Journal for Research in Mathematics Education
2009, Vol. 40, No. 5, 491–529

# Mathematical Knowledge for Teaching in Planning and Evaluating Instruction: What Can Preservice Teachers Learn?

Anne K. Morris and James Hiebert
*University of Delaware*

Sandy M. Spitzer
*Towson University*

The goal of this study is to uncover the successes and challenges that preservice teachers are likely to experience as they unpack lesson-level mathematical learning goals (i.e., identify the subconcepts and subskills that feed into target learning goals). Unpacking learning goals is a form of specialized mathematical knowledge for teaching, an essential starting point for studying and improving one's teaching. Thirty K–8 preservice teachers completed 4 written tasks. Each task specified a learning goal and then asked the preservice teachers to complete a teaching activity with this goal in mind. For example, preservice teachers were asked to evaluate whether a student's responses to a series of mathematics problems showed understanding of decimal number addition. The results indicate that preservice teachers can identify mathematical subconcepts of learning goals in supportive contexts but do not spontaneously apply a strategy of unpacking learning goals to plan for, or evaluate, teaching and learning. Implications for preservice education are discussed.

*Key words*: Content knowledge; Pedagogical knowledge; Preservice teacher education; Teacher knowledge

Preparing teachers to teach mathematics effectively is one of the most urgent problems facing those who wish to improve students' learning. What can preservice teachers learn during their preparation programs to increase the chances that they will become effective mathematics teachers? In this article, we explore the development of specialized knowledge-for-teaching-mathematics, a kind of content knowledge that increasingly appears to be critical for effective teaching and that could be acquired during teacher preparation.

The study we report brings together two streams of work that address the knowledge and skills needed by beginning mathematics teachers. A first stream focuses on the competencies needed to learn to teach effectively over time. Arguing that it

---

Preparation of this article was supported by the National Science Foundation (Grant #0083429 to the Mid-Atlantic Center for Mathematics Teaching and Learning). The opinions expressed in the article are those of the authors and not necessarily those of the Foundation.

is unrealistic to expect graduates of teacher preparation programs to enter the classroom as expert teachers, Hiebert, Morris, Berk, and Jansen (2007) propose that preservice teachers should acquire knowledge, skills, and dispositions that would enable them to study their teaching and gradually improve over time. The authors identify skills needed to analyze teaching in a deliberate and systematic way. These skills can be learned by preservice teachers, because the skills can be taught in the usual teacher preparation settings of university courses with linked field experiences (Jansen & Spitzer, 2009; Spitzer, Phelps, Beyers, Johnson, & Sieminski, in press).

If the teaching being analyzed is mathematics teaching, as opposed to, say, history or English teaching, what kind of content knowledge is required to carry out useful analyses? The second stream of work contributing to this study is the development of "mathematical knowledge for teaching" (MKT) (Ball & Bass, 2000; Ball, Thames, & Phelps, 2008). The concept of MKT provides the most promising current answer to the longstanding question of what kind of content knowledge is needed to teach mathematics well. Indeed, recent studies at the elementary school level provide initial data linking teachers' MKT with the mathematical quality of instruction (Hill, et al., 2008) and the level of students' achievement (Hill, Rowan, & Ball, 2005).

We believe that joining the analysis of teaching skills with aspects of MKT defines a portion of the competencies that preservice graduates can, and should, acquire. If this is true, then the development of these competencies creates a guide for a portion of the curriculum needed in teacher preparation programs. The development of such a curriculum depends, of course, on understanding the successes and struggles faced by preservice teachers in acquiring these competencies. This study was designed to describe in detail the challenges that preservice teachers face in applying MKT to analyze teaching.

## BACKGROUND

### *Skills for Analyzing Teaching*

In the absence of empirical and theoretical support for traditional forms of teacher preparation, Hiebert and colleagues proposed an alternative aim: Rather than attempting to produce skilled teachers upon graduation, preservice education should prepare teachers to learn from teaching when they enter the profession (Hiebert, Morris, & Glass, 2003; Hiebert et al., 2007). This is not a new concept. It has been endorsed in various ways for at least several decades (e.g., Feiman-Nemser, 1983; Hawkins, 1973; Schaefer, 1967). Theoretically, it draws from a variety of sources, including learning theory in which the goal for students is to learn to learn, that is, to become "intelligent novices" (Brown, Bransford, Ferrara, & Campione, 1983). In teacher preparation, preservice teachers are the students and they are learning to learn how to teach. In other words, the aim is for preservice teachers to acquire the skills needed to learn to teach in an intentional, systematic way.

It is difficult to imagine learning to teach in a systematic way without being able to analyze the effects of teaching on students' learning. In particular, the skills hypothesized to be essential include specifying learning goals for students, examining the alignment between instruction and achievement of these goals, and analyzing the evidence of students' learning in order to evaluate and then revise instruction to make it more effective (Hiebert et al., 2003; Hiebert et al., 2007; Jansen, Bartell, & Berk, 2009). These skills allow teachers to treat their lessons as experiments, as episodes of teaching that can be assessed by measuring how well they helped students reach the learning goals and, in turn, how they can be revised to be more effective next time.

The beginning point for analyzing teaching, serving as a prerequisite for the entire activity, is the specification of learning goals (Jansen et al., 2009). Without clarity about the learning goals, it is difficult to judge whether teaching is effective. To be clear about learning goals means to identify the learnings required to achieve the goals. In other words, clarity about learning goals requires analyzing and unpacking learning goals into their constituent parts. Consider, for example, the learning goal of understanding the addition of decimal numbers. Unpacking this learning goal would reveal at least the following constituent parts or, as we will call them, subconcepts:

1. A quantity is identified as "one."
2. The value of each digit in a decimal number is determined by its place in the numeral. Each place is associated with a unit of measure, and the size of the units increase or decrease by a factor of 10 as you move to the left or right, respectively.
3. When adding decimals, same-sized units are joined, and the sum will be of that unit.
4. When a quantity contains 10 or more of a particular unit of measure, those 10 can be exchanged for one unit of measure of the next larger size.

A deliberate recognition of these subconcepts would allow a teacher to analyze teaching in several critical ways: plan instructional activities that address these subconcepts, anticipate ideal student responses and anticipate ways that responses might fall short, construct assessment tasks that reveal students' understanding of each subconcept, and evaluate students' learning over the lesson(s) and look for evidence that students understood each of the subconcepts. Analysis of the subconcepts is necessary for teachers to improve their practice because it allows them to pinpoint the areas in which their instruction is successful and areas in which improvements are needed.

An important feature of the analysis-of-teaching skills proposed by Hiebert and colleagues (Hiebert et al., 2003; Hiebert et al., 2007) is that they shift much of the intellectual work of teaching to activity outside the classroom—planning lessons, including specifying learning goals, aligning the instructional activities with the learning goals, and anticipating student responses that show achievement of the goals, and then evaluating lessons by examining evidence of students' thinking and learning. Although effective teaching likely requires both skillful inside-the-classroom routines (Grossman & McDonald, 2008; Kazemi, Lampert, & Ghousseini, 2007; Lampert & Graziani, 2009) and thoughtful outside-the-classroom planning and evaluation, we believe that improvements in teaching rest, in part, on systematic and intentional application of outside-the-classroom skills. As noted earlier, the preservice nature of preparation programs suits them to teaching outside-of-classroom skills that can then be used when prospective teachers work in classrooms.

*Mathematical Knowledge for Teaching*

Building on the concept of pedagogical content knowledge (Shulman, 1986), Ball and colleagues (Ball et al., 2008; Hill, et al., 2008) describe four components of mathematical knowledge for teaching (MKT): knowledge of mathematics that most educated people acquire ("common content knowledge"); knowledge of mathematics that is unique to, and essential for, teaching mathematics ("specialized content knowledge"); knowledge that combines knowledge of content with knowledge of students; and knowledge that combines knowledge of content with knowledge of teaching. An example of the four components of knowledge at work can be fashioned from an ordinary school mathematics task: adding fractions with unlike denominators, say, $2/3 + 3/4$. Common content knowledge is involved in adding the fractions correctly to get the correct answer; specialized content knowledge is required to identify the subconcepts that must be understood to know why finding a common denominator is useful in calculating the answer, or to judge whether a nontraditional method a student might invent will always work; knowledge of content and students is needed, among other things, to predict the most common misconceptions that students will have; and knowledge of content and teaching is needed to decide how to help students correct these misconceptions.

The second component—specialized content knowledge (SCK)—is of special interest for us. This kind of knowledge falls largely outside of Shulman's (1986) pedagogical content knowledge because it does not draw directly on knowledge of students or teaching. It is content knowledge, but content knowledge of a particular kind. It is implicated in common teaching tasks such as choosing representations of mathematical ideas that reveal key subconcepts of the ideas, evaluating whether student responses show an understanding of key subconcepts, and justifying why arithmetic algorithms work. It involves unpacking or decompressing mathematical knowledge in order to make particular aspects of it visible for students or to identify the source of students' difficulties. At the core of many uses of SCK is the skill and knowledge required to unpack a mathematical concept or skill into its subconcepts.

*Joining the Two Streams of Work*

Specialized content knowledge (SCK) is of special interest here for several reasons. First, because it is content knowledge, not directly dependent on knowledge of students and teaching, it is a good candidate for preservice teacher education. Most preservice teachers have limited opportunities to build their knowledge of school students and classroom teaching so it can be difficult for them to develop aspects of MKT that depend on this classroom-based knowledge until they begin teaching. Consequently, it seems possible, and wise, to focus part of preservice teachers' attention on aspects of MKT that are well suited to preteaching experiences.

A second reason that SCK is of special interest is that it is tightly intertwined with the skill of specifying learning goals, a foundational skill for studying and improving teaching. In fact, SCK is precisely the kind of mathematical knowledge needed to become proficient with this skill. As illustrated previously for the learning goal of understanding the addition of decimal numbers, unpacking the goal into its subconcepts requires knowing the mathematics that fits together to yield this concept. Such knowledge lies at the heart of SCK (Ball et al., 2008; Hill et al., 2008). Knowing mathematics in this way enables, in turn, planning for teaching the concept by attending appropriately to the subconcepts as well as evaluating the effects of teaching on students' acquisition of the learning goal by locating the source of incomplete understandings. Said another way, SCK is required to identify the subconcepts of learning goals, and identifying the subconcepts of learning goals is essential for studying and improving one's teaching.

