Knowledge Needed by a Teacher to Provide Analytic Scaffolding During Undergraduate Mathematics Classroom Discussions

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Using case study analysis and a cognitive theoretical orientation, we examine elements of knowledge for teaching needed by a mathematician to orchestrate whole-class discussions in an undergraduate mathematics classroom. The instructor, an experienced teacher and mathematics researcher, used an inquiry-oriented curriculum to teach a differential equations course for the first time after teaching it with traditional lecture methods for many years. Examples of classroom teaching and interview data demonstrate that, despite having extensive teaching experience and possessing strong content knowledge, some instructors may still face challenges when trying to provide analytic scaffolding to move whole-class discussions toward a lesson's mathematical goals. We also hypothesize several component practices necessary for the successful use of analytic scaffolding. Our analysis focuses on the relationship between the instructor's pedagogical content knowledge and specialized content knowledge and his capacity to enact these component practices during whole-class discussions.

Key words: College mathematics; Teacher knowledge; Teaching practice

Mathematics teachers have a responsibility to direct and shape the learning opportunities of their students. Evidence is now emerging that curricula and teaching practices consistent with some recent efforts toward educational reform show promise for improving students' learning of mathematical skills with deeper conceptual understanding (Briars, 2001; Briars & Resnick, 2000; Fennema et al., 1996; Schoenfeld, 2002). However, movement toward widespread mathematics reform is slowed, at least in part, because of the significant challenges that teachers may face when teaching in reform-oriented ways (Ball, 1993; Chazan & Ball, 1999; Cohen, 1990; Heaton, 2000; Wagner, Speer, & Rossa, 2007; Williams & Baxter, 1996).

Research-based reform practices include the use of small-group activities, inquiry-oriented and discovery-oriented curricula, and deeper attention to mathematical communication and representation, all with an emphasis on valuing students' own mathematical ideas and reasoning. In support of all these activities, teachers are often expected to lead whole-class discussions of mathematical ideas (Ball, 1993; Lampert, 2001; NCTM, 1989, 1991, 2000), thereby promoting mathematical critique and argumentation leading toward the development, justification, and proof of important mathematical concepts. To provide this type of instruction, teachers must take the mathematical contributions that students generate and decide, in the moment, which contributed ideas should be pursued, how to pursue them, which examples should be used, or how even incorrect mathematical ideas can be used to advance the lesson's objectives.

Research has shown, however, that leading such discussions can place new and significant challenges on the shoulders of teachers (Ball, 1993; Heaton, 2000; Nathan & Knuth, 2003; Stein, Engle, Smith, & Hughes, 2008; Williams & Baxter, 1996). Pedagogical skills and knowledge that served them well in traditional lecture classrooms may not be sufficient for developing mathematical concepts as they arise from students' incipient and informal ideas. Although the consequences of these challenges can be readily observed (e.g., discussions fail to reach their intended objectives; discussions may be lively and engaging, but lack mathematical substance or progress), relatively little is known about how these challenges manifest themselves in the moment-to-moment work that teachers do while attempting to lead discussions in productive ways. In an effort to contribute to this area of research, we examined the nature of the knowledge that teachers could productively employ to make use of students' mathematical ideas to guide whole-class discussions in ways that further the mathematical goals for the class.

One challenge in leading whole-class discussions is negotiating what is often seen as a tension between the need to encourage and value students' ideas and the simultaneous use of those ideas to keep the discussion moving in a mathematically productive direction (Ball, 1993; Nathan & Knuth, 2003; Sherin, 2002a; Stein et al., 2008; Williams & Baxter, 1996). In particular, Williams and Baxter (1996) examined this tension by distinguishing between teachers' use of *social scaffolding* to support norms of discourse and student participation and *analytic scaffolding* to support progress toward the mathematical goals for the discussion.

Although some researchers have identified particular practices or tools that teachers can use to foster mathematically productive class discussions, we are concerned that such practices are useful only to the extent that teachers have the knowledge to use them effectively. In addition, although numerous studies have examined teachers' skills in leading discussions (e.g., Nathan & Knuth, 2003; Williams & Baxter, 1996), there has been little examination of how different types of knowledge may be implicated in teachers' use of analytic scaffolding to support whole-class discussions in mathematically productive ways.

Findings from our earlier studies, based solely on data from postinstruction interviews (Wagner et al., 2007), suggested that the challenges one teacher faced

in orchestrating discussions stemmed in part from his lack of familiarity with the different ways that students would approach or think about problems in the curriculum, or from difficulties he had in following the mathematical ideas that students contributed to the discussions. Ball, Lubienski, and Mewborn (2001), however, noted that despite the extensive body of research on teachers' knowledge, few researchers have attempted to examine the knowledge that teachers use in the real-time practices of teaching. In response, we examine data from the classroom and subsequent interviews with Professor Gage,[1] a mathematician teaching differential equations using an inquiry-oriented curriculum. Our analysis focuses on the role of pedagogical content knowledge and specialized content knowledge (Ball, Thames, & Phelps, 2008) in Gage's (in)ability to scaffold mathematical learning while orchestrating class discussions. We hypothesize the roles of specific component practices of such scaffolding and consider specifically the knowledge needed by a teacher to follow students' mathematical reasoning and to recognize, in the moment, how it may be used to further the learning goals for the class.

