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Predications of the Limit Concept: An Application of Repertory Grids

Steven R. Williams, *Brigham Young University*

This study uses repertory grid methodology together with a predicational view of human thinking to describe the informal models of the limit concept held by two college calculus students. It describes how their models, based on iteratively choosing points that get closer to the limiting value, are affected by experimental sessions designed to alter them. The informal models are based on the notion of actual infinity, which poses a severe cognitive obstacle to the learning of the formal definition of limit. The study also suggests that the predicational view of cognition, together with analysis of repertory grid data based on fuzzy set theory, can be useful in studying students' concept images of advanced mathematical concepts.

Key Words: Advanced mathematical thinking; Calculus/analysis; Case study methods; College mathematics; Conceptual knowledge

In a previous paper (Williams, 1991), I suggested that students establish and give meaning to informal notions of limit by way of metaphorical extensions from physical experience (e.g., moving along a path, approaching a wall, or drawing a graph getting close to an asymptote). Kaput (1979), indeed, has suggested that it is the motion metaphor that gives the limit notion "its primary meaning" (p. 294) and argues for the "basic, irreducible, and essential metaphoric nature of human thinking" (p. 289). More recently, Lakoff and Núñez (2000) proposed that the primary understanding of limits is rooted in what they call the "Basic Metaphor for Infinity" and is built up from that using other metaphors, including viewing numbers as points on a line and thinking of a path as the motion tracing that path. Thus, for Lakoff and Núñez, our basic understanding of limit is accomplished by idealizing a process very much like choosing points (or numbers) that approach a given point and watching the function values approach a limiting value. From this and what we know from previous studies of students' understanding of limit (summarized below), it is reasonable to conjecture that many students have a mainly experiential notion of limit, based on something like what Lakoff and Johnson (1999; Lakoff, 1987) call *image schemata*. By these, they mean kinesthetic and orientational structures that recur in everyday experiences, such as containers, paths, balance, part-whole relations, and so forth. These are built on mental cate-

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gories defined from gestalt perceptions, body movements, and mental imagery. Lakoff and Johnson argue that image schemata form a foundation for all conceptual understanding. From these inherently meaningful structures we reason by way of propositional extensions, metaphorical extensions, and categorical extensions to more abstract structures.

If this is true, how do these informal notions of limit relate to the formal $\varepsilon - \delta$ definition of limit? Lakoff and Núñez (2000) state:

> The point of this is not to eliminate the epsilon-delta condition from mathematics but, rather, to comprehend how we understand it. We understand it first in geometric terms using the notion of "approaching a limit." And we understand "approaching a limit" in arithmetic terms via the BMI [the Basic Metaphor of Infinity].… (p. 199)

The basic point here is that the mathematical approach to limit and the cognitive approach to limit are quite different. The mathematical approach, which makes use of universal and existential quantifiers, is designed to solve mathematical difficulties, not psychological ones.

The approach taken in this paper is that, much as we cannot understand young children's thinking about addition and subtraction solely in terms of binary operations and commutativity, we cannot understand calculus students' reasoning solely in terms of inequalities and universal quantification. Thus, the analysis in this paper begins not with the formal mathematical definition of limit, which has been developed to solve mathematical difficulties, but rather with the informal models that are invoked in giving meaning to the limit idea and at how these informal models arise and develop. Specifically, the purposes of the study reported here are (1) to introduce a theory of understanding, based on the idea of predication, that offers an alternative way of describing understanding that is sensitive to and begins with the informal notions brought to bear in complex domains, (2) to use this theory to gain insight into the informal notions of limit held by two representative students, and (3) to examine repertory grid methodology as a way of capturing the predications students bring to bear in understanding the concept of limit. In particular, I argue that the methodology provides a coherent picture of these students' understandings, a picture that helps explain and illuminate students' struggles to make sense of the limit idea.

After describing existing research on students' understandings of limit, I describe the theoretical and methodological backgrounds of the study and discuss the procedures for data collection and analysis. I then present two case studies, discussing along the way the understandings held by the two subjects, called Gerry and Jacob (both names are pseudonyms). In the concluding section, I discuss both methodological and theoretical contributions of the study.

RESEARCH ON STUDENTS' UNDERSTANDINGS OF LIMIT

Much of the work on students' understandings of the limit concept has been focused in three major areas. One body of work makes use of Tall and Vinner's (1981; Vinner, 1983) distinction between *concept image*—the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152)—and *concept definition,* the formal definition as understood and accepted in the mathematical community. The focus here is on the relationships between the informal limit notions held by students and the more formal and mathematically precise idea represented by the $\varepsilon - \delta$ definition, and on the typical misconceptions or informal conceptions held by students. Cornu's work on spontaneous conceptions (Cornu, 1981, 1983), the work of Schwartzenberger and Tall (1978) on informal meanings of technical language, Robert's work on limits of sequences (1982), and Monaghan's (1991) treatment of language use are typical of this work. All assume some underlying mental models or representations of limit held by students and seek to understand those models as they relate to more formally expressed mathematical definitions.

A second, closely related line of work focuses on obstacles to learning, seeking to understand why the coordination of concept image and concept definition can be so difficult. Such work has often used the notion of *cognitive obstacles* (Brousseau, 1983) to explain the difficulties students have in learning. Such obstacles can be the result of students' own psychological or social development, inadequate or misleading (though usually well-intentioned) instruction, or the nature of the concepts themselves (Cornu, 1991). These later obstacles have been called *epistemological obstacles* by Bachelard (1938) and discussed in the context of pedagogical theory by Brousseau (1983), who describes them as well-established pieces of knowledge that are useful in one arena of activity but not in another and that, therefore, stand in the way of proper functioning in that second arena. Other obstacles might include the unintended side effects of technology or the beliefs students hold about the nature of mathematical knowledge. Sierpinska's (1985, 1987) work on student's informal notions of limits and infinity, Cornu's description (1991) of epistemological obstacles related to limits, Davis and Vinner's (1986) discussion of "seemingly unavoidable misconception stages" in the learning of limit, Williams' (1991) work on informal models and the tenacity with which students hold them, Lauten, Graham, and Ferrini-Mundy's (1994) work on interactions with technology, and Szydlik's (2000) work on beliefs about mathematics and their affect on limit understanding provide examples of this genre.

A third major focus of work grows from a Piagetian tradition and is typified by the work of Dubinsky and his colleagues (1991; Cottrill et al., 1996) on reflective abstraction. This work assumes the existence of a *schema*—"a more or less coherent collection of objects and processes" (Dubinsky, 1991, p. 102), where *objects* are understood to be mental or physical objects and *processes* are mental actions performed on objects. Central to this work is the idea that processes can be *encapsulated* into objects that are then available to be operated on themselves, an idea also advanced by Sfard (1991). Central as well is the *genetic decomposition* of the limit concept, which is "a description … of the mathematics involved and how a subject might make the constructions that would lead to an understanding of it" (Dubinsky, 1991, p. 96). This approach begins with the mathematics, positing the existence of objects, processes, and schemata that lead to the development of the appropriate concept, and then enters a cyclical process of refinement, drawing on empirical data to test and adjust the decomposition. Dubinsky (1991) notes that his approach is complementary to the two approaches discussed above (i.e., concept image and cognitive obstacles): "One can think of reflective abstraction as trying to tell us what needs to happen, whereas the other notions attempt to explain why it does not" (p. 103).

Growing from these three bodies of work is a fairly consistent set of standard conceptions, or mental models, that student seem to have about limit. These conceptions include beliefs that a limit is a boundary, whether local or global (see Szydlik, 2000); that functions cannot reach their limits; and that limit is best described in terms of a dynamic process of points or numbers "getting close to" a limit point or number. Contrasted with this are various static views, ranging from informal and intuitive (e.g., "The limit of a function is L if whenever x is close to the limiting value s, the function is close to L." [Szydlik, 2000, p. 268]) to the formal $\varepsilon - \delta$ definition. These views of limit, which may be parts of more sophisticated mental models, recur throughout the literature on the understanding of limit and represent students' attempts to make sense of what is a very subtle and complex mathematical concept.