The alignment of the competencies involved in specifying learning goals and displaying SCK creates a fertile area for exploring the competencies that could define a high-leverage goal for mathematics teacher preparation. If mathematics teacher educators wish to rethink the critical competencies needed for teaching and to consider which competencies might be productively addressed during teacher preparation, then we believe the intersection of the analysis of teaching skills and MKT described previously is a fruitful site to explore in some detail. In particular, the knowledge and skill required to unpack learning goals into their constituent parts provide a critical site because these competencies are essential for all further analyses of mathematics teaching and are potentially learnable during teacher preparation.

Our analysis leads to the following research question for this study: How do preservice teachers unpack learning goals into subconcepts to plan for, and evaluate, teaching and learning? The goal of the study is to uncover the successes and challenges that preservice teachers are likely to experience as they try to acquire such knowledge and skill.

## METHOD

### *Participants*

The participants were 30 sophomore and junior undergraduates in the University of Delaware's (UD) K–8 teacher preparation program. The 4-year program for K–8 certification includes general studies (science, mathematics, social science, fine arts, English), additional courses in a selected discipline, professional studies (e.g., human development, educational assessment), 2 semesters of content-specific courses on methods of teaching with field experiences in K–8 classrooms, and student teaching. The participants' general studies mathematics preparation consists of 3 semesters of mathematics content courses for K–8 preservice teachers, followed by a mathematics methods course. The first two content courses focus on number and operations, and the third on geometry and algebra.

Thirty of the 45 preservice teachers in the third content course who had agreed to participate in research studies were randomly selected and then invited to participate in our study. Preservice teachers who could not participate were replaced by additional random selections.[1]

The mathematics content courses completed by the participants focus on the mathematical content that is studied in grades K–8 and introduce preservice teachers to the specialized content knowledge they might need as teachers. For example, preservice teachers learn to explain mathematical ideas in ways that reveal the key subconcepts of the ideas, use representations of mathematical ideas that reveal key subconcepts, justify why arithmetic algorithms work, and decompose concepts into subconcepts in the context of evaluating K–8 students' responses (e.g., based on a student's reported response involving the subtraction of whole numbers, preservice teachers form hypotheses about which ideas about the subtraction of whole numbers the student did and did not understand). Although the content courses introduce preservice teachers to the SCK they might need in teaching and include tasks similar to the tasks used in this study, there is no systematic work in the courses on unpacking mathematical learning goals. Hence, we expected considerable variation in performance.

### *Procedures*

Preservice teachers independently completed four written tasks during two 2-hour research sessions. The tasks were administered during the 3rd and 4th months of the semester to students enrolled in the third content course in the program. The mathematical content of the tasks involved fraction and decimal number concepts. These topics were chosen because the first two content courses are devoted to developing preservice teachers' understanding of fraction and decimal number concepts and we wanted to maximize the probability that the participants would understand the mathematical content of the tasks.

---

[1] UD mathematics education faculty and doctoral students frequently carry out research projects that investigate the knowledge, skills, and dispositions of the preservice teachers in the four mathematics teacher preparation courses. The studies are used to guide course improvements. To support this model of continuous improvement, participants are given a small number of points toward their course grade. At the beginning of each semester, preservice teachers are asked to indicate their willingness or unwillingness to participate in studies of this kind (on a human subjects consent form). Participants for a particular study are then randomly selected from the list of preservice teachers in the relevant course(s) who indicated a willingness to participate. Preservice teachers who opt out of participating can earn the same number of course points through other course activities. In general, very few of the preservice teachers indicate an unwillingness to participate. However, when preservice teachers are invited to participate in a given study, they sometimes have scheduling problems, work conflicts, and other obligations that prevent them from participating.

*Tasks*

*Anticipating an Ideal Student Response*. The Anticipating an Ideal Student Response task is shown in Appendix A. Part 1 of the task specified a learning goal ("Students will understand how to add fractions and will understand the concepts underlying this operation.") and specified four problems. Participants were asked to construct an ideal student response to each problem that would convince them that the student understood the learning goal. This task assessed whether preservice teachers would identify the subconcepts required to achieve the learning goal and use the identified subconcepts to construct responses that would provide evidence that the student understood each subconcept.

It is especially important to note that the preservice teachers were asked to construct an ideal student response for each problem:

> Imagine that this is the ***only*** problem that Sue will solve for you; i.e., this is the ***only*** evidence that you will use to judge whether Sue understands the concepts underlying the addition of fractions. Use the exact wording that you want Sue to use while she solves the task.

This required that the preservice teachers construct an ideal response for each problem—a response that referred to, and showed understanding of, each subconcept.

Part 2 of the Anticipating an Ideal Student Response task asked the participants to identify one of the four problems as the problem with the most potential to reveal students' understanding of the learning goal ("Which of the four problems will tell you the most about whether your students understand the concepts underlying the addition of fractions?"). This task assessed whether preservice teachers would identify the subconcepts required to achieve the learning goal and use the subconcepts to identify the problem that would be most likely to reveal students' understanding of each subconcept.

*Evaluating a Student's Incorrect Response*. The Evaluating a Student's Incorrect Response task is shown in Appendix B. The task involves reading a lesson transcript and evaluating a particular student's response. The learning goal for the lesson is the following: "Students will understand how to compare fractions with different denominators and numerators, and the concepts underlying this type of comparison." Preservice teachers read the transcript and responded to the following prompt: "Nikki gave an incorrect answer. What doesn't Nikki understand about the concepts underlying the comparison of fractions? List as many ideas as you can (up to five) that you think Nikki should understand, and doesn't." This task assessed whether preservice teachers would identify the subconcepts of the learning goal and use the subconcepts to assess whether Nikki's response showed evidence that she understood or did not understand each of the subconcepts. An unpacked learning goal provides a standard, which helps to identify missing ideas, misconceptions, or incomplete ideas in a student's response.

*Evaluating a Student's Correct Work*. The Evaluating a Student's Correct Work task is shown in Appendix C. Preservice teachers read Josie's responses to several tasks and evaluated what Josie did and did not understand about a specified learning goal ("Students will understand how to add decimals and understand the concepts underlying this operation."). Although Josie's responses are correct, her responses could be given with little or no understanding of the learning goal. This task assessed whether preservice teachers would identify the subconcepts required to achieve the learning goal and use the subconcepts to assess whether Josie's responses showed evidence that she understood or did not understand each of the subconcepts.

*Analyzing a Classroom Lesson*. The Analyzing a Classroom Lesson task is shown in Appendix D. Preservice teachers were asked to read the transcript of Mrs. Roland's classroom lesson, which had the following learning goal: "Students will understand how to add fractions and will understand the concepts underlying this operation." The transcript is divided into two segments. In Segment 1, Mrs. Roland explains the standard algorithm for adding fractions with like denominators. The students then solve problems that require adding two fractions with the same denominator. The students give correct answers but, given that they know the algorithm, their answers require little or no mastery of the learning goal. In Segment 2, Mrs. Roland asks the students to figure out how to add two fractions with unlike denominators. Students appear to productively struggle with the concepts of the learning goal, but do not find the correct sum.

After reading the transcript, preservice teachers evaluated the effectiveness of the lesson, revised one of the two segments, and provided a justification for the revision. This task assessed whether preservice teachers would identify the subconcepts of the learning goal and use the identified subconcepts to (a) evaluate students' understanding of each subconcept over the lesson and the effects of the instruction on those understandings (Part 1 of the task), and (b) revise instruction to better help students achieve the learning goal (Parts 2 and 3 of the task).

In Analyzing a Classroom Lesson, the task of identifying the subconcepts of the learning goal was deeply embedded in a teaching scenario. Consequently, we hypothesized that preservice teachers' performance would be influenced by other types of MKT (e.g., knowledge of content and students, knowledge of content and teaching) (Ball et al., 2008; Hill et al., 2008) as well as their beliefs about pedagogical practices. The transcript was written to highlight two principles of learning and teaching that have been shown to support students' achievement of learning goals that involve understanding mathematics: Instruction should, at some point and in some way, make the key mathematical relationships clear for the students; and instruction should, at some point and in some way, allow students to wrestle with the key mathematical ideas (Hiebert & Grouws, 2007). We were interested in whether, and in what way, preservice teachers would apply their SCK to unpack learning goals and then use this analysis to examine teaching and learning in the context of these two principles. Segment 1 violated both principles, and Segment 2 illustrated the second principle. Prior pilot studies indicated that the lesson transcript created a realistic classroom scenario for preservice teachers that tested their inclination to analyze the mathematics involved (identify the mathematical subconcepts and use this to evaluate the teaching) in the face of other aspects competing for attention (e.g., students' correct responses and/or their confusion).

### *Coding and Reliability*

The data were coded by the authors. To calculate reliability, two coders coded a subset of the participants' responses (4 to 10 of the responses) and calculated reliability as number of agreements divided by number of codes applied.

*Anticipating an Ideal Student Response Task.* We unpacked the learning goal of the task ("Students will understand how to add fractions and will understand the concepts underlying this operation") into six subconcepts that, we believe, are necessary to achieve the learning goal.

1. A quantity is identified as the quantity "one."
2. We obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts.
3. The numerator is the number of units of size $1/n$.
4. The addends must both be expressed in terms of the same-sized unit.
5. The addends must be joined.
6. The sum must be expressed in terms of a unit of size $1/n$.

For Part 1 of the task, we determined whether preservice teachers referred to these subconcepts when they constructed ideal student responses. It should be noted that other decompositions of the learning goal are possible. We regarded our list of subconcepts as one possible list. Because the goal of the study was to assess whether preservice teachers unpacked learning goals into any reasonable subconcepts, our intent was to add to the list if preservice teachers identified subconcepts that we had not identified. However, this did not occur for the list used in this task (or for any of the subconcept lists used in this study).

For each problem (Problems 1, 2, 3, and 4 of the task in Appendix A), preservice teachers were assigned a score of 0, 1, or 2 for each subconcept in the list. A score of 0 indicated that the participant did not mention the subconcept at all; 1 indicated the participant referenced or used the subconcept but in a way that could hide a lack of understanding (e.g., the participant verbally named the concept only, the participant used the concept to draw a correct picture but did not explain how or why she was drawing it that way, or the participant provided an incomplete or vague explanation); and a score of 2 indicated that the participant explicitly referenced or used the subconcept in a way that indicated a genuine understanding. To clarify the difference between scores of 1 and 2 on this task, we provide in Table 1 examples of responses to Problems 2 and 3 that would receive each score.

Interrater agreement for Problems 1, 2, 3, and 4, respectively, was 88%, 83%, 96%, and 96%. For each problem, each preservice teacher was also assigned a total score for reference to the subconcepts. Because there are six subconcepts, total scores for each problem could range from 0 to 12.