THEORETICAL PERSPECTIVE

Teachers' Knowledge and Practices

The work we present in this article lies at the intersection of research on teachers' knowledge and research on teachers' instructional practices, specifically as those practices support productive classroom discourse surrounding mathematical ideas. We take a cognitive approach to our analysis—an approach shared by other researchers who have examined teachers' knowledge and the roles it plays in shaping teaching practices (e.g., Borko & Putnam, 1996; Calderhead, 1991, 1996; Escudero & Sánchez, 2007; Schoenfeld, 2000; Sherin, 2002b). In such an approach, knowledge is seen as one of several factors influencing teachers' goals and the ways they work to accomplish those goals as they plan for, reflect on, and enact instruction.

It is clear that mathematical content knowledge is essential for teachers to lead mathematical discussions, but because teachers' mathematical content knowledge in itself is not strongly linked to student achievement (Ball et al., 2001; Wilson, Floden, & Ferrini-Mundy, 2002), researchers have turned their attention to additional kinds of knowledge. Researchers have found that other types of knowledge play substantial roles in teachers' practices and the learning opportunities they create for students (e.g., Fennema et al., 1996; Fennema, Franke, Carpenter, & Carey, 1993; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004). Of particular note are the influences of pedagogical content knowledge (Shulman, 1986) and specialized content knowledge (Ball et al., 2008; Hill, Ball, & Schilling, 2008).

Pedagogical content knowledge (PCK) is the label used to describe what (among other things) teachers know about which mathematical topics typically cause

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[1] Names used for the professor and his students are pseudonyms.

students difficulty, the nature of the difficulties they have, and how particular examples or explanations can be useful in teaching particular mathematical concepts (Shulman, 1986). Since its identification, researchers have found that PCK plays important roles in teachers' practices and the learning opportunities that such practices create for students. For example, teachers' knowledge of the different strategies that their students use to approach problems is positively correlated with student achievement (Fennema et al., 1996).

Mathematical knowledge for teaching (MKT) has been proposed as a specific form of knowledge, not necessarily developed in ordinary mathematics courses, that permits teachers to follow and understand the ideas and solution strategies that students generate (Ball & Bass, 2000; Hill et al., 2004, 2005). More recently, Hill et al. (2008) have shifted their use of terminology by referring to such knowledge as *specialized content knowledge* (SCK), a uniquely pedagogical subset of subject matter knowledge. SCK is defined as "the mathematical knowledge that allows teachers to engage in particular *teaching* tasks" such as following students' mathematical thinking, evaluating the validity of student-generated strategies, and making sense of a range of student-generated solution paths (Hill et al., 2008, p. 377). Teachers' mathematical knowledge of this specific type has been shown to be positively related to student achievement gains in elementary mathematics (Hill, et al., 2005). Definitions for PCK and SCK in the literature, however, are not always precise, nor is the distinction between the two always clear. We describe in a later section how we chose to operationalize these terms.

Mathematicians' Teaching Practices and Knowledge

One especially difficult aspect of research on teachers' knowledge has been distinguishing between the effects of teachers' (possibly weak) knowledge of mathematical content and (possibly weak) command of other teaching-related types of knowledge (e.g., PCK or SCK). To illuminate the role of other types of knowledge while reducing the potential interference of weak common content knowledge, we conduct our research on the teaching practices of college mathematics teachers, primarily professional mathematicians with earned doctorates in mathematics. Hill et al. (2008) refer to the mathematical knowledge that is used in teaching that is also used by those in other professions as *common content knowledge* (CCK). In our work, we interpret CCK as referring to the formal mathematical knowledge that mathematicians with advanced degrees have developed through study and/or research.

Very little has been studied to date concerning the knowledge and teaching practices of this particular subpopulation of teachers (Smith, Speer, & Horvath, 2007), and even less concerning attempts to change the practices of college mathematics instructors toward reform-oriented teaching. In addition to contributing to the scant research on college mathematics teachers, we gain the added bonus of studying teachers whose mathematical content knowledge is very strong, thus allowing a clearer picture of what other types of knowledge teachers at all levels

may need in order to adopt new teaching practices aligned with contemporary reform curricula.

Challenges of Orchestrating Discussions

Successfully orchestrating mathematically productive classroom discussions requires the use of skills different from those typically used in or developed through teaching via lectures (Ball, 1993; Heaton, 2000; Nathan & Knuth, 2003; NCTM, 1989, 1991, 2000; Stein et al., 2008; Wagner et al., 2007; Williams & Baxter, 1996). Without such skills, various pitfalls could be encountered. In particular, researchers have described a tension between maintaining the process of mathematical discourse and directing the content of the mathematical outcomes (Sherin, 2002a). On the one hand, focusing primarily on content over process can result in teachers being highly mathematically directive, and thus failing to support the students' development of their own reasoning and justification skills (Cohen, 1990). On the other hand, excessive focus on process over content may result in classroom discussions that appear to promote the kind of discourse recommended by mathematics reformers, yet fail to result in substantive mathematical outcomes (Nathan & Knuth, 2003).