These models of understanding arise from studies whose common feature is an analytical starting point in the mathematical definition of limit, and an assumption that a psychological understanding of limit should ultimately be based in the mathematical understanding of limit, as represented by the $\varepsilon - \delta$ definition. Much of the previous work on limit, indeed, relied upon comparing student's notions to accepted, formal mathematical descriptions of limit. Often, the historical development of limit was compared to students' own personal development of the idea, with the $\varepsilon - \delta$ definition being the ultimate goal in both cases. Other studies rely on a fairly in-depth analysis of limit from a mathematical viewpoint and go on to compare student's understandings to the results of these analyses (Davis & Vinner, 1986) or to suggest that the analyses yield probable trajectories for learning (Cottrill et al., 1996). In all cases, the big ideas of limit are external, and the focus is on the formal, complete, expert view of limit.

As mentioned above, the research presented in this paper takes a different approach. After a discussion of the theoretical and methodological background for the study, two case studies are discussed in detail, with attention to the informal models of limit held by two students, the nature of these models, and how these models develop during the course of a 7-week experiment.

THEORETICAL BACKGROUND

One way of understanding the ways students give meaning to the notion of limit is through Lakoff and Núñez's (2000) analysis of specific conceptual metaphors. Another approach, which takes metaphorical thinking as a special case, is provided by Rychlak (1988, 1991, 1994), who uses the term *predication* to describe the location of one item of meaning within a broader item of meaning. Rychlak's framing of predication and of cognition in general insists that all cognition, including learning, is a matter of bringing to bear broader precedent assumptions that then extend meaning to narrower target items. Rychlak calls this process *meaning extension*. Meaning extension occurs when a precedent premise is extended "by way of what are called inductions, deductions, implications, metaphors, analogical extensions, and so forth" (Rychlak, 1975, p. 253). For example, when we say "Broccoli is a vegetable," we place broccoli within the larger context of vegetables, so that meaning is extended to the item broccoli by our knowledge of vegetables in general. Similary, the proposition "Sego Lily bulbs are a vegetable" extends the broad meaning of "vegetable" to the narrow "Sego Lily bulb," providing information about Sego Lily bulbs (e.g., that they are edible). Both metaphorical thinking and traditional logical implications are special cases of the more general concept of predication.

It is worth pausing here to see how this is true. When we say, for example, that a life is a journey, we place the referent "life" in the wider meaning of "journey" and thus let what we understand of a journey give meaning to the idea of life. Similarly, in an example more specific to limits, when our mental model of "letting x approach the value 2" is metaphorically like the iterative choosing of numbers closer and closer to 2, we predicationally place the mathematical concept of "letting x approach 2" in the wider circle of meaning of iteratively choosing numbers. This iterative choosing of numbers may in turn be metaphorically based in other iterative physical processes and, thus, may be predicated on these other processes. Similarly, traditional logical implication is also a case of predication. When we say "If a square, then a rhombus," we place the referent "square" in the wider set of "rhombus," and the meanings we hold for "rhombus" are extended to squares. Our idea of rhombus, in turn, is predicated on other ideas that extend meaning to it.

It is also worth pausing to note a difference between this view of understanding and a predominant view that equates understanding with the quality or integrity of internal mental representations. Hiebert and Carpenter (1992) describe this viewpoint well:

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. (p. 67)

Rather than viewing understanding as a network of connections between discrete pieces of information, a predicational viewpoint suggests that understanding flows from core meanings. Thus the strength of understanding is not in number of connections but in the power of the core meanings that can serve as predicational "targets" and, hence, extend meaning to new ideas.

METHODS

The data reported here were gathered using repertory grid methodology and were collected as part of a larger study. The following sections provide a description of repertory grid methodology, then go on to describe the original study and provide the details of subject selection, data gathering, and analysis.

Repertory Grids

Repertory grid methodology was originally developed by George Kelly as a research tool for his Personal Construct Psychology (Kelly, 1955). As such, it is firmly based in Kelly's view of human thought, particularly upon the notion of *construct*. A construct can be thought of as a precedent assumption, or predicative category, used to extend meaning to a subject's world. As such, it is generally compatible with Rychlak's (1994) meaning extension. The basic technique is to elicit constructs used to extend meaning to a particular set of items and then to have the subject rate these items in terms of the elicited constructs. Pope and Keen (1981) point out that the technique has evolved into a "methodology involving highly flexible techniques and variable application" (p. 36). Thomas and Harri-Augstein (1985) list over 250 types of items used in the elicitation of personal constructs, among them physics concepts, statistics concepts, mathematical operations, and chemical bonds. In the literature of mathematics education, repertory grids have been used to capture affective knowledge structures, such as belief systems about teaching, learning, or motivation (Lucock, 1987; Owens, 1987; Hoskonen, 1999; Middleton, 1995, 1999) and conceptual knowledge structures in particular domains such as geometry, Logo programming (Lehrer & Koedinger, 1989), and teachers' knowledge of fractions (Lehrer & Franke, 1992).

Methodologically, Kelly (1955), described a construct as "a way in which two elements are similar and contrast with the third" (p. 61). It can thus be thought of as a predicative category used to distinguish among elements. A common method of eliciting a construct is to present subjects with three items chosen from a list of interest and to ask them to designate how two of the items are similar, and therefore different, from the third. In other cases, subjects may be asked to simply compare and contrast two items. The result in either case is a construct with two poles, each one represented by one of the contrasting items. To the extent possible, this construct is then named and described by the subject. Other constructs emerging from the items are elicited in this manner, after which the subject is asked to rate the other items in the list as to the degree to which each construct applies to them. Following the ranking of each item in terms of the elicited constructs, the subject is then given more items and the process is repeated until such time as the experimenter and subject feel there are no more differences or similarities to be noted. At the end of the session, a matrix of items by constructs has been created, with elements consisting of ratings of the items in terms of how well the constructs apply to each.

Gaines and Shaw (1986) have proposed an analysis of repertory grid data that uses fuzzy set theory to gain some indication of the relationship among constructs.

They interpret the constructs as predicates defining fuzzy sets and the item ratings for each construct as indicating membership in these sets. Thus the constructs are essentially determined by the degree of membership of each item in the fuzzy set. They go on to develop an analytic method that reveals the implication structure among these fuzzy predicates. The method provides us with a measure of the extent to which items defining the fuzzy predicate u also define the fuzzy predicate v. Thus, it provides a measure of the extent to which the predicate v extends meaning to the predicate u and is therefore particularly suited to studying *predication* structure as a more general case of the *implication* structure suggested by Gaines and Shaw.

Lehrer has written software that provides a representation of the implication structure among constructs on the basis of the Gaines and Shaw (1986) technique. Lehrer and Koedinger (1989) have used the analysis of implication structures successfully in analyzing data from studies on adult learners of Logo and on fourth-grade learners of geometry. Lehrer and Franke (1992) employed this method in studying teachers' knowledge of fractions. In essence, the software measures the degree to which construct u implies construct v by measuring the degree to which the items that are members of the fuzzy set defined by predicate u are also members of the fuzzy set defined by predicate v. In the analyses presented in this report, strengths of implications are computed for each pair of constructs and their negations. Implications that are vacuously true (such as when a false precedent will imply any statement) and hence provide little real information are eliminated. Among those that remain, whenever two constructs imply one another, they form a bi-implication or are logically equivalent in the fuzzy sense discussed above. On the other hand, when a construct A implies a construct B, but construct B does not imply construct A, or the implication is weak (see Lehrer & Koedinger, 1989, for a discussion of the technical details), then an asymmetric association is said to exist between the constructs. Whereas Gaines and Shaw would interpret this as an implication $A \rightarrow B$, the predicational interpretation is that B is a broader referent than A, and meaning is extended from B to A.