Table 1
*Examples of Responses to Part 1 of the Anticipating an Ideal Student Response Task That Would Be Given Scores of 1 and 2 for Each of the Six Subconcepts*

| Problem 2: Solve 1/4 + 3/8 by drawing a diagram on graph paper. | Score of 1 | Score of 2 |
|---|---|---|
| 1. A quantity is identified as the quantity "one." | Sue should explain the concept of "one." | I picked eight squares (or 16, 24, etc.) to be "one." |
| 2. We obtain units of size 1/*n* by partitioning the "one" into *n* equal parts. | First I colored in two out of eight squares to show 1/4, and then I colored in three out of eight squares to show 3/8. | Then a unit of size 1/4 will be two squares, because I have to divide the "one" into four equal parts to obtain a unit of size 1/4. A unit of size 1/8 will be one square, because I have to divide the one into eight equal parts. |
| 3. The numerator is the number of units of size 1/*n*. | First I colored in two out of eight squares to show 1/4, and then I colored in three out of eight squares to show 3/8. | So 1/4 will be two squares (or one unit of size 1/4) and 3/8 will be three squares (or three units of size 1/8). |
| 4. The addends must both be expressed in terms of the same-sized unit. | I can see from the picture that 1/4 = 2/8. | Now I need to measure both addends with the same-sized unit. Since 1/4 is two squares, I can see that two units of size 1/8 will fit into 1/4. So 1/4 is the same quantity as 2/8. This is true because one eighth is two times as small as one fourth. Therefore two times as many eighths fit into this quantity. |
| 5. The addends must be joined. | I counted up all the colored squares, and there were five. | Now I can combine the two addends. I know that I have two units of size 1/8 (two squares) and three more units of size 1/8 (three more squares). So, my sum will look like five squares. |
| 6. The sum must be expressed in terms of a unit of size 1/*n*. | Since five of eight squares are shaded, the sum is 5/8. | I can see that a unit of size 1/8 (that is, one square) will fit into the sum five times, so the answer is 5/8. |

| Problem 3: Solve 1/4 + 3/8 using the common denominator method. | Score of 1 | Score of 2 |
|---|---|---|
| 1. A quantity is identified as the quantity "one." | Sue would have to explain the concept of a "one." | When I add 1/4 + 3/8, some quantity had to be "one," and the same "one" was used for both fractions. |
| 2. We obtain units of size 1/*n* by partitioning the "one" into *n* equal parts. | For 1/4 + 3/8, the denominators tell you the number of pieces. | We obtained 1/4 by partitioning the one into four equal parts and 1/8 by partitioning the one into eight equal parts. |
| 3. The numerator is the number of units of size 1/*n*. | The numerators tell you how many pieces. | 1/4 means I have one unit of size 1/4, and 3/8 means I have three units of size 1/8. |
| 4. The addends must both be expressed in terms of the same-sized unit. | Next, you do 1/4 + 3/8 = 2/8 + 3/8. You need to have the same-sized pieces or the same denominators to add fractions. The fractions 1/4 and 2/8 are the same size. | 1/4 + 3/8 = 2/8 + 3/8. In this step, I converted 1/4 into an equivalent fraction. I needed to measure both addends with the same-sized unit. Since a unit of size 1/8 is half as big as a unit of size 1/4, two times as many units of size 1/8 will fit into a quantity. Since two units of size 1/8 fit into one unit of size 1/4, the two quantities 1/4 and 2/8 are the same size. |
| 5. The addends must be joined. | Now all I need to do is add the numerators or number of pieces. | We need to combine the two quantities. Since both 2/8 and 3/8 are measured with the same-sized units, we can add the number of units to find the sum of the two fractions. |
| 6. The sum must be expressed in terms of a unit of size 1/*n*. | So now I have five of eight pieces, so the answer is 5/8. | I have a total of five units, each of size 1/8, so the sum is equal to 5/8. |

For Part 2 of the task, we determined whether preservice teachers referred to the subconcepts when they explained why they picked a particular problem as having the most potential to reveal students' understanding of the learning goal. A score of 0 indicated they did not refer to the subconcept and 1 indicated they did. Interrater agreement was 98%.

*Evaluating a Student's Incorrect Response Task*. We decomposed the learning goal of the lesson ("Students will understand how to compare fractions with different denominators and numerators, and the concepts underlying this type of comparison.") into four component subconcepts that, we believe, are necessary to achieve the learning goal. The four subconcepts are shown in the rightmost column of Table 2. Next, we determined whether preservice teachers referred to the subconcepts when they analyzed Nikki's response. For each subconcept, preservice teachers were assigned a score of 0, 1, or 2. The meaning of the scores is the same as those described for the Anticipating an Ideal Student Response task. Interrater agreement was 100%. Each preservice teacher was assigned a total score for the Evaluating a Student's Incorrect Response task; because there are four subconcepts, total scores could range from 0 to 8.

*Evaluating a Student's Correct Work Task*. We identified the subconcepts that, we believe, are involved in achieving the learning goal ("Students will understand how to add decimals and understand the concepts underlying this operation"). The four subconcepts are shown in Table 2.

Preservice teachers responded to the prompt, "What ideas do you think Josie does not understand about the concepts underlying the addition of decimals?" For each subconcept, preservice teachers were given a score of 0 if they did not mention the subconcept at all in their analysis of what Josie did not know; 1 if they referenced the subconcept but only in the specific context of Josie's work; and 2 if they explicitly explained the subconcept in a way that expressed the generalized form of the subconcept and went beyond the particular context of Josie's work. To clarify the difference between scores of 1 and 2 on this task, we provide examples of responses that would receive each score for subconcept 4.

*Score of 1*: The only thing I would say Josie doesn't understand involves problem 1. She decides that she's going to count all of the little blocks, which is fine, but I think it's important that she knows that 10 of those little blocks equals .1 so she could have made it easier to find the answer if she counted that she has 10 little blocks which equals .1 and 3 left over which equals .03. So now all she needs to do is count the number of .1's she has (4) and the 3 little blocks left over [which is] .43.

*Score of 2*: I do not see any evidence that Josie understands that when a quantity contains ten or more of a particular unit, we can exchange ten of the units for one unit of the next larger size. On all three problems, Josie always counts up how many .01's there are. If there are 45 .01's, for example, Josie knows the answer is .45. This does not show that Josie recognizes the idea of a ten-for-one exchange. A student who understood this idea should be able to exchange 10 hundredths for 1 tenth and explain that this is allowed because a tenth is ten times as big as a hundredth. If Josie showed the ability to exchange 10 units for 1 unit for several different place values and several types of tasks (e.g., exchanging in the subtraction algorithm) and could explain the ideas behind this exchange, then I would have evidence that Josie understands the idea of a ten-for-one exchange.

Table 2
*The Four Tasks, Their Learning Goals, and the Subconcepts for Each Learning Goal*

| Task | Learning goal | Subconcepts |
|---|---|---|
| Anticipating an Ideal Student Response | Students will understand how to add fractions and will understand the concepts underlying this operation. | 1. A quantity is identified as the quantity "one."<br>2. We obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts.<br>3. The numerator is the number of units of size $1/n$.<br>4. The addends must both be expressed in terms of the same-sized unit.<br>5. The addends must be joined.<br>6. The sum must be expressed in terms of a unit of size $1/n$. |
| Evaluating a Student's Incorrect Response | Students will understand how to compare fractions with different denominators and numerators, and the concepts underlying this type of comparison. | 1. A quantity is identified as the quantity "one."<br>2. We obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts. As $n$ gets larger, $1/n$ gets smaller.<br>3. The numerator is the number of units of size $1/n$.<br>4. A quantity can be represented by different, equivalent fractions. For example, if the same "one" is used, a quantity that can be represented by the symbol 4/5 can also be represented by the symbol 8/10. Because the unit of size 1/10 is one half as large as the unit of size 1/5, two times as many units of size 1/10 will fit into the quantity (i.e., 8 units of size 1/10 will fit in, versus 4 units of size 1/5). |
| Evaluating a Student's Correct Work | Students will understand how to add decimals and understand the concepts underlying this operation. | 1. A quantity is identified as the quantity "one."<br>2. The value of each digit in a decimal number is determined by its place in the numeral. Each place is associated with a unit of measure, and the size of the units increase or decrease by a factor of 10 as you move to the left or right, respectively.<br>3. When adding decimals, same-sized units are joined, and the sum will be of that unit. In particular, it is most efficient if each unit of measure is added separately.<br>4. When a quantity contains ten or more of a particular unit of measure, those ten can be exchanged for one unit of measure of the next larger size. |
| Analyzing a Classroom Lesson | (Same as for Anticipating an Ideal Student Response) | (Same as for Anticipating an Ideal Student Response) |

Interrater agreement was 90%. Each preservice teacher was assigned a total score; because there are four subconcepts, total scores could range from 0 to 8.

Preservice teachers also responded to the prompt, “What ideas do you think Josie does understand about the concepts underlying the addition of decimals? List as many ideas as you can (up to four) that you think Josie does understand.” We first identified categories or types of responses by reading through all of the participants’ responses. We then determined how many participants gave each type of response. Interrater agreement was 94%.

*Analyzing a Classroom Lesson Task*. Because the learning goal of Mrs. Roland’s lesson is identical to the learning goal for the Anticipating an Ideal Student Response task, the same six subconcepts listed for the Anticipating an Ideal Student Response task were used to analyze responses to the Analyzing a Classroom Lesson task. Preservice teachers were assigned a score of 0, 1, or 2 for each subconcept. The meaning of the scores is the same as those described for the Anticipating an Ideal Student Response task. Part 1 was coded separately; Parts 2 and 3 were coded together because they are interrelated. Agreement between coders was 95%.

*Rationale for the coding*. The tasks were designed to assess whether preservice teachers applied a strategy of unpacking learning goals to plan for, and evaluate, teaching and learning. The four tasks require preservice teachers to recognize the value of conducting a mathematical analysis of teaching and learning situations. That is, the tasks test preservice teachers’ inclination to analyze the mathematics involved in the situation (identify the mathematical subconcepts and use them to evaluate and plan for teaching and learning). We took both the number of subconcepts mentioned and the explicitness with which they were mentioned as evidence of the extent to which preservice teachers attended to the mathematics in these teaching and learning situations. We interpreted higher scores to mean greater attention to the mathematical subconcepts of the learning goal and, in turn, a more active application of SCK to the analysis of teaching and learning.

*Order of the tasks*. The tasks were administered in a fixed order. In the first research session, participants completed the Evaluating a Student’s Incorrect Response task and then completed the Analyzing a Classroom Lesson task. In the second session, participants first completed the Anticipating an Ideal Student Response task, followed by the Evaluating a Student’s Correct Work task.