The response of the research community to this tension has been multifaceted. Some researchers have examined it through the lens of teachers' beliefs (e.g., Nathan & Knuth, 2003); others have considered teachers' "flexibility" in response to unexpected student contributions (Leikin & Dinur, 2007); and a substantial number have observed or constructed particular methods that teachers might use to lead discussions more productively (e.g., Rasmussen & Marrongelle, 2006; Sherin, 2002a; Stein et al., 2008). To date, however, we are unaware of much research on the role of teachers' knowledge in negotiating the instructional tensions described above. Leiken and Dinur (2007) attended to some issues of knowledge in their attempt to understand teachers' flexibility in their responses to students; however, they did not make teachers' knowledge a central focus of study. Rasmusssen and Marrongelle (2006) assumed that "pedagogical content tools" for developing students' mathematical ideas have a relationship to PCK and other types of knowledge, but their analysis attended primarily to the nature and use of the tools themselves, not to the nature of the knowledge that permitted them to be useful.

Social and Analytic Scaffolding

We find Williams and Baxter's (1996) constructs of *social scaffolding* and *analytic scaffolding* to be useful for analyzing the classroom tension that teachers may experience between maintaining a classroom of authentic mathematical discourse and ensuring that such discourse advances the mathematical objectives of the class. Social scaffolding refers to "the scaffolding of norms for social behavior and expectations regarding discourse" in the classroom; analytic scaffolding is the "scaffolding of mathematical ideas for students" (p. 24). In many

cases, it seems that teachers are more successful at using social scaffolding to encourage students to share their mathematical ideas than they are at using analytic scaffolding to shape the substance of the conversation. Anecdotally, Heaton (2000) described precisely this in her experience (without using the terms *social scaffolding* and *analytic scaffolding*), and Williams and Baxter (1996) and Nathan and Knuth (2003) provided classroom data demonstrating the lower quality of mathematical precision and substance that emerge when teachers emphasize social over analytic scaffolding.

We acknowledge that both forms of scaffolding are essential to mathematically productive classroom discourse, but because of the particular difficulty that analytic scaffolding seems to hold for teachers, we focus on it as our primary research interest. Specifically, we investigate the nature of the knowledge that teachers need so that they can provide successful analytic scaffolding, moment-by-moment, during whole-class discussions in their classroom. To do so, we hypothesize fine-grained component practices of analytic scaffolding and argue that successful use of them requires particular types of knowledge for teaching.

RESEARCH DESIGN

The broad goal of our analysis is to understand what aspects of teaching in inquiry-based ways college teachers of mathematics find challenging and why they find those aspects challenging. We focus our analysis on data from just one such teacher, Gage, one of several mathematicians whose teaching we have studied. We narrowed the scope of the analysis to the sources of some of the challenges that Gage faced specifically while orchestrating large-group discussions, and we focused it even more specifically on just the challenges he experienced related to analytic scaffolding. To carry out our analysis, we examined data from Gage's in-class instructional practices during select large-group discussions and from subsequent interviews tied to those specific classroom episodes.

Our Use of Analytic Scaffolding

The use of the construct *analytic scaffolding* in research on teaching is a relatively recent development. Although the term provides a more specific characterization of teachers' practices than was typically found in earlier work, it has not yet been refined to include detailed descriptions of specific instructional practices or work that teachers must do while providing this type of scaffolding.

As we conducted our analysis, we identified instances in which Gage was unable to provide analytic scaffolding; however, the specific difficulties he faced in the moment varied from one instance to another. We found it useful to deconstruct analytic scaffolding into a set of smaller practices as a way of identifying, at a finer grain size, the issues with which Gage struggled. Our development of this list of component practices of analytic scaffolding was a theoretical endeavor informed by our data.

We began by considering the desired outcome of analytic scaffolding and our particular interest in the core issue of how teachers make use of student contributions. We considered the objective of analytic scaffolding to be to guide students further toward the desired mathematical goal(s) by using selected student contributions. We then conducted a thought experiment, working backward from this end state to consider what teachers may need to do to be well equipped to provide the appropriate guidance. The result is a set of component practices that we hypothesize can lead to the desired objective of analytic scaffolding. We refer to the set as hypothetical because we have not conducted sufficiently detailed analyses to assert that these are the practices used by teachers who successfully provide analytic scaffolding. We do contend, however, that several of these component practices are precisely those with which Gage struggled.

To guide students toward the mathematical goal via discussion, a teacher must know the immediate mathematical goals for the particular discussion and the longer term goals for the course. The teacher must also be able to elicit contributions from the students as fodder for the discussion. Armed with mathematical goals and student contributions, much work still remains to be done to utilize those contributions in ways that create opportunities for students to learn the desired mathematical ideas. We hypothesize that teachers can move toward the objectives for analytic scaffolding by doing the following:

1. Recognize or figure out students' (correct or incorrect) mathematical reasoning;
2. Recognize or figure out if/how students' ideas (both correct and incorrect) have the potential to contribute to the mathematical goals of the discussion;
3. Recognize or figure out if/how students' ideas (both correct and incorrect) are relevant to the development of students' understanding of the mathematics; and
4. Prudently select which contributions to pursue from among all those available.

We do not claim that these exhaust the list of preconditions for guiding students further toward the desired mathematical goal(s). However, we assert that this set of practices can support the use of analytic scaffolding and that different kinds of knowledge may be needed to enact these component practices effectively.