The Original Study

Data reported here are from a larger project that investigated the changes that 10 second-semester calculus students experienced in their understanding of the limit notion as the result of a 7-week experimental treatment. These 10 students were selected from 341 students enrolled in second-semester calculus classes on the basis of answers to a questionnaire about their understanding of limit. On this questionnaire, students were asked to indicate whether each of six statements about limit were true or false, to tell which of the six best described limit as they understood it, and to write a description of limit as they understood it (see Figure 1). The 10 students were chosen to represent various alternative conceptions of limit, particularly those involving reachability of limits, dynamic or motion-oriented views of limit, and notions of limit as a boundary, as well as a spectrum of formality in their own descriptions of how they understood limit. All students had completed a traditional, large-section first-semester calculus class.

I. Please mark the following six statements about limits as being true or false:

1. T F A limit describes how a function moves as x moves toward a certain point.
2. T F A limit is a number or point past which a function cannot go.
3. T F A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.
4. T F A limit is a number or point the function gets close to but never reaches.
5. T F A limit is an approximation that can be made as accurate as you wish.
6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

II. Which of the above statements best describes a limit as you understand it? (Circle one)

1 2 3 4 5 6 None

III. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f as $x \to s$ is some number L.

IV. If possible, write down a rigorous definition of limit.

Figure 1. Initial questionnaire given to all students.

Subject Selection and Data Gathering

This report focuses on 2 of the 10 students from the original study, Gerry and Jacob. Gerry was chosen as a subject for this report because he had a clearly described and strongly held image, or base metaphor, for the limit process that emerged during interviews. Jacob was chosen as a contrasting case of a student with a less well-articulated base metaphor for limit, which nevertheless developed over the course of the study. The methodology employed here is shown to highlight the similarities and contrasts between Gerry and Jacob through its ability to capture their idiosyncratic knowledge.

Students in the study met individually with the investigator for five sessions spread over 7 weeks. During each session, students' definitions of limit were explored and discussed, so that their personal definitions of limit were modified and refined over the 7 weeks. During the first and fifth session, repertory grids were elicited. During the middle three experimental sessions, students were asked to respond to a series of tasks aimed at moving their alternative or informal concepts of limit toward a more formal concept. The focus was on altering students' views on reachability of the limit, a dynamic or motion-oriented view of limit, and a view of limit as a bound. The tasks students worked on included (a) typical textbook limit problems to assess students' technical competence; (b) discussing contrasting opinions about limits as voiced by hypothetical students, some of which voiced conceptions shared by the subjects; and (c) working problems designed to produce cognitive conflict and thus provide impetus for change. For example, one problem provided a table of values of the function $f(x) = x + 1 + \frac{1}{10^{20}x}$ and asked the students whether they could determine the limit as x approached 0. Subsequent discussion focused on the sense in which a limit could or could not be found by evaluating the function at numbers that got closer and closer to a given value. The tasks included answering written questions, and students were often asked clarifying questions or probing questions in an effort to better understand their views of limit. The questioning and interviewing that surrounded these tasks form another corpus of data against which the predicational relationships from the repertory grids are discussed. Thus, the data reported here are primarily from the first and fifth sessions, where grids were elicited, with clarifying evidence coming from transcripts of the middle three sessions.

A series of 10 written statements about limits were used in the elicitation of a repertory grid. They were composed based on students' statements gleaned from the literature, from a pilot questionnaire, and from pilot work with calculus students. The aim was to provide students with statements in which each of three misconception themes were present: a limit is unreachable, a limit involves motion, and a limit is a boundary. The statements were made as broad as possible to encourage the elicitation of other constructs that students saw as important to their understanding of limit. The statements were also designed to differ in their degree of formality; some more closely approximated the $\varepsilon - \delta$ definition , and others were more informal. The limit statements, referred to hereafter as *items*, were subjected to pilot testing with calculus students and were substantially rewritten. They were also reviewed by six graduate students in mathematics to ensure that all three themes occurred in the items and that no important themes were missed. The items are listed in Figure 2.

Constructs were elicited following the procedures described in general above, using the 10 limit statements as items. A subject was given two of the items and asked to describe how the two items were alike or different. The subject responded with verbal descriptions of what was alike or different, which the interviewer recorded. Clarification was asked for as needed, such as when the verbal descriptions of two constructs seemed to be the same. The interviewer chose a few of the subject's words to use as a label for the emergent construct and, when offered by the subject, a label to stand for the opposite construct. When all similarities and differences between one pair of items were written as constructs, all 10 items were rated by the subject on a 5-point scale as being more like the emergent pole of each construct or its opposite. This procedure was then repeated for a second pair of items, and the process continued until the interviewer and the subject both agreed that they were unlikely to elicit new constructs. Pairs of items were presented to

Item 1:

A limit is a sort of estimate of a given value the function attains within a given amount of tolerance. You can get better and better estimates by restricting x, so the tolerance gets smaller and smaller, but a function never reaches its limit.

A limit is really an approximation, not an exact number. If you plug in numbers close to s, you can get close to the limit, but not beyond it.

Item 2:

A limit is a number or point that the function has values close to but never exactly equal to. If you take the limit of $f(x)$ as $x \rightarrow s$, you can make the function as close as you want to the limit, but it will never actually equal the limit, just like x never actually equals s.

When you take a limit, you don't care if x is ever really equal to s, just that it's close. Same with $f(x)$. It doesn't really matter if $f(x)$ is bigger or smaller than the limit, but just that it's close. If $f(x)$ ever equals the limit, you don't really have a limit.

Item 3:

The limit is the maximum (or minimum) of a function as x approaches some number. As you get close to that number, the values of the function are trapped by the limit number. For example, when a function grows really fast but then levels off to an asymptote, the limit is the value of the line.

So a limit is a point or a number past which values of the function will not go; in fact, the values never even reach the limit, but they do get close.

Item 4:

A function f has a limit L as $x \rightarrow s$ if the values of numbers near s are near L. Specifically, for any tiny interval you draw around L, you can find an interval around s so that all x values in the interval around s have function values somewhere in the interval around L.

Item 5:

A limit means that when x moves closer to some number s, $f(x)$ is moving closer to the limit. The function gets infinitely close to the limit but never touches it. It's like an asymptote that the function might cross over a few times (or even infinitely many times) but will get closer and closer to.

Item 6:

When a function moves toward a certain number and gets closer and closer to it, that number is the limit. So a limit is a number or point that a function grows toward but doesn't go past.

It's like walking halfway to a wall, then halfway again, and so forth. You keep moving closer, and the wall is like the limit. Eventually, you reach the limit, just like you reach the wall.

Continues

Item 7:

What's important about limits is the idea of "closeness." When you say limit as x approaches s, it means that if x is *close to* s, then $f(x)$ is *close to* the limit. That's what the definition is trying to say.

The idea of the definition is proving you can get *as close as you want:* I say I can make $f(x)$ as close as you want to the limit by making x close enough to s, and I prove it by telling you how close x has to be to s whenever you tell me how close you want $f(x)$ to be to the limit. That's what all the delta-epsilon stuff is about.

Item 8:

You can't evaluate a limit by just plugging in points close to the number, because you can only plug in a finite number of points, and that isn't enough to tell you what the function is really doing. It might be different when you get closer to the point.

You really have to prove that you can get as close as you want to the number and the function is still close to the limit. That's why you need the limit theorems.

Item 9:

Finding a limit is a lot easier than understanding the definition. When you need to find a limit, you just plug the number in. Like, to find the limit of $f(x)$ as x approaches 0, you plug 0 into $f(x)$. If it doesn't work, you do some algebra, try to cancel some stuff out, and then try again.

The definition talks about "getting close" and all that, but when you work the problems, the limit turns out to be what you get when you plug the value in.

Item 10:

If you want to picture a limit, picture x moving closer and closer to some number and the point on the graph above it moving along the graph, getting closer and closer to the limit. You're just approaching a point on the graph that the function goes through.