This order was selected for two reasons. First, this combination allowed both research sessions to be scheduled for approximately the same amount of time. Second, the Analyzing a Classroom Lesson task and the Anticipating an Ideal Student Response task had the same learning goal, and we believed the latter task might affect preservice teachers’ performance on the former task. Creating four ideal student responses to show understanding of the learning goal might affect how preservice teachers would analyze the effects of a lesson with that learning goal. Therefore, the Analyzing a Classroom Lesson task was given first, and the two tasks were given during different research sessions.

## RESULTS

### *Anticipating an Ideal Student Response*

Part 1 of the Anticipating an Ideal Student Response task specified a learning goal and four problems and asked the participants to construct four ideal student responses (one response for each problem) that would convince them the student understood the learning goal. Recall that the instructions for each of the four problems in the task were, "Imagine that this is the ***only*** problem that Sue will solve for you; i.e., this is the ***only*** evidence that you will use to judge whether Sue understands the concepts underlying the addition of fractions. Use the exact wording that you want Sue to use while she solves the task." As shown in Appendix A, each of the four problems used a different representation: Problems 1, 2, 3, and 4 used fraction pieces, graph paper, the common denominator algorithm, and pennies, respectively.

Note that there is a difference between solving 1/4 + 3/8 = ? using the given representations and constructing an ideal student response that shows a student understands each of the subconcepts when the student is using each of the representations. For some of these representations, the task of the "solver" and the task of the "constructor of an ideal student response" are more similar. In the graph paper problem, for example, the solver has to explicitly think about or use many of the subconcepts during the solution process: He or she has to identify a quantity to represent "one," create fractional units, and find the answer using his or her constructed units. The solution itself requires thinking about the subconcepts (if one is able to do so). In Problem 3 (the common denominator method), the solution can be carried out with no reference to, or understanding of, any of the subconcepts. The solver can think, "1/4 + 3/8 = 2/8 + 3/8 = 5/8." This means it is less likely that the solver will generate an ideal student response for this problem as compared to the graph paper problem (see Table 1). In the case of fraction pieces, the solver does not have to attend to, or even understand, the concept of one in order to solve the problem. With the fraction pieces, the solver can think, "This piece is 1/4. This piece is 1/8. If I join 1/4 and 3/8, I see (by just looking at the physical size of the materials) that five of the 1/8 pieces fit into the sum. So, the answer is 5/8."

Similar statements can be made about other subconcepts and problems. For example, fraction pieces may encourage the solver to apply subconcept 4 (the addends must both be expressed in terms of the same-sized unit). Fraction pieces provide support for the application of this idea because it is relatively easy to find a common unit that fits into both addends by laying fraction pieces on top of each other and comparing their relative sizes. Solving the pennies problem does not make this subconcept visible because it is easier to join the addends without finding a common unit and then to apply subconcept 6 (the sum must be expressed in terms of a unit of size $1/n$).

Because it is unnecessary to think about each subconcept in order to solve 1/4 + 3/8 = ? with particular representations, constructing four ideal student responses requires (a) identifying the subconcepts by unpacking the learning goal, and then (b) making sure that the student refers to each of the subconcepts in each of the four ideal student responses. Thus the construction of four ideal student responses requires a deliberate unpacking of the learning goal.

We needed to interpret preservice teachers' responses on the Anticipating an Ideal Student Response task carefully because, on some of the four problems in this task, just solving the problem would likely yield an identification of some subconcepts (e.g., subconcept 1 for the graph paper problem), whereas solving other problems would likely keep the subconcepts hidden (subconcept 1 for the algorithm problem). Consequently the best measure of unpacking learning goals to identify subconcepts on this task was preservice teachers' relative performance across the four problems. Preservice teachers who scored higher on supportive tasks (tasks that required the use of a subconcept just to solve the problem) and lower on nonsupportive tasks were unlikely to be deliberately unpacking the learning goal. For problems that require thinking about a given subconcept just to solve the problem, it is possible that a preservice teacher could construct a student response that refers to the subconcept without explicitly unpacking the learning goal. But for problems that do not require thinking about a given subconcept during the solution process, it is unlikely that a preservice teacher would construct a student response that refers to the subconcept unless the preservice teacher had consciously identified the subconcepts involved in achieving the learning goal.

The results from Part 1 of the Anticipating an Ideal Student Response task support two conclusions.

1. The majority of preservice teachers understood, or at least used or referred to, the relevant mathematical subconcepts for at least one problem, and therefore appeared to have the ability to identify individual subconcepts of the learning goal.
2. Despite the fact that the individual subconcepts were accessible to them, preservice teachers did not, in general, apply a strategy that involves unpacking the learning goal to identify subconcepts and using the subconcepts to construct ideal student responses (i.e., responses that provide evidence that students understand the component ideas of the learning goal).

Evidence for the first claim can be found in Table 3. The first data column of Table 3 shows the percentages of preservice teachers who received a score of 2 for each subconcept on at least one of the four problems. As Table 3 shows, for five of the six subconcepts, approximately 50% or more of the preservice teachers explicitly referred to, or used, the subconcept in a way that indicated they understood the subconcept. The second data column of Table 3 shows that, for each of the six subconcepts, the percentage of preservice teachers who received a score of 1 or 2 was 83% or greater. These results support the claim that the majority of preservice teachers understood, or at least used or referred to, the relevant mathematical subconcepts, and therefore showed an ability to identify the individual subconcepts of the learning goal.

Table 3
*Percent of Preservice Teachers Receiving a Score of 1 or 2 for Each of the Six Subconcepts on at Least One Problem in Part 1 of the Anticipating an Ideal Student Response Task*

| Subconcept | Percent who received at least one "2" | Percent who received at least one "1" or one "2" |
|---|---|---|
| 1. A quantity is identified as the quantity "one." | 83 | 83 |
| 2. We obtain units of size 1/*n* by partitioning the "one" into *n* equal parts. | 63 | 100 |
| 3. The numerator is the number of units of size 1/*n*. | 47 | 100 |
| 4. The addends must both be expressed in terms of the same-sized unit. | 10 | 90 |
| 5. The addends must be joined. | 60 | 100 |
| 6. The sum must be expressed in terms of a unit of size 1/*n*. | 50 | 100 |

Another analysis provides information on individual preservice teachers' understanding of all six subconcepts. For each preservice teacher, we identified the highest score he or she received for each subconcept across the four problems. For example, if a preservice teacher received scores of 0, 2, 1, and 2 for Subconcept 1 on the fraction pieces, graph paper, algorithm, and pennies problems, respectively, then the highest score for the preservice teacher for subconcept 1 was a 2. We then totaled the preservice teachers' highest scores for each of the six subconcepts to obtain the *total highest score*. Total highest scores for each preservice teacher could range from 0 to 12. Because this score shows whether preservice teachers used the subconcepts, with understanding, on any of the problems, it is a good indicator of whether the preservice teachers could access the six subconcepts and apply them in a meaningful way. The average total highest score was 8.87 ($SD = 1.93$). We interpret this average to show that the preservice teachers could identify, with understanding, many of the mathematical subconcepts.

The results also suggest, however, that despite possessing the relevant knowledge, the preservice teachers did not spontaneously unpack the learning goal in order to apply this knowledge to construct ideal student responses to each problem. As previously described, each preservice teacher was given a score of 0, 1, or 2 for each subconcept for each problem. These scores were then totaled to obtain a total score for each preservice teacher for each problem. If preservice teachers explicitly analyzed the learning goal, we would expect mean total scores for each problem to be similar across problems; an explicit analysis of the learning goal would allow one to construct a response for each problem that referred to each subconcept. Instead the means were significantly different ($F(3, 116) = 46.57, p < .0001$). Post hoc analyses using a Tukey HSD test showed the average total score for the algorithm problem ($M = 0.73$, $SD = 1.01$) was significantly lower than the average total scores for the fraction pieces problem ($M = 5.27$, $SD = 2.26$), graph paper problem ($M = 6.53$, $SD = 2.33$), and pennies problem ($M = 6.13$, $SD = 2.62$), with $p < .01$ for each of the three pairwise comparisons. The representation that does not require thinking about the subconcepts in order to solve the problem—namely, the algorithm—yielded a significantly lower total score than the other representations.

Moreover, the average total highest score of 8.87 (defined previously) was significantly higher than the average total scores for each problem ($F(4, 145) = 60.26$, $p < .0001$; $p < .01$ for each pairwise comparison of the mean total highest score versus the mean total score for each problem). This result suggests preservice teachers did not apply available knowledge of the individual subconcepts to construct ideal student responses for each problem. In summary, the data in Table 3, the total highest scores, and the total scores for each problem suggest preservice teachers could refer to each subconcept but did not spontaneously unpack the learning goal and use the identified subconcepts to determine what a student would have to say and do in each context to demonstrate mastery of the learning goal.

Additional evidence for the two conclusions described previously is provided by comparing scores for each subconcept across problems. For example, 50% and 63% of the preservice teachers received a score of 2 for Subconcept 1 (a quantity is identified as the quantity "one") on the graph paper and pennies problems respectively, problems that require attending to the "one" during the solution of the problem. But on problems that did not require explicit attention to this idea in order to solve it, preservice teachers received lower scores: only 17% received a score of 2 for Subconcept 1 on the fraction pieces problem, and no preservice teacher received a score of 2 on Subconcept 1 for the algorithm problem. For Subconcept 2 (we obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts), the percentages who received a score of 2 ranged from 0% for the algorithm problem, in which deliberate attention to the subconcept is unnecessary in order to solve the problem, to 50% for the pennies problem in which the solver must partition the one to make fractional units.

The differences in responses across the problems suggest preservice teachers can identify a subconcept when a context is supportive. By *supportive* we mean a context that draws their attention to the subconcept, which reveals the subconcept or makes it visible while examining the problem. But preservice teachers do not spontaneously initiate a strategy that identifies all of the subconcepts and uses the unpacked ideas to construct student responses that would provide evidence of conceptual understanding.

Part 2 of the Anticipating an Ideal Student Response task asked preservice teachers to think about what a problem requires from students and to relate that to an analysis of the learning goal. The four problems were designed to be more or less likely to reveal students' understanding of the learning goal. We believed the graph paper problem (Problem 2) and pennies problem (Problem 4) would be most revealing: Students must construct a "one," create fractional units, show they understand the meaning of the numerator and denominator, physically join the quantities, and find the answer using their constructed units. Fraction pieces (Problem 1) do more of the work for students, and students do not have to attend explicitly to the concept of one or create their own units. The algorithm (Problem 3) can be carried out with no understanding of the learning goal. Forty-three percent of the preservice teachers selected the pennies problem as the problem that would tell them the most about whether students understood the concepts underlying the addition of fractions, 23% chose the graph paper problem, 20% chose the common denominator algorithm problem, and 13% chose the fraction pieces problem.