We use *recognize* in our description of the component practices to denote situations in which teachers are already familiar with the ways that students think about and come to understand the mathematics. In other words, their existing PCK may include knowledge of how students think about the specific ideas at hand and/or typical students' difficulties with the topic. If that is the case, then to "recognize . . . students' mathematical reasoning" the teacher can match what a student contributes to that known information about student reasoning. At other times, even if teachers are not familiar with the particular ways of reasoning that students offer, they may be able to "figure out" what the students are suggesting

and thinking. Therefore, recognizing draws heavily on a teacher's PCK, whereas figuring out requires that a teacher do some mathematical work in the moment. Drawing on their SCK, PCK, and CCK may come into play in similar ways for the remaining practices as well.

In the following analyses, we focus only on the first three of these component practices. In particular, we examine specific classroom episodes in which whole-class discussions somehow went mathematically awry and failed to achieve the desired mathematical goals (in our judgment and/or in the instructor's judgment). This seems to have occurred because of the instructor's inability[2] to recognize or figure out either the students' reasoning (analytic scaffolding component practice 1), the mathematical potential within their contributions (analytic scaffolding component practice 2), or the relevance of students' contributions to the development of their mathematical understanding (analytic scaffolding component practice 3). We focus on these three practices because when we examined the classroom discussions that failed to reach a satisfactory conclusion, the challenges that Gage faced were primarily connected to these first three practices. In the episodes we analyzed, the challenges he faced while enacting the first three practices meant that he did not reach the point of being able to enact the rest of the process to achieve the desired goal of the discussion.

We use these episodes to examine the kinds of knowledge that are needed to support these essential components of analytic scaffolding. We believe that this research contributes to our understanding of the knowledge that teachers use in real-time classroom practice, at a finer grain size than many other analyses of teachers' knowledge, and in support of analytic scaffolding that has been shown to entail notable challenges for teachers.

The Setting

Data collection occurred at a university in the Midwest. Gage was teaching an undergraduate course in differential equations using the Inquiry-Oriented Differential Equations (IO-DE) materials developed by Rasmussen (2002). Gage had 17 years of university teaching experience at the time and had taught Differential Equations[3] in the past using a traditional text and more traditional instructional methods, without making use of small-group work or much whole-class discussion.

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[2] We acknowledge, as other researchers have (e.g., Simon & Tzur, 1999), a concern for the limitations of deficit studies that focus primarily on what teachers cannot do rather than on the knowledge they possess and on which they can build. To the degree that our ultimate concerns are for teacher development and growth, however, supporting teachers in building the knowledge they need requires an understanding of the direction in which development needs to proceed. By studying teachers with strong mathematical content knowledge, we are better able to detect directions for growth in other areas of knowledge that may otherwise be overlooked. Our point is not simply to demonstrate that Gage lacks something, but to characterize the nature of what he may need to know in addition to his strong content knowledge and his evident skills in social scaffolding.

[3] We use the words Differential Equations to refer to a mathematics course and the words differential equations to refer to the equations in such a course.

Of the 19 students in the class, most were pursuing majors or minors in mathematics and/or biology, chemistry, or physics.

The IO-DE curriculum consists of student and instructor materials developed from research on student learning and includes several series of problems, activities, and accompanying Java applets designed to guide students through discovery of the core concepts of a dynamical systems approach to Differential Equations. Students are expected to work collaboratively in small groups on problems and to participate in whole-class discussions through which they acquire graphical, numerical, and analytic techniques for analyzing, interpreting, and solving differential equations. Problems and activities were designed to challenge and encourage ways of thinking about the mathematics and to lead students to discover important ideas. Although the top-level content areas of a traditional course in Differential Equations are readily visible in the course materials, the particular "path" through which students encounter them differs from that of most traditional texts.

Data Collection Methods

During the semester, video and audio data were collected during almost all meetings of Gage's class. One camera positioned at the back of the room captured video and audio of the whole-class discussions. Students were seated in groups of four around tables; during small-group work, that camera was directed toward one of the student tables. A second camera was positioned at another student table during both group work and whole-class discussions. In addition, two microphones were placed on student tables to capture whole-class conversations as well as student–student conversations. For the purposes of the current analysis, we relied primarily on the video recordings of the whole-class discussions and the audio recordings of interviews with Gage.

After almost all these classes, interviews were conducted with Gage by one of the authors, a mathematics education researcher who attended each of these classes. All interviews during the semester were conducted by the same author–researcher and audio-recorded. The purpose of the interviews was to capture Gage's recollections of the class and the thoughts that informed the decisions he made during the class. During these interviews, Gage typically, and without being prompted, focused his comments on the aspects of the class with which he was least satisfied and which were sources of the most frustration. As a result, the interview data often included Gage's impressions of how particular classes or portions of classes fell short of his goals for the lesson and his recollections of his thoughts in the moment that shaped the particular instructional decisions he made.

At times, he requested feedback and/or advice, and sometimes the discussions included collaborative planning with the author–researcher for the next class. In contrast with other kinds of interviews (e.g., clinical interviews) for which the objective is to refrain from influencing the interviewee, these discussions likely shaped some of Gage's decisions about his teaching. The discussions during which Gage sought assistance, rather than being problematic for analysis, became rich sources of data on the kinds of knowledge he lacked.