The function goes through the limit point, and the points are just moving along the graph toward the point. There's no restriction on how close they can get and eventually, when x reaches the number, the function will reach the limit.

Figure 2. Items used in eliciting repertory grids.

all subjects in the same order. The first four pairs (7-10, 2-6, 3-10, 4-9) were chosen because they seemed to represent opposite points of view on motion, reachability, boundedness, and formality. After these first four pairs, pairs 3-6, 6-10, and 8-10 were presented to the subjects according to a list of pairs chosen at random prior to all interviewing. The same list of pairs was used for each subject; however not every subject saw the same number of pairs because the number of constructs elicited varied across subjects, and not all item pairs elicited new constructs. After all constructs were elicited, subjects were asked to rate all 10 items on two constructs supplied by the researcher: whether the item was *true* and whether the subject *liked* the item. Beyond the obvious presence of the misconception themes in the items themselves, every attempt was made to insure that the constructs elicited were not suggested to the student. This interview typically lasted about an hour, with the subjects examining five to eight pairs of items.

The results of the above procedure was a matrix in which each row represented one construct and consisted of a series of ten rankings (one for each item) on a 5-point scale. A rank of "5" for an item indicated that the emergent (left-hand) pole of the construct applied strongly to that item, whereas a rank of "1" implied that the opposite pole of the construct applied strongly to the item.

Initial analysis was aimed at obtaining information about (a) the features of the limit notion that have organizational salience for mathematics students (i.e., the constructs) and (b) the relationships between those features. Analysis based on Gaines and Shaw's (1986) logic of fuzzy predicates was used to gain insights into students' predicational structures surrounding limits.

RESULTS AND DISCUSSION

In this section, I discuss in detail the understandings of limit held by Gerry and Jacob and how meaning is extended to the limit notion for each student. I also discuss how their understandings evolved and solidified over the course of the experiment.

Gerry's Initial View of Limit

Gerry's initial description of what it meant to say "the limit of f as $x \to s$ is L" was given as follows: "As one gets closer to s the function begins getting closer to its value at s. This allows one to get an idea of what $f(s)$ will be without actually finding $f(s)$." Gerry chose #4, "A limit is a number or point the function gets close to but never reaches" from the list of six statements as best describing a limit as he understood it and chose #3, "A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values" as the *only* false statement among the six statements in Figure 1.

Gerry seemed to hold conflicting views of what a limit is. It is probable that in responding to the questionnaire, Gerry viewed his task as *describing* limit, rather than *defining* it. In the language of Tall and Vinner (1981), Gerry was working solely with concept image and not with concept definition. Tall and Vinner note, "As the concept image develops it need not be coherent at all times.… At different times, seemingly conflicting images may be evoked" (p. 152). Certainly this was true of Gerry as he struggled with verbalizing an understanding that was based largely on kinesthetic and image-laden understandings. In addition, it is possible that Gerry was focusing on different aspects of his image of limit as he considered the different statements. For example, although he initially accepted the statement above that a limit was a number the function got close to but never reached, it is likely that he really understood this statement as reflecting his description of the *process* of finding a limit, not as a description of how a function can behave. In summary, Gerry's initial understanding of limit can be characterized as somewhat inconsistent and verbalized in ways that failed to adequately express it.

Initial repertory grid. During the initial grid elicitation session with Gerry, the following constructs were elicited. Related comments are given in detail here to give the reader a sense of how the constructs were expressed. The labels in bold are the constructs eventually used to rank all 10 items.

1. **Closeness**. I think both [items #7 and #10] are trying to zero in on the closeness. That the *x* gets close.

2. **Goes through vs. Just near**. It seems to me like this one [#10] says the function kind of goes through the point, where this one [#7] says that it takes you near it.

3. **Precise value vs. Approximation/Not precise**. This one [#4] says if you plug in the number, you get a vague number. You get something close to it, but you never actually get it. But this one [#9] says you stick in the number and you get it exactly. This one's more precise.

4. **Stopping vs. Continuing**. This one [#6] seems to say that the function stops at the limit, and this one [#10] seems to say the function keeps going through.

5. **Don't actually know it vs. Definitely find it**. This one [#10] says that you definitely can find the limit, and this one [#8] says that you can't find the limit concretely but you have to prove it exists. But you really don't know what it is.

The results of the rankings for all seven resulting constructs (including **True vs. False** and **Like vs. Dislike**) are shown in Figure 3. The items and constructs have been rearranged to show the similarities between constructs and the relationships between constructs and item rankings. This rearrangement is based on a clustering technique employing additive trees (Sattath & Tversky, 1977) but is displayed here only to highlight how items differentiated between constructs. Here, a score of "5" represents high ranking for the emergent pole, and a score of "1" represents a low ranking on the emergent pole (or a high ranking on the opposite pole).

Although the primary focus of this report is on the predicational structure, it is worth noting a few obvious relationships. First, it is interesting that Gerry ranks items 4, 7, and 9 as "true, liked, and dealing with closeness." In addition, both truth and closeness have high rankings on items that imply the limit is reachable (Items 6, 9, and 10). That all of these are seen as liked and true is particularly interesting because Items 4 and 7 most closely represent a formal $\varepsilon - \delta$ definition, whereas Item 9 reflects a practical orientation to doing limit problems. This supports a later argument that Gerry is struggling with the relationship between theoretical and practical views of limit. It is also interesting that Gerry associates the formal definition with approximations rather than with exact values. By contrast, Item 9 was seen as associated with providing definitive answers.

Construct Emergent Pole "5"	Item Numbers 1	5	2	3	8	6	4	7	9	10	Construct Opposite Pole "1"
Goes through	1	1	3	1	2	1	2	1	5	5	Just Near
Precise value	1	2	3	2	2	4	1	1	5	3	Approxi-mation/ Not Precise
Closeness	3	2	4	1	1	2	5	5	5	5	
True	1	1	1	1	1	5	5	5	5	5	False
Stopping	4	2	2	4	3	5	1	2	3	1	Continuing
Don't actually know it	4	2	3	2	5	5	4	4	1	1	Definitely find it
Like	5	4	1	1	1	3	4	5	5	1	Dislike

Figure 3. Gerry's initial repertory grid arranged to show relationships between constructs and item rankings.

Initial predicational structure. If the constructs elicited in a repertory grid give an idea of the major meanings used to understand the limit notion, the predicational structure shows how those meanings are related (see Figure 4). If, as seems reasonable, Gerry would like and view as true those ideas most central to his view of the limit notion, then it is clear that "closeness" is one such fundamental idea. The bi-implication between the constructs "True" and "Closeness" form one of three major clusters in Figure 4. Moreover, the opposite of the "Closeness" construct, equated by Gerry with "stopping at the limit," is disliked. Gerry's feeling that "stopping at the limit" was somehow antithetical to closeness makes sense in light of the fundamental model that Gerry eventually expresses. For Gerry, it is vital to his model that limits be two-sided (this will be expanded later). Finally, there is a third cluster, in which Gerry equates getting a precise value for a limit by "plugging in" a value (i.e., when the function is continuous) with the fact that a function goes through its limit point (instead of just getting near). These imply that a precise value can be found for the limit. This is keeping with Gerry's association of Items 4 and 7, those resembling the $\varepsilon - \delta$ definition, with approximate values.