We coded whether preservice teachers referred to the subconcepts when they explained why they picked a particular problem as having the most potential to reveal students' understanding of the learning goal. A score of 0 indicated they did not refer to the subconcept and 1 indicated they did. The average score was 1.57 out of a possible 6 ($SD = 1.25$). Twenty-three percent of the preservice teachers referred to no subconcepts, but 77% referred to one or more subconcepts to justify their choice. The following responses are representative.

> The fraction strips [pieces] are the clearest means of determining whether or not the learning goal was achieved. Sue is able to match up 1/4 to 2/8 to determine they are equivalent fractions. Then she can add the 2/8 and 3/8 and it is clearly represented as 5/8 in the fraction strips [pieces]. It shows she understands the relationship between 1/4 and 2/8 and why she needs to use common denominators to add fractions.

> I chose problem 2 [graph paper] because it forces students to tell you all they know about adding fractions. It told me that they know that 1 has to be partitioned/cut up to find the size of the fraction. Then it requires them to draw out the representation of each fraction, put them together, and figure out the correct measuring unit needed to find the final answer. I think this will tell me the most about the students' understanding because they have to show and explain all the steps. Students can get away with not knowing how to add fractions with the common denominator method, but not this one.

> I chose problem 3 [algorithm] because it actually has the children thinking about making common denominators whereas the other problems involving pictures they do not have to make common denominators they just have to be able to correctly relate their graphical representation of the answer to their [one]. And having common denominators is an important concept to know when adding fractions.

> I chose problem 4 [pennies] because I feel that students will need to use many concepts and ideas to solve this problem. In order to solve this problem, students need to know how to make a [one] and from there, divide the [one] into 1/8[ths] and 1/4[ths]. Students will also have to understand that when you are adding two fractions, they need to have a common denominator. Some students might have difficulty converting 1/4 into 1/8ths. This problem, I feel incorporates the ideas of all the other three problems, which is why I believe problem 4 will tell teachers the most about students' understanding.

Table 4 supports the conclusion that the features of the problem influenced whether and how preservice teachers unpacked the learning goal and used the analysis to evaluate a representation's capacity for revealing student understanding.

Table 4 shows that all the preservice teachers who selected the fraction pieces problem as the most revealing problem referred to the subconcept that addends should be expressed in terms of the same-sized units. Fraction pieces do appear to encourage a solution that involves using the same-sized units for both addends, whereas graph paper solutions, for example, do not. (In a graph paper solution, it is easier to first join the addends, and then to express the resulting quantity in terms of a single unit.) Forty-three percent of the preservice teachers who chose the graph paper problem mentioned the idea of one, which is required to successfully complete the graph paper solution, whereas 0% of those who chose the fraction pieces mentioned the idea of one, which is not required when using fraction pieces.

The average score of 1.57 (out of 6) suggests that preservice teachers' analyses of the learning goal were limited, however. Although almost all of the preservice teachers referred to each of the subconcepts in Part 1 of the task (a score of 1 or 2 in Table 3), the percentages of preservice teachers who referred to each of the subconcepts in Part 2 were as follows:

1. A quantity is identified as the quantity "one": 40%.
2. We obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts: 37%.
3. The numerator is the number of units of size $1/n$: 7%.
4. The addends must both be expressed in terms of the same-sized unit: 50%.
5. The addends must be joined: 17%.
6. The sum must be expressed in terms of a unit of size $1/n$: 7%.

Table 4
*Percent of Preservice Teachers Referring to the Six Subconcepts When Explaining, on Part 2 of the Anticipating an Ideal Student Response Task, Why They Chose a Problem as Having the Most Potential to Reveal Student Understanding*

| | Among those who chose the problem as most revealing, percent who referred to the subconcept | | | |
|---|---|---|---|---|
| Subconcept | Fraction pieces ($n$ = 4) | Graph paper ($n$ = 7) | Algorithm ($n$ = 6) | Pennies ($n$ = 13) |
| A quantity is identified as the quantity "one." | 0 | 43 | 33 | 54 |
| We obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts. | 50 | 43 | 17 | 38 |
| The numerator is the number of units of size $1/n$. | 25 | 14 | 0 | 0 |
| The addends must both be expressed in terms of the same-sized unit. | 100 | 14 | 67 | 46 |
| The addends must be joined. | 25 | 29 | 0 | 15 |
| The sum must be expressed in terms of a unit of size $1/n$. | 25 | 14 | 0 | 0 |

The optimal response to Part 2 is for preservice teachers to carry out a complete analysis of the subconcepts of the learning goal, and then to relate that analysis to the capabilities of each of the problems for eliciting an understanding or lack of understanding of each subconcept. Instead, preservice teachers identified a subset of the subconcepts, and chose a representation that had the potential to reveal students' understanding of those subconcepts.

*Evaluating a Student's Incorrect Response*

Table 5 shows the percentages of preservice teachers who received scores of 0, 1, and 2 for each of the four subconcepts on the Evaluating a Student's Incorrect Response task. The average total score out of a possible 8 was 2.27 ($SD$ = 0.58). No preservice teacher received a score of 2 on any subconcept. However, all preservice teachers referred to at least one subconcept in their analysis: 7% referred to one subconcept, 60% referred to two subconcepts, and 33% referred to three subconcepts when describing what Nikki did not understand.

Based on the results for Part 1 of the Anticipating an Ideal Student Response task, we can predict that preservice teachers will fail to identify relevant subconcepts to evaluate a student response when the context does not draw their attention to that subconcept. Performance on the Evaluating a Student's Incorrect Response task is consistent with this prediction. Nikki's response indicates that she did not understand the concept of one, for example. Because Mrs. Smith's lesson involved fraction pieces, preservice teachers might not have thought about the concept of one when analyzing Nikki's response. Although the results from the Anticipating an Ideal Student Response task indicate that 83% of the preservice teachers could refer to the idea of one in other contexts (Table 3), only 23% of the participants referred to this subconcept when they assessed Nikki's response (Table 5).

Table 5 also shows that preservice teachers primarily referred to concepts involving the numerator and the denominator, ideas that Nikki most obviously lacked. This result again supports the conclusion that preservice teachers can identify a subconcept in a supportive context. It also suggests preservice teachers

Table 5
*Percent of Preservice Teachers Receiving Scores of 0, 1, or 2 on the Four Subconcepts of the Evaluating a Student's Incorrect Response Task*

| | Percent who received each score | | |
|---|---|---|---|
| Subconcepts | Score 0 | Score 1 | Score 2 |
| A quantity is identified as the quantity "one." | 77 | 23 | 0 |
| We obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts. As $n$ gets larger, $1/n$ gets smaller. | 0 | 100 | 0 |
| The numerator is the number of units of size $1/n$. | 10 | 90 | 0 |
| A quantity can be represented by different, equivalent fractions. | 87 | 13 | 0 |

tended simply to comment on what was visible or explicit in Nikki's response as opposed to constructing a list of component ideas required to achieve the learning goal, and then comparing these ideals or standards to Nikki's response.

As another example of this tendency to react to the immediate context, only 23% referred to the concept of a one when analyzing Nikki's response (Table 5); attention to this subconcept requires going beyond Nikki's response to identify what is missing in the response. In contrast, 67% of the preservice teachers wrote that Nikki needed to understand that just because the numerator is larger, smaller, or equal does not mean the fraction is larger, smaller, or equal and/or that she needed to understand that she has to look at the whole fraction, not just the numerator. The latter ideas are directly connected to what Nikki did; she assumed the number with the larger numerator was larger.

*Evaluating a Student's Correct Work*

The task asks participants to identify the ideas that Josie does understand about adding decimal numbers and then the ideas Josie does not understand about adding decimal numbers. We present first preservice teachers' responses regarding what Josie does not understand. In this context, preservice teachers tended to make few references to the subconcepts of the learning goal (Table 6). The average total score was 0.77 ($SD$ = 0.68) out of a possible 8. Thirty-seven percent of the preservice teachers referred to no subconcepts (out of four), 50% referred to one subconcept, and 13% referred to two subconcepts. This can be contrasted with the preservice teachers' analyses on the Evaluating a Student's Incorrect Response task for which 7% referred to one subconcept (out of four), 60% referred to two subconcepts, and 33% referred to three subconcepts. Apparently, preservice teachers were more likely to decompose a learning goal by identifying the individual subconcepts when analyzing an incorrect student response.

No preservice teacher received a score of 2 on any subconcept when they evaluated Josie's correct responses. That is, no preservice teacher explained a subconcept in a way that expressed the generalized form of the subconcept and went beyond the particular context of Josie's work.

Table 7 lists the most common responses to the prompt, "What ideas do you think Josie *does* understand about the concepts underlying the addition of decimals? List as many ideas as you can (up to four) that you think Josie does understand." As in the Evaluating a Student's Incorrect Response task, preservice teachers appeared to limit their analyses by sticking closely to the exact nature of the student's response. For example, 53% of the preservice teachers said Josie understood that the individual blocks represented 0.01 and the sticks represented 0.1. This claim was based on Josie's statement: "I used a long stick to show .1 and a little block to show .01." But as Table 6 shows, 90% of the preservice teachers received a score of 0 for subconcept 2 when analyzing what Josie should know and did not, which suggests that preservice teachers did not go beyond Josie's statement to ask whether Josie understood the more general ideas associated with her statement—that the

Table 6
*Percent of Preservice Teachers Receiving Scores of 0, 1, or 2 on the Four Subconcepts When Analyzing What Josie Did Not Understand About Addition of Decimals on the Evaluating a Student's Correct Work Task*

| Subconcepts | Percent who received each score | | |
|---|---|---|---|
| | Score 0 | Score 1 | Score 2 |
| 1. A quantity is identified as the quantity "one." | 90 | 10 | 0 |
| 2. The value of each digit in a decimal number is determined by its place in the numeral. Each place is associated with a unit of measure, and the size of the units increase or decrease by a factor of 10 as you move to the left or right, respectively. | 90 | 10 | 0 |
| 3. When adding decimals, same-sized units are joined, and the sum will be of that unit. In particular, it is most efficient if each unit of measure is added separately. | 90 | 10 | 0 |
| 4. When a quantity contains 10 or more of a particular unit of measure, those 10 can be exchanged for one unit of measure of the next larger size. | 53 | 47 | 0 |

Table 7
*Responses to What Josie Understands About Addition of Decimals on the Evaluating a Student's Correct Work Task Given by at Least 20% of the Preservice Teachers*

| Josie understands . . . | Percent of preservice teachers who identified each idea |
|---|---|
| The individual blocks represent .01 and the sticks (or 10 blocks, or stick of 10) represent .1 | 53 |
| 10 little blocks make one stick, or ten .01s make one .1 (10 times relationship for tenths and hundredths places only) | 50 |
| How to represent decimals with money or lengths, or how to represent money or lengths with decimals | 47 |
| Place value | 37 |
| How to represent decimals with base 10 blocks | 33 |
| How to add by counting the quantities | 20 |

value of each digit in a decimal number is determined by its place in the numeral, that each place is associated with a unit of measure, and/or that the size of the units increase or decrease by a factor of 10 as you move to the left or right, respectively.