Data Selection and Analysis

The methods used for gathering data generated a substantial quantity of video and audio recordings (approximately 30 hours of classroom video and 18 hours of recorded interviews). Because Gage's efforts to orchestrate whole-class discussions varied in their degree of success, we reviewed a large number of class and interview recordings and noted the discussions that seemed especially challenging for Gage (from his and/or the researchers' perspectives). In particular, we selected a class for analysis if (a) during interviews Gage expressed relatively high levels of frustration and/or described the class as especially challenging, or (b) the classroom video data suggested that the large-group discussions were problematic in some way (e.g., the main mathematical point did not surface clearly, or the discussion occupied significantly more class time than was recommended in the curriculum). We chose to focus our analysis on four classes that contained discussions in which Gage's challenges appeared to stem from the absence of effective analytic scaffolding. Episodes of whole-class discussions from two of the four classes that we analyzed are presented in this article. These episodes are by no means representative of the full set of Gage's practices and are not meant to be illustrative of his teaching as a whole.

For each episode of whole-class discussion that occurred during the selected classes, we conducted a five-part analysis. The different aspects of the analysis were conducted in a cyclic fashion, moving back and forth between the class video data and the interview data, but for rhetorical simplicity, we describe them separately here. First, we did a fine-grained analysis of the video transcript (in the spirit of Schoenfeld, 2000, 2008). The purpose of this phase of the analysis was to generate a detailed chronology of the episode that included the mathematical ideas students contributed to the discussion, student questions and/or comments, Gage's questions and/or requests for student contributions, and Gage's comments and/or evaluations of student contributions. The product from this portion of the analysis was a detailed description of the (lack of) progress toward the mathematical goals for the problem at hand from the researchers' point of view (informed by our knowledge of the curriculum and content).

Second, we used the interview transcript data to examine what occurred from Gage's point of view during the selected class discussions. The goals for this phase of analysis were to capture Gage's description of what had occurred during the episode, his recollection of what he was thinking during the episode, and his opinions about how his and the students' specific contributions had (or had not) helped to move the discussion toward the mathematical goal. Constructing the description was relatively straightforward because Gage's habit in the interviews was to revisit what had occurred and critique how he handled various parts of the discussion, thus giving us a narrative of the episode from his perspective along with his thinking and decision-making.

The third aspect of the analysis also made use of the interview transcripts, but at this point the goal was to identify instances in which knowledge (i.e., CCK, SCK,

and PCK) appeared to play a role in Gage's thinking and practices. We took a grounded-theory approach (Strauss & Corbin, 1990) to this phase of the analysis. The process began with each author coding the same transcript independently. We examined each line of transcript, looking for instances in which Gage explicitly stated something he knew (or wished he knew) about the mathematics (CCK) or about the teaching/learning of the specific mathematical ideas (PCK). We looked for instances in which Gage described thinking about and/or figuring out something about the mathematics the students were discussing. We coded these as instances that involved the use of his SCK. We also looked for instances in which, based on our own knowledge of the content and curriculum, we believed there were implicit issues related to PCK, CCK, or SCK. For example, there were instances in which Gage discussed a particular student's contribution to the conversation in ways that suggested he did not recognize it as a typical way that students think about the ideas. We coded such instances as related to (an absence of) PCK.

We then compared our two codings and, through competitive argumentation (VanLehn, Brown, & Greeno, 1982), finalized our coding for that first transcript and refined our criteria for assigning the codes. This approach to coding then became the basis for analysis of other transcript data. In the next section, in conjunction with the presentation of our analyses, we provide additional details to illustrate how we distinguished among types of knowledge.

In the fourth and fifth aspects of the analysis, we focused on alternative ways that the discussion could have proceeded and the knowledge that might have supported those alternate possibilities. To begin this part of the analysis, we examined recurring cycles during which Gage asked a question or made some other request for contributions to the discussions, one or more students made a contribution, and then Gage responded. We characterized the student contributions as being either potentially useful for moving the discussion in a mathematically productive direction (e.g., Gage could potentially use the contribution as the basis for some analytic scaffolding) or not useful (e.g., the ideas offered appeared unlikely to contribute or relate to the mathematical goals for the discussion). We based these characterizations on our knowledge of the content and curriculum, testing our conjectures about whether a contribution was potentially productive by trying to generate lines of mathematical reasoning and teacher actions (i.e., questions to be asked, points to be emphasized) that could connect the student's contribution to the desired mathematical outcome for the discussion. This aspect of the analysis produced a set of hypothetical discussion trajectories that represented plausible alternatives to what Gage did in the episodes that could have provided more analytic scaffolding for the discussion. We illustrate this aspect of the analysis in the later sections in which we present our data.

The final part of the analysis centered on the knowledge that could enable a teacher to see the specific student contributions as potentially productive. There were two steps to this part of the process. For each episode, we first generated a list of the knowledge (CCK, SCK, and PCK) that someone could use to recognize or figure out that a particular student contribution was a potentially productive

contribution to the discussion. This was primarily a thought experiment, based on our earlier analyses of the student contributions and the hypothetical discussion trajectories we had developed. We then relied on a process-of-elimination approach to determine what specific knowledge Gage might have lacked that might have contributed to the difficulties he had in recognizing contributions as potentially productive. We examined the transcript data from Gage's discussion of the episode, looking for evidence that Gage did possess the specific aspects of knowledge (e.g., particular elements of CCK, SCK, PCK). For example, if recognizing a particular contribution as potentially productive relied on knowing something specific about student thinking, we looked for instances in which Gage spoke about that type of student thinking. To focus on the source of Gage's difficulties, we eliminated from consideration the aspects of knowledge that he appeared to have at his command. The aspects of knowledge that remained were ones that Gage did not appear to be using at that particular moment. Thus, we believed this knowledge was plausibly linked to the difficulties he had in enacting the specific component practices of analytic scaffolding that were part of our possible alternate responses to the student contributions. We also looked for instances in which Gage stated explicitly that he lacked some knowledge and/or in which he said he wished he knew more about something. We took the fact that Gage repeatedly mentioned particular issues during his interviews (e.g., absence of knowledge of student thinking for particular topics) as evidence that those issues were major influences on his decisions and practices.