In summary, Gerry's initial predicational structure for limits seems to emphasize closeness (although it is not clear, at this point, what closeness means to Gerry) as well as the finding of an exact value for a limit as central ideas. It also seem that although both of these ideas are important to Gerry, they are not necessarily related. "Closeness," on the one hand, was a foundational construct for Gerry in that it largely determined the truth of statements about limits, and statements seen as not dealing with closeness tended to be disliked. On the other hand, the process

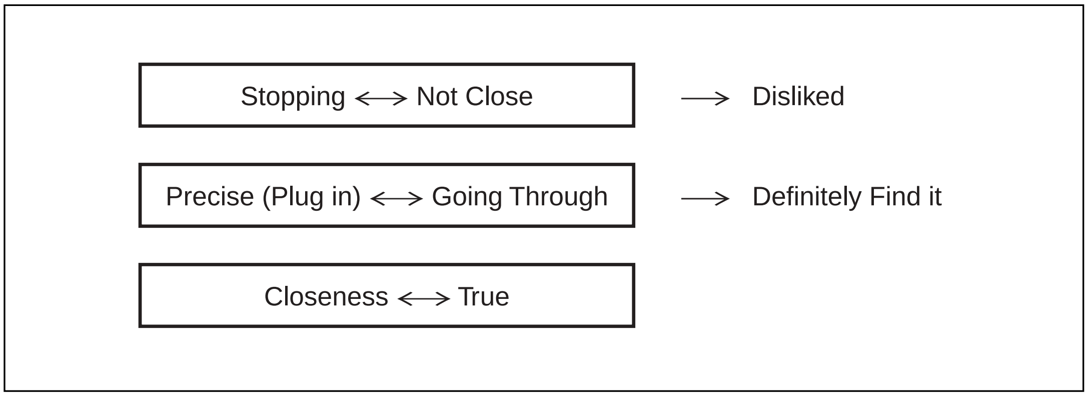

Figure 4. Gerry's initial predicational structure.

of *finding* limits was something else entirely; it dealt with issues of evaluating functions at the points of interest and perhaps inspecting their graphs to determine the values at which they "went through" their limit point. It is interesting to note that these two aspects echo the two very real divisions in Gerry's classroom life between understanding limits in theory and being able to find limits in order to do homework and test problems.

In the intervening five weeks, Gerry attended the three experimental sessions described earlier. These sessions offered Gerry an opportunity to refine his thinking on each of the three misconception themes and provided him with anomalous examples of limit problems that were best understood from a more formal, structural viewpoint. At the fifth session, his "definition" of limit was again elicited, as was a second repertory grid.

Gerry's Emerging View of Limit

In the fifth interview session, Gerry settled on this description for what it meant to say "the limit of f as $x \to s$ is L:"

> As one gets closer to s the function begins getting closer to a value. If you approach s from both sides and the value that the function is getting close to is the same, then that's the limit.

Notice that this description does not mention the value of the function at s, as did Gerry's first description. However, Gerry did add the "both sides" restriction. This is indicative of his emerging model of limit. Gerry considered questionnaire statements 1, 2, and 4 (see Figure 1) as false, and statements 3, 5, and 6 as true. He rejected notions of limit being a bound or being unreachable, and he felt uncomfortable with the idea of x "moving" toward a certain point as expressed in statement 1. He did recognize the truthfulness of statement 3, essentially a restatement of the $\varepsilon - \delta$ definition; but he chose statement 6, "A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached" as best describing his understanding of limit. This also was consistent with his base metaphor for limit, which had emerged as the sessions progressed.

Gerry discussed several times in interviews what he called his "sandwich" view of limit. In a discussion of how he "plugged in" a number to a continuous function to find the limit, he said: "… because you can plug them in but *if you think about them,* if you think about the two sides coming in from both sides, it kind of, kind of fits the same way [as my definition]." Later, he clarified this and gave it a name:

Gerry: Well, you're kind of sandwiching your number between the values that you're choosing, so you never get to the number but you keep closing it in.

Interviewer: So you're choosing points?

Gerry: Yeah, you're sandwiching in your—the number that you're getting close to.

Gerry took as his fundamental metaphor for limit this "sandwiching" notion, in which the value of the limit is approached from both sides and eventually trapped by the sides of the sandwich. He almost always described this process by saying that the number was "not reached" but understood that the function could take on the value of the limit. This viewpoint remained strong for Gerry as the sessions progressed. Although the third interview session was expressly aimed at discounting the notion that $\lim_{x \to s} f(x)$ could be found by evaluating the function at points that got successively closer to s, Gerry admitted in the final session that he still saw that viewpoint as valid.

Final repertory grid. During the final elicitation session with Gerry, the following constructs were elicited.

1. **Very very close**. They [Items #7 and #10] are both saying that there's no restriction on how close you get. You can get very very close.

2. ***X*'s getting closer/sandwich**. They're [#7 and #10] both saying that your x is getting closer to the number around which you're taking your limit. And, uh, I think that's what they're both saying, that your x's are moving in. Again, it's like your sandwich.

3. **Never goes through vs. Does go through**. This one [#3] says that the function never goes through the limit. And this one [#10] says that it does.

4. **Just plug in number vs. Close in upon it**. OK, these are like totally opposite. This one's [#9] saying that your limit—you just plug in and that's it. I mean it has none of this sandwich stuff or anything. That's just it—that's the limit. Where this one [#4] says—you know, you have to close in upon it.

5. **Reach vs. Not reach**. This [#6] one says that you reach the limit but you don't go any further than that. And this one [#10] says you reach the limit but you go through it. [This was judged to be the same as #3 above]. They both say you can reach the limit.

6. **Can get exact number vs. Can just prove it's close**. This one [#10] says you can actually find an exact number for the limit and this one [#8] says you can never find an exact one. You can just prove that it's about that.

In addition, Gerry was asked to rank all 10 items on the constructs **True vs. False** and **Like vs. Dislike**. The results of the rankings for all eight constructs are shown in Figure 5. Again, the items and constructs have been rearranged to show the similarities between constructs and the relationships between constructs and item rankings.

Construct Emergent Pole "5"	Item Numbers										Construct Opposite Pole "1"
	2	3	5	1	8	4	7	6	10	9	
X's getting closer/ sandwich	2	3	5	2	1	5	3	4	5	1	
Never goes through	5	5	5	5	2	5	3	5	1	1	Does go through
Just plug in number	2	3	1	1	3	1	2	1	2	5	Close in upon it
Can get exact number	5	5	3	1	1	1	1	4	5	5	Can just prove it's close
Reach	1	1	1	1	1	2	2	5	5	3	Not reach
Very very close	2	1	3	2	1	5	5	4	5	3	
Like	2	1	2	3	1	4	5	4	4	5	Dislike
True	2	1	3	3	3	5	5	4	4	4	False

Figure 5. Gerry's final repertory grid arranged to show relationships between constructs and item rankings.

As with Gerry's initial repertory grid, Items 4 and 7, representing a formal definition, are seen as true and are liked. Moreover, Items 6, 9, and 10, which represent graphical and practical concerns, are also seen as true and are liked. In general, there seems to be less polarization in this repertory grid. However, the fine structure of the grid is more easily seen in terms of its predication structure.

Final predication structure. At least two major changes relative to the initial grid are apparent in predicational structure given by the final repertory grid (see Figure 6). First, foundational concepts have pulled together into a tighter structure. A cluster of four constructs (**Liked, True, Very Very Close, *X*'s Moving In**) linked by bi-implications form a core that for Gerry represents the fundamental model of limit—his "sandwich" model. This gives a consistent picture of his base metaphor of *x*'s being chosen closer and closer (indeed, very very close) to the target value, and the limit being the number that the functional values approach.