In summary, preservice teachers were more likely to attempt to unpack a learning goal when a student gave an incorrect response. This supports the conclusion that preservice teachers can identify individual subconcepts when the context is supportive: Incorrect student responses indicate something is wrong, and probably prompt an unpacking of the mathematical ideas to identify the source of the difficulty. Correct student responses do not explicitly call for such an analysis. However, preservice teachers' evaluations of both correct and incorrect student responses were also similar in one respect. In both cases, preservice teachers limited their analysis of student responses by sticking closely to the exact nature of the response. They did not go beyond the response to identify all the subconcepts that might be missing, incorrect, or incomplete. The overlap in the mathematical ideas involved in the Anticipating an Ideal Student Response task and the Evaluating a Student's Incorrect Response task suggests that this tendency did not reflect a content knowledge deficit or an inability to identify individual subconcepts. Instead, it suggests that preservice teachers did not intentionally unpack the learning goal in order to think explicitly about what a student should or could be doing to show understanding, and then compare that to what a student is actually doing.

### *Analyzing a Classroom Lesson*

Part 1 of this task assessed preservice teachers' ability to evaluate instruction and student responses by linking them to key subconcepts of the learning goal. The results suggest that when preservice teachers evaluated whether the lesson was helping students understand the learning goal, they did not link their analysis to the learning goal. As Table 8 shows, the percentages of preservice teachers who failed to refer to each of the six subconcepts ranged from 90% to 100%. When preservice teachers revised a segment of the lesson and justified their revision (Part 2/3), they were slightly more likely to refer to the learning goal (see the right column of Table 8). Whereas 90% of the participants referred to no subconcepts in Part 1 of the task, 50% of the preservice teachers referred to one, two, three, or four subconcepts in Part 2/3 of the task. Thus, the activity of revising a lesson was more likely to lead to some unpacking of the learning goal.

The average total highest score (defined in the results for the Anticipating an Ideal Student Response task) on the Analyzing a Classroom Lesson task was 1.07 ($SD$ = 1.31) out of 12. The average total highest score for the Anticipating an Ideal Student Response task, which had the same learning goal as the Analyzing a Classroom Lesson task, was significantly higher ($M$ = 8.87, $SD$ = 1.93, $t(58)$ = 18.32, $p < .0001$). This discrepancy highlights the difference between identifying individual subconcepts when the context is supportive, and doing so when there are other features competing for attention.

Table 9 compares preservice teachers' performance on the Anticipating an Ideal Student Response task with their performance on the Analyzing a Classroom Lesson task. The percentages of preservice teachers who received a highest score of 2 for a subconcept on the Anticipating an Ideal Student Response task but received a highest score of 0 for the same subconcept on the Analyzing a Classroom Lesson task was 40% or higher for five of the six subconcepts. The percentages of

Table 8
*Percent of Preservice Teachers Receiving a Score of 0 for Each Subconcept of the Analyzing a Classroom Lesson Task*

| Subconcept | Percent who received a "0" in Part 1 | Percent who received a "0" in Part 2/3 |
|---|---|---|
| 1. A quantity is identified as the quantity "one." | 97 | 90 |
| 2. We obtain units of size $1/n$ by partitioning the "one" into $n$ equal parts. | 90 | 70 |
| 3. The numerator is the number of units of size $1/n$. | 100 | 90 |
| 4. The addends must both be expressed in terms of the same-sized unit. | 97 | 80 |
| 5. The addends must be joined. | 100 | 80 |
| 6. The sum must be expressed in terms of a unit of size $1/n$. | 100 | 90 |

preservice teachers who received a highest score of 1 or 2 for a subconcept on the former task but received a highest score of 0 on the latter task ranged from 67% to 90% for all subconcepts. This result supports the conclusion that preservice teachers possessed the relevant knowledge but did not spontaneously unpack the learning goal in order to apply this knowledge to evaluate and revise instruction.

As noted earlier, in Analyzing a Classroom Lesson, the task of identifying and using the subconcepts of the learning goal to evaluate and improve instruction was deeply embedded in a classroom scenario. Unlike the other tasks, this task asked the preservice teachers to conjecture about cause–effect links between instructional features and student responses while relating the instruction and the student responses to an analysis of the learning goal. This required preservice teachers to keep their attention focused on the mathematical analysis of the situation in the face of other important and sometimes competing aspects of teaching. Consequently, we expected the Analyzing a Classroom Lesson task to be the most challenging for preservice teachers.

## DISCUSSION

We began with the goal of uncovering the successes and challenges that preservice teachers might experience as they acquire the knowledge and skills needed to unpack mathematical learning goals. Before summarizing what we found, we want to review why these competencies are important.

Table 9

*Comparison of Preservice Teachers' Scores for Each Subconcept on Part 1 of the Anticipating an Ideal Student Response Task and the Analyzing a Classroom Lesson Task*

| | Percent of preservice teachers in each category | |
|---|---|---|
| Subconcept | Received at least one score of 2 on the Anticipating an Ideal Student Response task, and a score of 0 on all parts of the Analyzing a Classroom Lesson task | Received at least one score of 1 or 2 on the Anticipating an Ideal Student Response task, and a score of 0 on all parts of the Analyzing a Classroom Lesson task |
| 1. A quantity is identified as the quantity "one." | 77 | 77 |
| 2. We obtain units of size 1/*n* by partitioning the "one" into *n* equal parts. | 40 | 67 |
| 3. The numerator is the number of units of size 1/*n*. | 40 | 90 |
| 4. The addends must both be expressed in terms of the same-sized unit. | 7 | 70 |
| 5. The addends must be joined. | 47 | 80 |
| 6. The sum must be expressed in terms of a unit of size 1/*n*. | 43 | 90 |

Expertise in teaching is not acquired during preservice education but develops over time, as teachers learn from their own experience (Hiebert et al., 2007; Ma, 1999; Sowder, 2007). Models of effective teacher learning involve teachers studying the effects of their own practice (Gallimore, Ermeling, Saunders, & Goldenberg, 2009; Huang & Boa, 2006; Lampert & Graziani, 2009; Stigler & Hiebert, 1999). Central to these models is an analysis of how the particulars of classroom instruction influence the thinking and learning of students. Useful analyses of teaching–learning links are dependent on clear descriptions of what is to be learned—clear descriptions of the mathematical learning goals for the lesson. How did the instructional activities help or hinder students' achievement of the learning goals?

Unless teachers are clear about what they intend students to learn, it is difficult even to begin examining how instruction might have helped students learn it. More than that, it is difficult to plan instructional activities that would be helpful. So, both planning for instruction and evaluating its effects depend on clear descriptions of learning goals. Being clear about learning goals means unpacking them to identify their constituent parts. What mathematical concepts and skills feed into the target learning goal(s)? What do students need to know and be able to do in order to achieve the goal(s)? Being able to answer these questions would seem to be a prerequisite for intentionally and systematically learning from one's practice.

As we argued previously, the knowledge and skills needed to unpack learning goals are primarily mathematical (Ball et al., 2008). This makes them prime candidates for learning goals in preservice education. Although there are good reasons to embed them in teaching contexts, for purposes of both learning and assessment (Blömeke, Felbrich, Müller, Kaiser, & Lehmann, 2008a), they are not dependent on extensive experience with school students and real-time teaching situations (Li & Kulm, 2008). In addition, the attention to school students' learning goals is already a part of teacher preparation programs in many countries (Blömeke et al., 2008b), so the development of competencies needed to carefully analyze learning goals fits well within many current programs.

To reiterate, unpacking mathematical learning goals defines a set of competencies that (a) are essential for learning from one's own teaching experience and improving one's effectiveness over time, and (b) can potentially be acquired during preservice education. What do our findings show about the challenges facing preservice teachers in developing these competencies?

In simplest terms, the results indicate that preservice teachers, who already have had some experience with decomposing mathematical concepts into subconcepts, can identify mathematical subconcepts of learning goals in supportive contexts but do not spontaneously apply a strategy of unpacking learning goals and then using this information to plan for, or evaluate, teaching and learning. As described earlier, supportive contexts included contexts in which the subconcepts became visible simply by solving the mathematical problem (e.g., the graph paper problem in the Anticipating an Ideal Student Response task) or by examining the most apparent features of a student's incorrect response (e.g., the role of numerator and denominator in the Evaluating a Student's Incorrect Response task). In other words, supportive contexts were those that revealed the relevant subconcepts without intentionally having to search for them. Nonsupportive contexts were those in which the relevant subconcepts remained hidden (as often is the case when students give correct responses) and when other factors competed for attention (as was the case with pedagogical factors in the Analyzing a Classroom Lesson task).

Based on these findings, it appears that unpacking mathematical learning goals is not a tendency that comes "naturally" for preservice teachers (Murray, 1996). Although preservice teachers in this study displayed some of the specialized mathematical knowledge needed for teaching, they did not always use this knowledge to analyze the teaching and learning situations. Opportunities to acquire the relevant mathematical knowledge *and* the disposition to use it (Blömeke et al., 2008a) will need to be deliberately planned. This is likely to require multiple learning opportunities along with considerable practice, perhaps moving from supportive contexts to nonsupportive contexts. We expect that an important ingredient in these learning opportunities will be explicit discussions of the value of unpacking learning goals for studying, and improving, teaching.