In the end, multiple sets of classroom and interview data corresponded to each of our claims. In this article, we illustrate our claims with examples that we believe best illustrate our findings with the least amount of background information. That is, we selected classroom episodes for which it would be relatively simple to describe the mathematical ideas at stake and the discussion that preceded those episodes.

CASE 1: WHEN STUDENTS CONTRIBUTE GOOD IDEAS

In the first portion of our analysis, we consider two episodes from one of Gage's classes early in the semester. In the first episode, Gage interrupted and redirected a conversation between several students who were wrestling with a mathematical problem, explicitly suggesting that the issues they raised were not relevant at the time. We use this episode to examine how some forms of PCK might have enabled Gage to see the students' discussion as, in fact, immediately relevant, even essential, to the development of students' understanding of the mathematical ideas in the problem. Had he recognized some students' contributions as potentially productive to the discussion, Gage might have been able to use those contributions to direct the conversation toward the mathematical goal of the lesson.

In the second episode, we analyze a subsequent exchange that reveals how certain elements of PCK and SCK may be needed for an instructor to direct a stalled discussion in a productive direction. Gage found himself poorly prepared to move a

discussion forward because he had expected the students to handle the issue easily, but instead, they struggled. In the discussion that followed, even though Gage had listened to a student's contribution well enough to summarize it for the class, he was nevertheless unable to recognize or figure out the value of the contribution for illuminating the exact mathematical point that he had attempted to make just minutes earlier—a point that Gage himself later described as passing unacknowledged by the class. Each of these episodes demonstrates how the absence of certain types of knowledge may make it difficult for an instructor to provide analytic scaffolding, even during a class discussion in which students put forward sensible and useful mathematical ideas.

Episode 1

Context. In the first episode, the students had been asked to model simple population growth of a continuously reproducing species of fish in a lake with unlimited resources. The students worked in groups, proposing and discussing population versus time graphs for different initial population values, and then shared and discussed their results with the whole class. There appeared to be agreement that the curves should suggest increasing (perhaps exponential) growth over time. After comparing several solution curves, the students also seemed to agree that the rate of change of the population at any time should depend only on the size of the population at that time. Thus, all solution curves had the same slope for any given value on the vertical (population) axis, and so every solution curve could be obtained by horizontally shifting any other solution curve. Immediately thereafter, Gage presented them with the following written problem:

> This situation can also be modeled with a rate of change equation, $dP/dt =$ *something*. What should the "something" be? Should the rate of change be stated in terms of just P, just t, or both P and t? Make a conjecture about the right-hand side of the rate of change equation and provide reasons for your conjecture.

The activity was designed to lead students to recognize that $dP/dt = kP$ was a reasonable model, with P representing the fish population at time t, and k being a constant of proportionality.

Narrative and transcript. Based on their analysis of the graphs, the students argued correctly that the rate of change of the population (dP/dt) was determined only by P, the size of the population. They did not, however, readily see the implications of that argument for the construction of their differential equation. To the contrary, one student argued, "Anything that's a function of P, assuming P is a function of t, can be written as a function of t," implying that dP/dt could be expressed in terms of t rather than P. Soon after, another student entered the debate, introducing into the discussion a role for the initial population:

Joy: If *P* was represented as a function of *t*, then wouldn't it change with a different initial population? Because, if we were saying that the rate of change is dependent only on the population, then that would be shifting the graph back and forth.

Immediately thereafter, three additional students joined the conversation, each discussing the relationship of the "initial population" to the problem at hand. (In the interest of space, we omit the transcript of that discussion here.) After allowing the conversation to continue for several minutes, Gage interrupted the students. By suggesting that the content of their discussion was not relevant at the moment, he opened the conversation to other ideas:

Gage: Can, can I ask, at least, I mean, we are trying to make a decision kind of, you know, what should be right here in this equation, and right now, I think we're kind of a little bit further down the road, you know. We're kind of, *P* of *t*, and initial populations, and, and, I'm not saying it's—. I'd like to maybe hear a couple other people's comments on it. Matt?

Even though the conversation moved forward at this point, students continued to make reference to the "initial population," but its role did not become a focused point of discussion.