Second, a new relationship has emerged that reflects Gerry's ambivalence about reaching the limit. One the one hand, there is an implication chain **Just Plug In** $\rightarrow$ **Reach** $\rightarrow$ **Very Very Close**. This has face validity, since if the limit of a continuous function can be obtained by finding the functional value, it is clear that

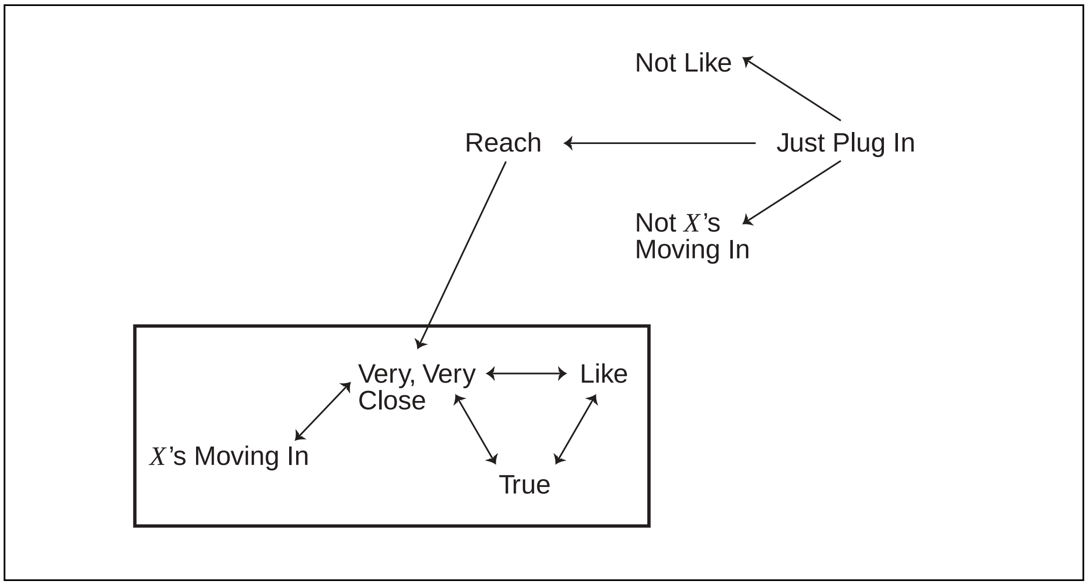

Figure 6. Gerry's final predicational structure.

the function reaches its limit and also that there are no restrictions on how close one can get to the limit point. In this sense, "plugging in" to obtain the limit value seems to be a special case of the more general "sandwich" notion. On the other hand, "just plugging in" numbers is also disliked and is seen as antithetical to the ***X*'s Moving In** construct—that is, as antithetical to the sandwich model.

Thus, for Gerry, "plugging in" to get a limit is not really part of his core concept of what a limit is—it doesn't involve "sandwiching." On the other hand, being able to "just plug in" clearly implies that the limit is reached, and if it is reached, the end result of the "sandwiching" process must be the same point.

Here the complexity of Gerry's meaning-making process is demonstrated nicely by the predicational structure. On the one hand is his base metaphor of x values being chosen to close in on the given value from both sides. On the other hand, as he freely admits, he does limit problems without recourse to that metaphor, by evaluating the function at the point of interest (possibly after some algebraic manipulation to yield a suitable continuous function with the same limit) or by inspection of the graph. He has tried to take account of a new aspect of limit, highlighted in the experimental sessions he attended, by incorporating limits that are reached—but this includes all those continuous functions for which the theoretical grounding is not necessary. In fact, "just plugging in" results in limits that are reached and is seen as contrasting starkly with his theoretical model. That he recognizes this contrast is indicated by his not liking the "just plugging in" idea.

In summary, what interviews reveal as fundamental in Gerry's experience throughout the five sessions is, in fact, reflected in the final predicational structure as constructed from his repertory grid. The contrast between theoretical understanding of the limit and practical actions involved in evaluating limits is still present, but there has been some refinement. The “sandwich” model, which is, in fact, an informal model that is based largely on a metaphorical moving of pointers along the graph, has come to occupy a central part of Gerry’s view of limit. At the same time, the relationship between this model and the practical aspects of finding limits has also been highlighted. The mediator between the theoretical sandwich model and the practical methods of evaluating limits is the notion of “reaching.” Getting close and eventually reaching the limit seems to lend meaning both to the model and to everyday practice. Thus, the notion of the function reaching its limit is ambivalent for Gerry. On the one hand, Gerry was convinced by the experimental sessions that some functions take on their limiting value. On the other hand, his sandwich metaphor does not require that the function do so. Thus the notion of reaching a limit point enriches both theory and practice for Gerry, but it does not reconcile them.

We now look briefly at a second student, Jacob, in order to more fully illustrate the explanatory power of repertory grids and predicational relationships. Jacob differs from Gerry in having no clearly described, overarching image of limit, like Gerry’s “sandwich” notion, and also in developing a somewhat more sophisticated view of limit. He shares with Gerry a strong, robust, dynamic view of limit, which he begins to question by the end of the session.

Jacob’s Initial View of Limit

Jacob’s initial description of what it meant to say “the limit of f as $x \to s$ is L” was given as “The limit of $f(x)$ as $x \to s$ means that, as you have a function $f(x,)$ if you keep plugging in numbers (x) closer and closer to s, you will finally achieve the limit (L) as $x \to s$.”

Jacob chose the statement, “A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached,” on the initial questionnaire as best describing his understanding of limit. Initially, and on the surface, Jacob’s view seems to have much in common with Gerry’s—an essentially dynamic view of limit characterized by considering a sequence of points that get closer and closer to the limit value. Indeed, by the time of the first meeting, when he was asked to again describe his view of limit, he chose the statement “A limit describes how a function moves as x moves toward a certain point” as best describing how he understood limit.

It is clear that a dynamic view predominated Jacob’s early reporting of limit ideas. He did express some confusion over “reaching” a limit, as many of the other nine subjects did, but even with that his view was fairly sophisticated:

> I understand that what they’re looking for is not the point when you plug it in, but I do understand that when it’s continuous and you do plug in the point, that is what the numbers, as you pick them close, are convergent to. So, I know that if it’s not continuous and you substitute in the number, and you get a number out, that that’s not necessarily the limit. I understand that it’s the points, that it’s the neighborhood around there as you get infinitely close, but, whether—I don’t know whether it reaches its limit or not. I don’t remember if we were taught [that].

By our second meeting Jacob had established a pattern of answers that remained stable for the rest of the 7 weeks. He indicated that statements 1, 3, and 5 on the questionnaire (see Figure 1) were true, that statements 2, 4, and 6 were false, and he chose statement 3, "A limit is a number that the *y*-values of a function can be made arbitrarily close to by restricting *x*-values," as best describing his understanding. Thus, Jacob espoused a view of limit that strongly resembled the formal, "static" definition (represented by statements 3 and 5) but at the same time had a strong dynamic flavor as well (statement 1).

The following constructs were elicited from Jacob during the initial session:

1. **Closeness**. [Both #7 and #10 say that] as *x* moves closer and closer to some number, the graph approaches the limit.

2. **Can't reach vs. Can reach**. [#7, in contrast to #10, says] you get closer but you never really actually reach it.

3. **Growing toward**. They're both [#2 and #6] talking about going toward a function but not going past … growing toward the limit.

4. **Plugging in vs. Area around limit**. This one [#9] is just talking about plugging in and this one [#4] is talking about the area right around [the limit].

5. **Can't go past**. [Both #3 and #6 say] you can't go past the limit.

6. **Prove you can get close**. [Both #8 and #10 say] you have to prove you can get as close as you want to the number.… there's no restrictions on how close you can get.

Figure 7 presents Jacob's initial predication structure, including his rankings on **Like vs. Dislike** and **True vs. False**. Two linked clusters of constructs are immediately apparent. The first, linking false and disliked statements with those about "plugging in" and "reaching," gives a clear indication of Jacob's beliefs that limits are **not** about just plugging in a value, and they cannot be reached. The second cluster, one that extends meaning to the first, links the idea of closeness with the idea of "growing toward" a limit. This makes two things clear: that Jacob views closeness in terms of numbers or points "growing toward" the limit and also that false and disliked statements lack this view of closeness. This provides further evidence that Jacob initially has a view of limit similar to Gerry's sandwiching idea. A second relation, that of "can't go past" implying "can't reach," is sensible and helps to establish that, for Jacob, the issue of "reaching" is still relevant. "Reaching" the limit is associated with false statements, and "not going past" the limit seems to share conceptual space with growing toward the limit and being concerned with closeness—for Jacob, both dynamic notions. This supports and enriches Jacob's expressed ambivalence about reaching a limit. He understood that the limits of continuous functions could be found by evaluating the function at a value, but was not sure if, in general, limits were reachable.

