Percentage of sample using a multiplicative operation model in correct responses to various division problems at each interview stage.

Consistency of Students' Intuitive Models

It is obvious from Figures 1–6 that most students were not consistent in the intuitive models they used correctly at any one time. Problem characteristics, such as semantic structure and the specific numbers used, seemed to influence which intuitive model would be employed. At each interview, there were some students who used the same intuitive model correctly on all problems, but there were others who used as many as three different models.

Nonetheless, students showed a consistent progression of intuitive model used from interview to interview within each problem. For example, consider the first small-number, equivalent-groups multiplication problem. Of the 17% of responses employing direct counting successfully at Interview 1, only 2% used direct counting at Interview 2; the other 15% employed repeated addition successfully. Of the 66% of responses showing successful use of repeated addition at Interview 3, 29% changed to a multiplicative operation model at Interview 4, 35% remained with repeated addition, and 2% failed to solve the problem. On all 12 multiplication problems, in only 3% of the cases did students successfully use a more primitive intuitive model to solve a problem that they had successfully solved at the previous interview, and only 2% failed to solve it.

For the division problems, the data confirm our earlier claim that repeated subtraction is a more primitive model of division than repeated addition. On all 12 division problems, there was not one case in which a successful use of repeated addition was followed by successful use of repeated subtraction at the next interview.

DISCUSSION

Summary of Results

The present study has extended previous research on young children's intuitive models of whole-number multiplication and division by widening the range of semantic structures included and by including the effect of maturation. Our definition of an intuitive model as an internal mental structure corresponding to a class of calculation strategies—as opposed to a class of solution strategies, modeling strategies, or semantic structures—has clarified the literature and proved to be reliably applicable in practice.

Among students in Grades 2 and 3, we have been able to clearly identify three intuitive models for multiplication (direct counting, repeated addition, and multiplicative operations) and four for division (direct counting, repeated subtraction, repeated addition, and multiplicative operations). As the names imply, these are basically only four different models. The direct counting model is little more than primitive counting applied to a correct interpretation of the given word problem. The repeated addition and subtraction models arise when students devise more efficient counting procedures to take advantage of the equal-sized groups in the problem. Both in effect create a sequence of multiples. The operation model represents the use of multiplication as a binary operation whose output is the final number in the sequence of multiples.

Other classifications of students' calculation strategies are, of course, possible. For example, the repeated-addition and repeated-subtraction models could well be broken into two parts: those strategies that use sequences of multiples and those that use known addition or subtraction facts. No doubt this is an important transition.

Also important is how students represent the problem situations within the various intuitive models. We claim only that the suggested classification is valuable for the purpose of studying overall changes in students' understanding of multiplication and division prior to and during formal schooling.

Our data show a consistent progression in the intuitive models used by students in Grades 2 and 3, from direct counting to repeated addition or subtraction to multiplicative operations. There was also a steady increase in performance. In particular, although many students were successfully solving multiplicative problems by the end of Grade 2, instruction in Grade 3 saw a considerable increase in the use of the operation model as well as in the overall success rate. Furthermore, although instruction was reportedly limited to multiplication and only illustrated in equivalent group situations, the use of operation models and the overall success rate also increased in other semantic structures and in division problems.

Previous findings that problem difficulty varies with semantic structure have been confirmed. In particular, comparison problems were relatively difficult and Cartesian product problems extremely difficult. We also found a clear variation in the intuitive models successfully employed in different problems. However, the structure of the preferred intuitive models did not necessarily correspond to the semantic structure of the problems: All intuitive models were employed across all problems. Many of the observed differences in preferred model were readily explained by the size of the numbers, the particular multiples involved, or the presence of superficial verbal cues.

We did not expect to find such a strong preference for the repeated-addition model of division across all semantic structures. This phenomenon appears to be a result of the close connection that students see between division and multiplication problem situations before they receive instruction in division. The same close connection is evidenced by students' spontaneous use of an operation model for division shortly after instruction in multiplication.

Our findings are in clear contrast to the one model of multiplication and the two models of division proposed by Fischbein et al. (1985), models that are essentially reflections of three common semantic structures. We found no evidence that Grade 2 and 3 students solve equivalent-groups, partition, and quotition problems using intuitive models reflecting these three semantic structures, or that they use only models corresponding to these three semantic structures to solve problems with other semantic structures. Instead, it would seem that they use a different set of intuitive models (direct counting, repeated addition/subtraction, and multiplicative operations), which they can apply to both multiplication and division problems of various semantic structures. One consequence of this finding is the need to differentiate clearly between the equivalent-groups semantic structure and the repeated-addition intuitive model.

We might ask why the same intuitive models can be used for all semantic structures. The reason appears to lie in the fact that in every multiplicative situation, "*there must be equal-sized groups*" (Confrey, 1994, p. 307; italics in original). Therefore, fundamental to processing a multiplicative situation effectively must be the recognition of the appropriate equal-sized groups. This step is not always an easy one; for example, it is not at all obvious in a Cartesian product situation. It is the equal-groups structure that allows the use of repeated addition/subtraction or multiplication, not the semantic structure. The intuitive model employed to solve a particular problem therefore does not reflect any specific problem feature, but rather the mathematical structure that the student is able to impose on it.