Interpretation and analysis. Experience with this curriculum has demonstrated that understanding the direct dependence of differential equation on *P* but not on *t* is a conceptual challenge for students to overcome (Rasmussen & Marrongelle, 2006). It seems sensible that if dP/dt can be written in terms of *P*, and *P* can be written as a function of *t*, then dP/dt ought to be expressible in terms of *t*. Making sense of why this is not the case is a primary learning goal of the activity.[4]

Joy correctly noted the class members' agreement that "the rate of change is dependent only on the population," but the full point of her contribution is not clear. One plausible interpretation of her words is that if *P* is expressed in terms of *t*, then different initial populations would result in different expressions of $P(t)$ shifted vertically on a graph. The class, however, had already noted from their analysis of the graphs that potential expressions for $P(t)$ were all horizontal shifts of each other. Whatever Joy's precise point, she and the students immediately following her all wrestled with the role of an initial population in the differential equation and its solutions. This, in fact, is the central conceptual challenge to students in this

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[4] The distinctions among $dP/dt = f(P)$, $dP/dt = f(P, t)$, and $dP/dt = f(t)$ are subtle, but essential. Saying that dP/dt should be expressed only in terms of *P* ($dP/dt = f(P)$) means that the rate of change of a population at any time is determined solely by the size of the population. This is the case in the situation under consideration, and a standard assumption for simple models of population growth. Including *t* in the differential equation along with *P* ($dP/dt = f(P, t)$) would suggest that the same population size, for example, might result in different rates of population growth on different days of the week. This would make no sense, given the stated ideal conditions (e.g., unlimited resources). Finally, expressing dP/dt solely in terms of *t* ($dP/dt = f(t)$) would indicate that the rate of change of the population is independent of the size of the population, which is not a reasonable assumption for any living species.

activity: distinguishing between the rate of change of a single population or solution curve determined by an initial population and the differential equation governing the family or system of population curves regardless of the initial population. In other words, realizing that an initial condition is irrelevant to the question posed (i.e., that the differential equation must hold for all possible initial conditions simultaneously) is a challenging learning objective for the students as they work through this activity.

   The students' belief that the initial condition was relevant indicated their need to work through some subtle and important mathematical distinctions. However, believing that the ongoing discussion was not contributing to the mathematical progress of the class, Gage interrupted the discussion and elicited other ideas in an effort to redirect the conversation. Of particular note, Gage believed that the students' discussion about "*P* of *t* and initial populations" was getting "a little bit further down the road." We interpret this to mean that Gage had followed the students' discussion sufficiently closely and that he understood the mathematical arguments students were making (analytic scaffolding component practice 1). However, his CCK led him to realize that knowledge of the initial conditions was needed only to determine a *particular solution* to the differential equation, a problem taken up only after the differential equation was solved. The goal of the activity, however, was to determine the nature of the differential equation, and not its solutions. As a result, Gage judged that because the initial conditions were not mathematically relevant to the solution of the problem, the students' emerging ideas would not contribute to the mathematical goals of the activity (component practice 2). Thus, he tried to redirect the conversation away from the matter. He did not see, however, that the conversation he interrupted occurred because the students did not understand that their concerns were "further down the road." Because he did not have the PCK to enable him to recognize that the issues were relevant to the development of students' understanding (component practice 3), he actually moved the focus away from the mathematical ideas that were most relevant to the students' learning at the moment. Gage expressed his frustration with the students' concerns for initial conditions during an interview later that day:

Gage: [T]his whole discussion of initial condition, a specific *P* of *t*. I'm not able to turn this into a distinction between some kind of a system, description of a system, and solutions to the system. Those ideas. I'm not able to, to make that happen.

   Gage's decision to interrupt and redirect the students' conversation thus stemmed from his desire for them to pay attention to the behavior of the system rather than the behavior of individual solutions. From his perspective, since knowledge of the initial conditions was unnecessary to solve the problem, steering them away from such considerations was intended to direct them toward what he knew to be the better path. We believe that PCK related to the students' struggles at this point would have provided Gage with other possible responses that involved directly pursuing the students' concerns with initial conditions. The fact that a significant

number of students referred to the role of initial conditions suggested that they needed to clarify the matter for themselves.

A premise of inquiry-oriented instruction is that, whenever possible, instructors should work with the students' ideas, moving the students from "where they are" toward the instructional goals. In this instance, from a purely mathematical perspective, the students' attention to initial conditions was unhelpful for solving the problem, so Gage steered them away from it. Nevertheless, learning the distinction between "a system and solutions to the system," as Gage desired, necessarily involves learning that initial conditions are relevant to describing individual solutions and not the system itself. It might be suggested that Gage could have benefited from the availability of different pedagogical tools or strategies with which to respond to the students' ideas. We believe this to be unlikely. All the evidence indicates that, in Gage's judgment, the content of the students' suggestions was misdirected, supporting our conclusion that he was limited by his need for specific PCK. Thus, Gage halted a potentially productive conversation that appropriate analytic scaffolding might have fostered.

Episode 2

Context. About 20 minutes after the first episode, the students had narrowed their choices for models of the fish population growth to two: $dP/dt = P$ and $dP/dt = e^t$. Even though the class members had, by this time, ostensibly agreed that dP/dt should be a function of P only, they were not yet in agreement that the second model should be eliminated. Students commonly suggest the latter model, in part because they typically have prior experience with exponential population models and also because of the conceptual challenge in identifying the difference between the two models. Because $P(t) = e^t$ is a solution to both differential equations, understanding how the two models differ and why the first (but not the second) is a reasonable model for population growth can be difficult.

Narrative and transcript. Gage offered a suggestion intended to help students recognize that the two models were not "the same." He asked them to consider another function that satisfied one of the models, but not the other:

Gage: If you take e^t, then if you differentiate it, you get it back. Some—. So are they the same or not? [14 seconds silence] How about, uh, you know, something so, like 2 times e^t? [8 seconds silence] $P(t)$ is 2 times e^t?