We are optimistic about the potential benefits for preservice teachers of acquiring the competencies needed to unpack mathematical learning goals. As Hill, Ball, and colleagues have demonstrated, it is in the details of mathematical knowledge for teaching (MKT) that connections are found to the work of teaching and the learning of students (Hill et al., 2005, 2008). We believe the results presented here describe, with more detail than previously offered, the nature of one component of MKT—specialized content knowledge (Ball et al., 2008). Because this knowledge is measurable, it is possible to evaluate and improve a program's effectiveness in helping preservice teachers acquire it. As preservice teachers develop the tendency to unpack mathematical learning goals and use this information to plan and evaluate instruction, we believe they will be on the road to continued learning and increasingly effective teaching.

## REFERENCES

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83–104). Westport, CT: Ablex.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407.

Barnett, C., Goldenstein, D., & Jackson, B. (Eds.). (1994). *Fractions, decimals, ratios, and percents: Hard to teach and hard to learn?* Portsmouth, NH: Heinemann.

Blömeke, S., Felbrich, A., Müller, C., Kaiser, C., & Lehmann, R. (2008a). Effectiveness of teacher education. *ZDM, 40*, 719–734.

Blömeke, S., Paine, L., Houang, R. T., Hsieh, F.-J., Schmidt, W. H., Tatto, M. T., et al. (2008b). Future teachers' competence to plan a lesson: First results of a six-country study on the efficiency of teacher education. *ZDM, 40*, 749–762.

Brown, A. L., Bransford, J. D., Ferrara, R. A., & Campione, J. C. (1983). Learning, remembering, and understanding. In J. H. Flavell & E. M. Markman (Eds.), *Handbook of child psychology: Vol. 3. Cognitive development* (4th ed., pp. 77–166). New York: Wiley.

Feiman-Nemser, S. F. (1983). Learning to teach. In L. S. Shulman & G. Sykes (Eds.), *Handbook of teaching and policy* (pp. 150–170). New York: Longman.

Gallimore, R., Ermeling, B. A., Saunders, W. M., & Goldenberg, C. (2009). Moving the learning of teaching closer to practice: Teacher education implications of school-based inquiry teams. *The Elementary School Journal, 109*, 537–553.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*, 184–205.

Hawkins, D. (1973). What it means to teach. *Teachers College Record, 75*, 7–16.

Hiebert, J., & Grouws, D. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Charlotte, NC: Information Age.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education, 58*, 47–61.

Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An "experiment" model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education, 6*, 201–222.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*, 430–511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*, 371–406.

Huang, R., & Bao, J. (2006). Towards a model for teacher professional development in China: Introducing Keli. *Journal of Mathematics Teacher Education, 9*, 279–298.

Jansen, A., Bartell, T., & Berk, D. (2009). The role of learning goals in building a knowledge base for elementary mathematics teacher education. *The Elementary School Journal, 109*, 525–536.

Jansen, A., & Spitzer, S. M. (2009). Prospective middle school mathematics teachers' reflective thinking skills: Descriptions of their students' thinking and interpretations of their teaching. *Journal of Mathematics Teacher Education, 12*, 133–151.

Kazemi, E., Lampert, M., & Ghousseini, H. (2007). *Conceptualizing and using routines of practice in mathematics teaching to advance professional education*. Chicago: Spencer Foundation.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal, 109*, 491–509.

Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM, 40*, 833–843.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Murray, F. B. (1996). Beyond natural teaching: The case for professional education. In F. B. Murray (Ed.), *The teacher educator's handbook: Building a knowledge base for the preparation of teachers* (pp. 3–13). San Francisco: Jossey-Bass.

Schaefer, R. J. (1967). *The school as the center of inquiry*. New York: Harper & Row.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157–223). Charlotte, NC: Information Age.

Spitzer, S. M., Phelps, C. M., Beyers, J. E. R., Johnson, D. Y., & Sieminski, E. M. (in press). Developing pre-service elementary teachers' abilities to identify evidence of student mathematical achievement. *Journal of Mathematics Teacher Education*.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the worlds' teachers for improving education in the classroom*. New York: Free Press.

## Authors

**Anne K. Morris**, School of Education, University of Delaware, Newark, DE 19716; abmorris@udel.edu

**James Hiebert**, School of Education, University of Delaware, Newark, DE 19716; hiebert@udel.edu

**Sandy M. Spitzer**, Department of Mathematics, Towson University, 7800 York Road, Towson, MD 21252; sspitzer@towson.edu

## APPENDIX A

### Anticipating an Ideal Student Response Task

*Instructions*

You have just completed several lessons in your fifth-grade math class with the following learning goal:

> "Students will understand how to add fractions and will understand the concepts underlying this operation."

You want to give the students a problem that will assess whether the learning goal was achieved. You are considering the following problems. (Your students are familiar with all of these materials from previous instruction.)

*Problem 1*: Solve 1/4 + 3/8[a] using your fraction pieces.
*Problem 2*: Solve 1/4 + 3/8 by drawing a diagram on graph paper.
*Problem 3*: Solve 1/4 + 3/8 using the common denominator method.
*Problem 4*: Solve 1/4 + 3/8 using pennies with 24 pennies equal to 1.

### PART 1

Sue is a member of your class. Imagine that Sue has mastered the learning goal. Explain how Sue would respond to each of the four problems: What would Sue have to do and **say** (exactly) to convince you that she understands the concepts underlying the addition of fractions?

*Problem 1*: Solve 1/4 + 3/8 using your fraction pieces.

I will be convinced that Sue understands the concepts underlying the addition of fractions if she **says** and does the following while solving this problem:

(Imagine that this is the ***only*** problem that Sue will solve for you; i.e., this is the ***only*** evidence that you will use to judge whether Sue understands the concepts underlying the addition of fractions. Use the exact wording that you want Sue to use while she solves the task.)

*Problem 2*: Solve 1/4 + 3/8 by drawing a diagram on graph paper.

I will be convinced that Sue understands the concepts underlying the addition of fractions if she **says** and does the following while solving this problem:

[a] Although one does not "solve" expressions, this wording was used, in this instance and in others described in this article, because it was familiar to students.

(Imagine that this is the ***only*** problem that Sue will solve for you; i.e., this is the ***only*** evidence that you will use to judge whether Sue understands the concepts underlying the addition of fractions. Use the exact wording that you want Sue to use while she solves the task.)

*Problem 3*: Solve 1/4 + 3/8 using the common denominator method.

I will be convinced that Sue understands the concepts underlying the addition of fractions if she **says** and does the following while solving this problem:

(Imagine that this is the ***only*** problem that Sue will solve for you; i.e., this is the ***only*** evidence that you will use to judge whether Sue understands the concepts underlying the addition of fractions. Use the exact wording that you want Sue to use while she solves the task.)

*Problem 4*: Solve 1/4 + 3/8 using pennies with 24 pennies equal to 1.

I will be convinced that Sue understands the concepts underlying the addition of fractions if she **says** and does the following while solving this problem:

(Imagine that this is the ***only*** problem that Sue will solve for you; i.e., this is the ***only*** evidence that you will use to judge whether Sue understands the concepts underlying the addition of fractions. Use the exact wording that you want Sue to use while she solves the task.)

## PART 2

You decide to give the students ***one*** problem that will assess whether the learning goal was achieved. Which problem will tell you the most about whether your students understand the concepts underlying the addition of fractions? Select one problem and then explain why you think this problem is the best one of the four.

*Problem 1*: Solve 1/4 + 3/8 using your fraction pieces.
*Problem 2*: Solve 1/4 + 3/8 by drawing a diagram on graph paper.
*Problem 3*: Solve 1/4 + 3/8 using the common denominator method.
*Problem 4*: Solve 1/4 + 3/8 using pennies with 24 pennies equal to 1.
____ Problem selected.

Why did you choose this problem? Why will it tell you the most about the students' understanding of the concepts underlying the addition of fractions?

## APPENDIX B

### Evaluating a Student's Incorrect Response Task

(The transcript of Mrs. Smith's lesson was adapted from Barnett, Goldenstein, & Jackson (1994), pp. 45-47.)

*Instructions*

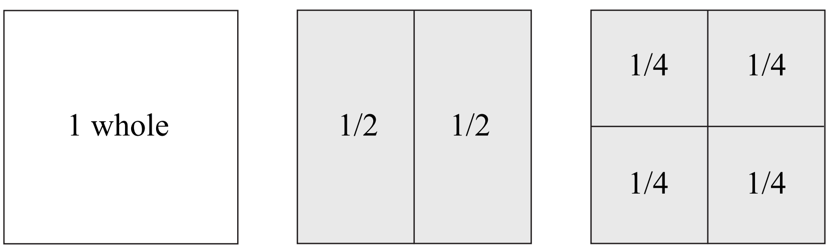
The following transcript took place in a fourth grade classroom. First read the transcript. Then answer the question that appears after the transcript.

### TRANSCRIPT OF MRS. SMITH'S LESSONS

Mrs. Smith, a fourth grade teacher, has planned three lessons on fractions. She has the following learning goal for her students:

"Students will understand how to compare fractions with different denominators and numerators, and the concepts underlying this type of comparison."

Mrs. Smith begins by giving each student a fraction kit. The denominators of the fractions in the kits are 1, 2, 4, 8, and 16. For example, the fraction kits include the following representations:

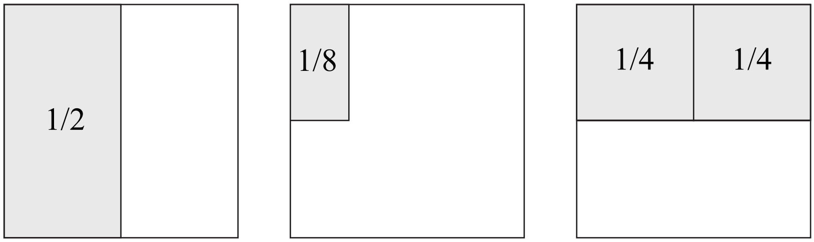



In the first lesson, Mrs. Smith focuses on identifying and orally naming various fractions.

*Mrs. Smith*: Place your "whole" [from the fraction kit] on your desktop. Now show me 1/4 of a whole by placing 1/4 on top of it.

*Students*: All the students place a one-fourth piece on the square representing 1 whole.

By the end of the lesson, all the students can correctly name and use the fraction kit to show unit fractions like 1/2 or 1/8 and nonunit fractions like 2/4 or 5/8. The students show these amounts with the fraction kit like this:




In the second lesson, Mrs. Smith focuses on equivalent fractions.

*Mrs. Smith*: How many eighths would be equal to 1/4? Figure the answer out, and write it down. I will come around and see what you have discovered.

*Students*: All the students use their fraction-kit pieces to determine the answer. They first place a 1/4 piece on the whole, and then place two 1/8 pieces on top of the 1/4 piece. Every student writes, "1/4 equals 2/8."