In summary, Jacob's initial predicational relationships indicate a preference for a dynamic view of limit, which he also sees as associated with closeness and not "reaching" the limit. As will be shown below, the issue of "reaching" eventually becomes resolved for Jacob, and there is some movement toward a less dynamic view of limit by the end of the 7-week experience.

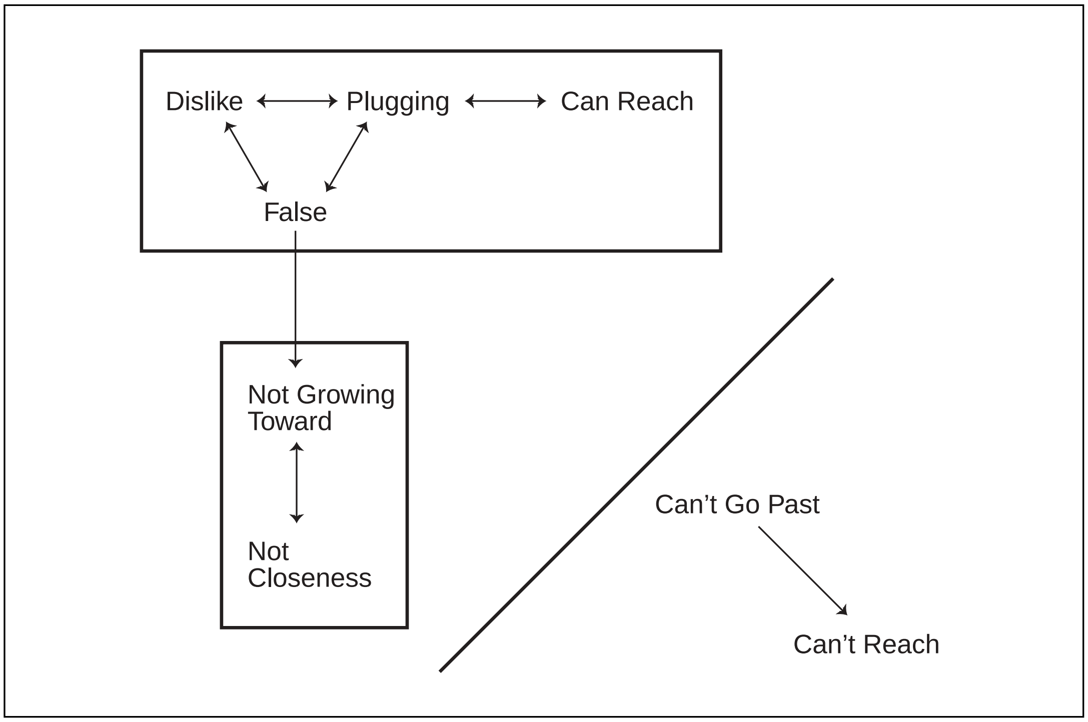

Figure 7. Jacob's initial predicational structure.

Jacob's Emerging View of Limit

As mentioned above, Jacob's answers to the questionnaire in Figure 1 were relatively stable from the second session to the end of the 7-week period. He seemed to have settled on a view of limit that combined elements of a formal static view with a dynamic feel of values of x and values of $f(x)$ moving toward specified values. His own definition from the final session reflects an increased sophistication in his view:

> As $x \to a$, the function will approach a certain limit, but doesn't necessarily have to reach it (depending on continuity, etc.). Plugging in numbers "far out" from the limit won't necessarily approach it.

This reflects Jacob's coming to understand, like Gerry, that the idea of "reaching" the limit was not relevant from a formal viewpoint and depended on the continuity of the function. It also reflected the idea that limits were concerned with what was happening locally, not "far out" from the limit value. Thus, Jacob was beginning to consider what was happening in small intervals around the limit point, as is reflected in his predicational relationships.

The following constructs were elicited from Jacob during the final session:

1. **Always reaches vs. getting close to**. [In Item # 10] They're saying it always reaches the limit, and here [#7] they're talking about closeness, getting close to.

2. **No restriction on how close**. [Both #7 and #10 say that] there's no restriction on how close you can get.

3. **Reach vs. Can't reach**. This one [#6] says you can reach it; this one [#2] says you don't ever reach it.

4. **Can't go past**. The both [#2 and #6] say you can't go past [the limit].

5. **Plugging in numbers vs. Understanding intervals**. One's [#9] just plugging in numbers, and the other [#4] is actually understanding the intervals.

6. **Can tell from function vs. Can't get close enough by plugging**. [In #10] they're kind of assuming you can tell what happens just by looking at the function, what happens when you get very close. And [#8] is kind of saying you can't get close enough [that way] because it might be different when you get closer to the point.

7. **Following graph to a point**. This one [#10] gives me the picture of following the graph right along … that the limit is just a point on a graph, no big deal. This one [#6] … makes it seem like something funny is going to happen right before the limit.

Jacob's predicational relationships at the end of the 7 weeks are shown in Figure 8. There are three points of focus. The first is a cluster of bi-implications that tie false statements with statements involving plugging in numbers, failing to prove that there are no restrictions on how close the function gets to its limit, and not being able to tell from examining the function what its limit is. A second cluster suggests that "liked" statements are both true and deal with understanding intervals (as opposed to just plugging in numbers). The combination of these implications and bi-implications suggest that Jacob now values an understanding of intervals, as opposed to just plugging in numbers, and does not value statements that rely on just plugging in numbers, without some argument about the function being able to get close "without restriction" to the limit value. However, he also links "can't get close enough by plugging in numbers" with false statements, indicating that he still believes that his initial view of limits being approached by plugging values in the function has some validity. A third cluster, involving statements about reaching the limit, or the dynamic view of limit (following a graph to a point) is unrelated to statements seen as either true or false. This may mean that the dynamic view of limit is still significant but he had not yet reconciled that significance with what he was coming to believe as true.

This view is supported by Jacob's reactions to working the problem involving the funtion $f(x) = x + 1 + \dfrac{1}{10^{20}x}$ mentioned earlier. He decided that, although it did not really alter his fundamental way of looking at limit, he did have to be more careful about describing what "far away" meant—that is, he had to be more careful about guaranteeing that he really understood the behavior "close enough" to the limit value. Without deciding what "close enough" means, he did at least begin to recognize that this was an issue. It seems that Jacob came to understand the major issues that gave rise to the current definition of limit without necessarily completely resolving them. In this way, his understanding was more sophisticated that Gerry's. At the same time, his hold to the dynamic notion was still strong. After deciding that the experimental session had not really changed his dynamic way of looking