Learning Multiplication and Division of Whole Numbers

Our results allow us to form a tentative picture of how young children's intuitive understanding of whole-number multiplication and division of whole numbers evolves.

It would appear that young students acquire a sequence of increasingly efficient intuitive models that are applicable to whole-number multiplicative situations. The structure of each model derives from the previous one. Students do not simply switch from one model to the next, but rather develop a widening repertoire of models. Which one (or more) of all the available intuitive models is called into play to solve a particular problem depends on several factors, including previous experience of and instruction in that problem situation and knowledge of the relevant number facts.

In fact, three factors seem to develop in a parallel and interrelated fashion:

1. Students progress in their ability to interpret word problems, even without specific instruction. They learn to build concrete models of a widening variety of multiplicative contexts and to visualize them more and more easily. Students therefore become able to solve problems with an increasing range of semantic structures, although they might only use direct counting.

2. Students begin to recognize the equal-sized-group structure in many problem situations. This enables them to develop repeated addition and subtraction strategies first and then multiplicative operations, and to apply them to a widening range of problems.

3. The corpus of easily retrievable number facts extends. The students probably learn first to skip count by twos and fives, but may later learn other sequences. They also learn to add without concrete modeling and later start memorizing multiplication facts. The cognitive processing load of each strategy is gradually reduced, and it becomes more likely that students will be able to apply to any particular problem the most efficient calculation strategy they know. (This assumes, of course, that the number facts are meaningful and have not simply been learned by rote.)

We conjecture that students first learn a new strategy to solve problems where the situation is familiar and the relevant number facts are well known. They would then gradually adopt their new strategy for other problems as their understanding of multiplicative situations and their numerical skills improve. As a result, the most efficient calculation strategy available to a student gradually becomes more refined and more widely used—eventually developing into a new, more sophisticated intuitive model. However, at least during the early learning period, different problems may be solved using different intuitive models.

Learning Multiplication and Division of Rational Numbers

At first glance, there would seem to be little connection between whole-number and rational-number multiplication. As Fischbein et al. (1985) remarked, "One cannot intuitively conceive of taking a quantity 0.63 times" (p. 6). However "0.63 times something" means "partition it into 100 equal-sized groups and take 63 of them," so the equal-sized-group structure characteristic of whole-number multiplication is still present when dealing with rational numbers. We would therefore expect to find a close link between students' intuitive models for whole-number and rational-number multiplication. Research similar to the longitudinal study reported in the present paper is needed to identify and study the intuitive models that students develop for rational-number multiplication and division.

The poor performance of older students on rational-number multiplicative problems has been interpreted as inappropriate application of whole-number knowledge (Bell et al., 1989; Fischbein et al., 1985; Greer, 1994). One explanation for this behavior is that students may never have had the opportunity to develop intuitive models of rational-number multiplication and so may not be aware of the equal-group structure of all multiplicative situations. In the circumstances, the students have no choice but to try to apply their whole-number models.

Implications for Teaching

The present study raises several questions about traditional approaches to teaching multiplication and division of whole numbers in elementary school. First, the standard curriculum takes no advantage of the informal understanding of multiplicative situations that many students have developed well before Grade 3. Second, multiplication is usually introduced before division and separated from it, whereas children spontaneously relate them and do not necessarily find division more difficult than multiplication. Children would surely benefit if teachers provided them with opportunities to solve multiplicative word problems as early as the first year of schooling and if they linked multiplication and division much more closely.

As we see it, the teacher's task is to assist students to widen their repertoire of calculation strategies. This can take place at three levels. At one level, students may need assistance in modeling some semantic structures so that they can apply direct counting successfully to them. At the next level, students who can solve a variety of multiplicative problems by direct counting may be encouraged to use the equal-groups structure to develop more efficient strategies involving repeated addition. Teachers can also help students to develop their facility with addition and repeated addition at the same time. At the third level, when students can use repeated addition in a wide variety of semantic structures, the idea of a multiplicative operation can be encouraged. Children can be helped to further improve their calculation efficiency through activities designed to develop multiplicative number sense (including the memorization of number facts), and more difficult semantic structures such as the Cartesian product can be investigated.

It would also seem possible to include multiplicative word problems involving rational numbers much earlier than is presently the case, as suggested by Fischbein

(1987)—even at the same time or soon after whole-number problems. For example, Confrey and Smith (1995) describe a broad category of measurement situations that appear familiar to young children and easily extend into rational numbers, but which are currently neglected in the school curriculum. Behr, Harel, Post, and Lesh (1994) show how rational-number arithmetic can be approached in such a way as to make the connection with whole numbers explicit. The result could be a far greater awareness of the equal-groups structure of multiplicative situations and the development of powerful intuitive models that apply to both whole numbers and rationals.

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