*Mrs. Smith*: Everyone discovered the correct answer! Try this one. How many sixteenths would equal 3/8?

*Students*: Every student places three 1/8 pieces on the whole, and then places six 1/16 pieces on top of the three 1/8 pieces. Every student writes, "3/8 = 6/16."

By the end of the second lesson, all the students can solve simple equivalency problems without using the pieces.

In the third lesson, Mrs. Smith believes the students are ready to compare fractions with different numerators and denominators.

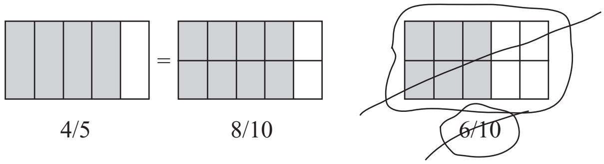
*Mrs. Smith*: I would like you to think about this problem, and then write your solution in your journals. Mrs. Smith writes the problem on the board:

> "Which would you rather have—6/10 of a dollar or 4/5 of a dollar? Explain your reasons for choosing your answer."

Mrs. Smith then walks around the room and reads what the children have written in their journals. She reads the following entries:

*Cindy*: If I had 6/10, I would have 2 more than 4/5. I would choose 6/10 so I could have more money.

*Chris*: 4/5 = 8/10. 8/10 is greater than 6/10. Of course, I would take 4/5 of a dollar. Wouldn't you? He includes an illustration:



*Nikki*: I want 6/10. It is bigger. Nikki draws the following picture to justify her answer:

*Instructions*

Nikki gave an incorrect answer. What doesn't Nikki understand about the concepts underlying the comparison of fractions? List as many ideas as you can (up to five) that you think Nikki should understand, and doesn't.

## APPENDIX C

### Evaluating a Student's Correct Work Task

*Instructions*

Josie's teacher, Mrs. Hagle, has just completed a lesson on adding decimals with her third-grade class. The goal of the lesson was: Help students understand how to add decimals and understand the concepts underlying this operation.

During the lesson, the teacher had asked the students to complete the following problems. Josie's work on each problem is shown. After the lesson is finished, Ms. Hagle looks over Josie's work to assess her understanding of the learning goal.

Read over Josie's work, shown on page 2, and then answer the following questions. Keep the learning goal in mind as you consider Josie's responses.

1. What ideas do you think Josie does understand about the concepts underlying the addition of decimals? List as many ideas as you can (up to four) that you think Josie does understand. Be as specific as you can. If you think she doesn't understand anything about adding decimals, write None. How can you tell from Josie's work that she understands these ideas?
2. What ideas do you think Josie does not understand about the concepts underlying the addition of decimals? List as many ideas as you can (up to four) that you think Josie should understand, and doesn't. Be as specific as you can. If you think she fully understands adding decimals, write None. How can you tell from Josie's work that she does not understand these ideas?

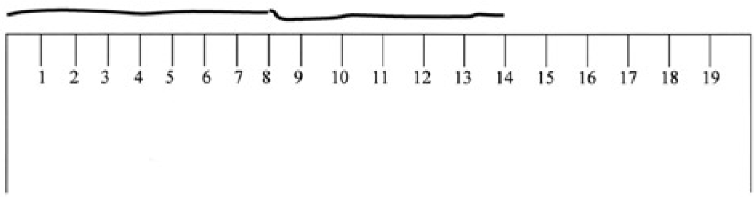
*Josie's Work*

*Problem 1*: Show .26 + .17 with Base-10 Blocks.

*Josie wrote*: I used a long stick to show .1 and a little block to show .01. So then I put out 2 long sticks and 6 little blocks to do .26 and I also put out 1 stick and 7 blocks to show .17, like this: [Shows two longs and six unit cubes on the left, and one long and seven unit cubes on the right.] Then I counted up all the little blocks and I saw that there were 43 little blocks in all so the answer is .43. So .26 + .17 = .43.

*Problem 2*: Heather had two pieces of string. One piece is .08 meters and the other piece is .06 meters. How much string does Heather have altogether?

*Josie wrote*: I remembered that .01 meters is the same as a centimeter. So I got some yarn and a ruler to help me do this problem. I used the ruler to make one piece of yarn that was 8 centimeters and then I did another one that was 6 centimeters. Then I put them together, and I used the ruler to find out how much it was, like this:




And then I saw it was 14 centimeters altogether so I knew the answer would be .14 meters.

*Problem 3*: Joey had $0.29 in his piggy bank. He earned $0.25 selling lemonade. How much money does he have now?

*Josie wrote*: I knew he had 29¢ so I took out 29 pennies. Then he got 25¢ more so I put out 25 more pennies. When I counted them up it was 54 pennies so I wrote $.54 for the answer.

## APPENDIX D

### Analyzing a Classroom Lesson Task

*Instructions*

Ms. Roland taught the following lesson for her fifth graders on adding fractions. She had already taught the meaning of the fraction symbol using a variety of physical materials. The particular goal for this lesson was:

Students will understand how to add fractions and will understand the concepts underlying this operation.

Read the transcript of the lesson and then answer the questions that follow. Always keep in mind the learning goal that Ms. Roland intended for her students.

### TRANSCRIPT

*Ms. Roland*: Good morning everyone. Today we will continue our work with fractions. We are starting a section on adding and subtracting fractions. We've talked before about what fractions mean but not how to add or subtract them.

*Ms. Roland:* Segment 1 [It's really important to understand how to add fractions, not just follow a rule. Most of the fractions we will work with today have the same denominators. That's important. Because when we add fractions, we first check that the denominators are the same and then we add the numerators. Adding fractions follows an easy rule but you should understand why it works. Don't just memorize a rule without understanding why. When we add fractions, we are adding parts of wholes. In previous lessons we worked on the meaning of fractions and you should be able to use that to help you understand how to add fractions. Do you think you can do that?

*Ms. Roland:* Good. OK. Let me show you how that works. Here's the first problem. (Ms. Roland writes on the board: 1/5 + 3/5 = ___ .) You might be tempted to add 1 plus 3 to get 4 and then add 5 plus 5 to get 10, but that wouldn't give you the right answer. Remember, when you add fractions, you just add the top numbers. Those are the numerators. The bottom numbers, or denominators, stay the same. That's pretty easy to remember, right? (Ms. Roland points to the 3 and 1 on the board and writes 4, and then points to the 5 and the 5, and writes 5 to make 4/5.)

*Ms. Roland:* OK, let's make sure everyone knows how to do these problems. Get into your groups and work out the answers to the problems on this worksheet. Get out your fraction pieces because they might help you see what's going on. It's important to understand the rules. (Ms. Roland distributes a worksheet with the following problems: 2/4 + 1/4; 1/6 + 3/6; 1/2 + 1/2; 1/3 + 1/3; 2/5 + 1/5. Students each have an envelope with same-sized colored laminated circles cut into pie pieces of halves, thirds, fourths, fifths, and sixths. Ms. Roland circulates around the room for about 15 minutes and encourages students to use the fraction pieces.)

*Ms. Roland:* I think most of you are done now so let's check a few of these. Who would like to tell us what they found? Tom?

*Tom*: I got 3/4 for the first one. I used the fraction pieces.

*Ms. Roland*: Could you show everyone how you did that?

*Tom*: I took the red pieces and I added 1 red piece and 2 red pieces and got 3 red pieces, so 3/4.

*Ms. Roland*: Great, Tom. Anyone else? Marie?

*Marie*: I got 4/6 for the second one. I did it the same as Tom. But I used the blue pieces because 6 of them make a whole. There were 4 pieces altogether so I wrote 4/6. (Ms. Roland circulates around the room for about 8 minutes and sees that the majority of students are getting the right answers to the problems.)] End of Segment 1

*Ms. Roland*: Segment 2 [Most of you look like you understand how to add fractions when the denominators are the same. What happens if the denominators are different? (Ms. Roland writes the following problem on the board: 1/2 + 1/3 = ___ .) What do you think the answer would be? Why don't you work on this problem for a while and see what you come up with? I'll give you a little time. Remember, you can draw pictures or use the fraction pieces or use whatever you know about fractions. (After about 5 minutes, Ms. Roland asks whether anyone had an answer.)

*Sasha*: I drew a candy bar and divided it into 2 pieces and then I drew another candy bar and divided that one into 3 pieces. Then I shaded 1 piece in each bar. That gave me 2 pieces and there were 5 pieces altogether. So that made 2/5.

*Jason*: I used the fraction pieces. I took 1 of the yellow pieces for the 1/2 and 1 of the green pieces for the 1/3. When I put them together, it looked almost like 1. So, I think it's 1.

*Harry*: I drew a picture and it looked like it was almost 1 to me too.

*Sally*: I agree with Sasha. I drew 2 circles for 1/2 and shaded 1 of them. Then I drew 3 circles for 1/3 and shaded 1 of them. There are 2 circles shaded and 5 in all, so it has to be 2/5.

*Jason*: But it can't be 2/5. If you look at the fraction pieces, 1/2 is bigger than 2/5 and you still have to add on 1/3. So, the answer has to be more than 2/5.

*Ms. Roland*: OK. There are lots of interesting answers here. Did anyone get 5/6? (No one responds.) OK, I still like how many of you are drawing pictures and using the fraction pieces. Some of you are estimating how much it might be and that's great. I know this problem is a little difficult for us now. Please think about it tonight and we'll continue to work on it tomorrow.] End of Segment 2

*Ms. Roland*: I'd just like to sum-up what we should have learned today and make sure you understand how to add fractions. When we add fraction problems like the ones we started with, we first check that the denominators are the same and then we add the numerators. That's why one half plus one third wasn't so easy—the denominators weren't the same. Adding fractions follows an easy rule but you should understand why it works. Don't just memorize a rule without understanding why.

PART 1: Circle one number to rate the lesson as helping or not helping students understand the concepts underlying the addition of fractions.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Helping | | | | Not Helping |

What evidence did you use to make this judgment? Please indicate what information in the transcript you weighed most heavily to determine your rating.

PART 2: _______ If you could revise one of the two segments to help students understand more fully the concepts underlying the addition of fractions, which of the two segments would you choose? The segment should be the one that has **potential** for helping students, even if it does not help them in its current form. Choose the segment that, with some revision, **could best help** students understand the concepts underlying the addition of fractions.

1. Describe how you would revise this segment. Describe **exactly** what you would change by writing what Ms. Roland should say and do.

2. Explain why you think this change would help students understand more fully the concepts underlying the addition of fractions.

PART 3: Look back at the segment you did not choose to revise. Explain why you didn't choose this segment. Be as specific as you can.