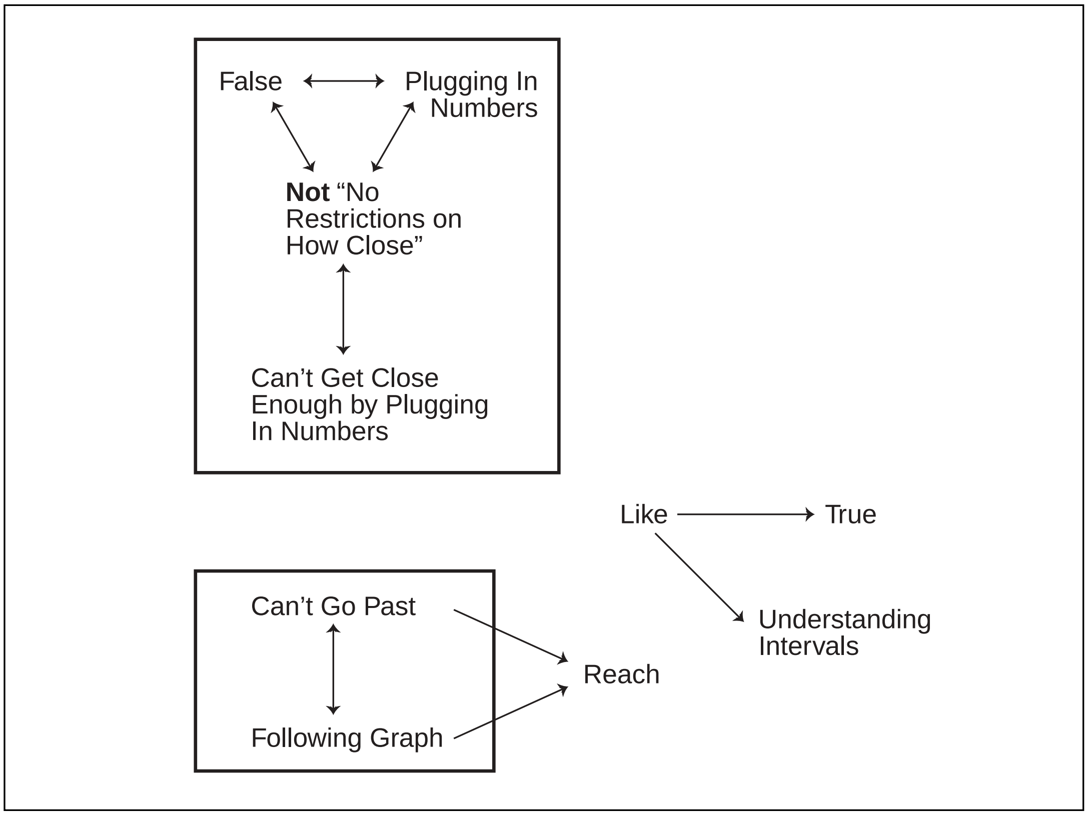

Figure 8. Jacob's final predicational structure.

at limits, he was asked what it would take to make him give up the notion of limit he held. He replied, "I don't know. That's just kind of the way I look at it. If you show me an easier way. . . ." In the final analysis, although he made some changes in his view of limit, he still clung to what was for him a powerful underlying motion metaphor.

SUMMARY AND CONCLUSIONS

The analysis above offers insight into Gerry's and Jacob's individual thinking about limits and so suggests some theoretical directions to be explored in the area of students' understanding of limits. It also provides some insights into repertory grid methodology and predication theory as a way of capturing cognition in a complex domain. These topics are discussed in this final section.

Gerry's and Jacob's Understanding of Limits

In looking at the approaches both Gerry and Jacob take to limits, one is struck by the similarity between their descriptions of limits and the basic metaphor for limit discussed in Lakoff and Núñez (2000). In both cases, the natural approach of examining values that get "closer and closer" to the limit forms a foundation for their understanding. Moreover, even though the experimental sessions were specifically designed to create some cognitive discord with this notion, both Gerry and Jacob continued to believe in it and still claimed it was their fundamental way of understanding limits. In this, they were not alone; nine of the 10 subjects in the study also remained convinced that this dynamic view of a limit was essentially correct. The remaining student could see it as problematic but had no competing scheme with which to replace it. In general, absent the mental action of iteratively choosing points and evaluating the function, students seem to have very little with which to frame a theory of limits.

For both Gerry and Jacob the notion of "reaching" a limit carried some ambivalence. Both understood that continuous functions "reached" their limits in the sense of taking on the limit value, but both still felt unsure whether, in the process of taking a limit, it was accurate to say that the limit was actually reached. In Lakoff and Núñez's (2000) basic metaphor for limit, the "reaching" is accomplished by metaphorically extending the iterated choosing of points to a "final resulting state" (p. 159) in which the limiting value is clear. This final resulting state is equated with "actual infinity" and is also the critical step in the basic metaphor for infinity; it is the cognitive leap from finite to infinite. It is precisely at this point where Gerry and Jacob are not sure of themselves.

The necessity for this cognitive leap helps to explain at least two phenomena apparent in Gerry's and Jacob's accounts of limit as well as the corpus of literature on limits. Given the conceptual difficulty of the notion of infinity, whether actual or potential (Tirosh, 1991), it is not surprising that Jacob, Gerry, and their thousands of counterparts in calculus classes stumble over whether a limit is reached. The notion of reaching a limit rests on the fundamental distinction between actual and potential infinity. Moreover, the $\varepsilon - \delta$ definition, and indeed most of modern mathematics, categorically rejects the concept of actual infinity (Tirosh, 1991). Thus with the concept of limit, students' informal conceptions that depend on coming to grips with actual infinity meet mathematical formalism head on. Given that the avoidance of actual infinity within modern mathematics is motivated by the desire to avoid rather subtle and mathematically complex paradoxes, it is not surprising that few students see the need to reject their informal model. This explains in part the conceptual difficulties surrounding limit consistently reported in the literature, the robust nature of those difficulties, and the problems associated with teaching and learning the $\varepsilon - \delta$ definition. Actual infinity may thus be the most important cognitive obstacle to learning the formal definition.

Issues Related to Theory and Methodology

Certainly no one method can capture the richness and variety that characterizes human thinking. Any method will necessarily obscure some aspects as it highlights others. Had Gerry and Jacob engaged in other kinds of tasks, different kinds of knowledge and relationships may have emerged. At the same time, such a methodology may also have obscured the fundamental models and metaphors that Gerry and Jacob used to make sense of limits. The methodology employed here, combining repertory grids with an interpretation drawing on predication and metaphorical extension, seems well suited to getting at these fundamental models and, at the same time, sensitive enough to capture growth over the course of the 7-week experimental sessions.

Gerry, for example, distinguished between practical and theoretical aspects of limit throughout the sessions, but both aspects were refined as Gerry attended the experimental sessions. Gerry began to be aware of these distinctions as he came to understand differences between how he thought about limits and how he actually evaluated limits. This change is reflected in the differences between the initial and final predication structures. Similarly, although Jacob held a dynamic view of limit throughout the sessions, his view also matured to include a greater attention to verifying what is meant by "getting close enough."

Finally, this technique allowed aspects of the limit notion that were personally meaningful for both students to emerge, rather than only those which were part of the researcher's agenda. Thus, instead of beginning with a mathematical analysis of limit, and the assumption of certain mental objects and processes (as in, e.g., genetic decomposition), this method begins with students' own reports of what is significant about limit. For example, the division between practical and theoretical notions of limit, which was fundamental not only for Gerry but for all students in the larger study, became a major organizing theme in his predicational structures. Moreover, whereas the "sandwich" model had not completely emerged during the first elicitation, it was strongly represented in the second grid and formed a core of the associated predicational structure. The ability to let constructs emerge from the subjects themselves has long been seen as a strength of repertory grid techniques, and this study confirms their value in addressing idiosyncratic mental models. Of course, interpretation of the grids, and the predicational relationships that come from them, is still necessary. Indeed, interpretation is a necessary part of any attempt to understand another person's thinking. Whether we choose to approach this task through asking verbal questions, giving written tasks, observing problem-solving behaviors, or any of a myriad of other methods, we will always be interpreting tangible data and inferring thinking patterns from it. Thus, although the methodology of this study is not free of interpretation, it nevertheless allows for a greater chance for students' idiosyncratic thinking to emerge.

In this regard, it should be noted that the methodology offered some insight beyond what was obtained by analysis of verbal data (e.g., interviews). As an example, Jacob's final words in the interview seem to suggest that he had not given up his dynamic view of limit and would not do so until someone could show him an "easier" way. Yet, the predicational relationships from his last grid suggest that although he had not abandoned a dynamic view, it had been distanced from the core of his thinking—the "true" and "liked" statements. This subtle difference, which emerged in the predicational analysis, suggests that such analysis might help discover more stable understandings and beliefs than the students may be able to express through answers to interview questions. This methodology, then, seems particularly suited as a means to supplement and enrich verbal protocols as evidence of understanding.

Repertory grid methodology, viewed as a means of capturing predication, offers promise for exploring students' informal approaches to understanding. In this case, coupled with interview data, it was helpful in identifying a key stumbling block to understanding the limit concept.

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Author

Steven R. Williams, Department of Mathematics Education, Brigham Young University, Provo, UT 84602; williams@mathed.byu.edu