By suggesting that the students consider $P(t) = 2e^t$, Gage hoped that they would notice that $P(t) = 2e^t$ satisfied only the first of the two differential equations, thereby helping them to begin to distinguish between the two models. His suggestion, however, was met with lengthy silence.

The silence met by Gage's suggestion to consider $2e^t$ was finally broken when a student asked for clarification of what was meant by the suggestion that the two models were "the same." In response, Rob observed that even though one could

"just substitute between P and e^t," he believed that to be "a special circumstance … where that happens to work." Another student, Matt, asked Rob if he could come up with an example that did not work the same way. Rob's initial suggestion was found not to be helpful, and Gage redirected the conversation.

Sue: I really don't understand . . . what they mean by "the same."[5]
Gage: OK. Can somebody may—, uh, phrase what may be meant by "these are the same"? Rob?
Rob: I think what was said is, OK, let's say you say $P(t)$ is where P is equal to e^t. Then if you take the derivative you'll get dP/dt is equal to e^t. But I think that's, I, while I can't deny the truth of that because you can just, by going back to the original equation, you can just substitute between P and e^t, you can derive the equation that says P is equal to e^t. I think that's a specific circumstance, that, you know, where that happens to work.
Gage: Uh, sorry, where what happens to work?
Rob: That P is eq—, P is the same as e^t.
Matt: Can you come up with one where it doesn't work?
Gage: Yeah, can you come up with something where it wouldn't?
Rob: Well, let's say your, let's say your $P(t)$ was $P + t$.

[Transcript of this portion omitted]

Gage: OK, so Rob is saying that, you know, this kind of feels like some particular instance where something is happening, but we can't right now come up with something, ah, that kind of supports, supports that. Melanie?

   Interpretation and analysis. Rob's contribution was directly related to the point that Gage was trying to make by asking students to consider $P(t) = 2e^t$, and because of this, it held the potential to move the discussion forward. $P(t) = 2e^t$ satisfies the first differential equation, but not the second, thereby presenting itself as the example requested when Matt and Gage asked Rob if he could "come up with one where it doesn't work." It does "happen to work," however, that $P(t) = 2e^t$ satisfies both $dP/dt = P$ and $dP/dt = 2e^t$, thereby paralleling the situation under consideration. Because of this connection to the issue with which students were struggling, with proper analytic scaffolding, Rob's suggestion could have been used in a number of ways to steer the conversation in the direction that Gage had hoped. For example, if Gage had asked Rob to explain specifically "what happens to work," Rob's suggestion could have been highlighted and clarified for the whole class, and the class could have investigated whether other functions might "work" in the same way. If the class needed further impetus, Gage might have taken the opportunity to reintroduce $P(t) = 2e^t$ and ask the class directly if it worked in the same way that $P(t) = e^t$ did. This might have allowed the class to distinguish between the two differential equations under consideration and see that $P(t) = 2e^t$ satisfies only one

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   [5] Throughout the transcript excerpts, an ellipsis (". . .") refers to a brief pause of less than a second or two. Lengthier pauses are represented by bracketed ellipses ("[. . .]"). A long dash at the end of a word indicates an abrupt stop or interruption of a sentence, usually preceding a change of phrasing. No omissions to the transcripts have been made except where explicitly indicated.

of them. The point of this thought experiment is not to say what *would* have happened if Gage had acted differently, but merely to exemplify the good use to which Rob's suggestion might have been put.

Instead of asking Rob to clarify what "happens to work," Gage followed up on Matt's suggestion and asked Rob for an example "where it doesn't work." Unfortunately, Rob made the unhelpful suggestion, "say your $P(t)$ was $P + t$," resulting in, as one student put it, "a recursive definition for P." After clarifying the trouble with Rob's suggestion (transcript omitted here), Gage summarized what Rob had said and called on another student, thereby putting closure on the discussion of Rob's contribution and encouraging the conversation to move in a new direction.

Although Gage asked Rob for clarification, his apologetic tone suggested that he asked the question because he did not understand Rob's mathematical point. His subsequent summary of Rob's contribution included only the most surface-level details of the contribution ("this kind of feels like some particular instance where something is happening"), suggesting that even though he was attending to Rob's words, he was unable to figure out that Rob's observation about the "something" that "happened to work" was closely related to the point Gage had attempted to make by suggesting $P(t) = 2e^t$ just minutes earlier. In the postclass interview, in fact, Gage expressed frustration that his suggestion of $P(t) = 2e^t$ was not appreciated by the class, saying, "Nobody reacted to it. You know . . . OK, it doesn't make any sense to them. They didn't know what to do with that." Certainly, Gage might have been more proactive in encouraging the class to consider the significance of his suggestion when he first proposed it, but Sue's question served to derail Gage from the direction in which he was trying to move. The subsequent opportunity to reconsider $P(t) = 2e^t$ in light of Rob's observations was lost, not because of any evident lack of useful pedagogical tools or strategies on Gage's part, but because Gage did not recognize the opportunity when it presented itself.

What contributed to Gage's inability to recognize the potential in Rob's observation? First, we eliminate some obvious possibilities. It is highly unlikely that Gage was simply distracted or inattentive to Rob. He asked Rob to clarify his meaning, he repeated to Rob a pointed question posed by another student, and he ended the discussion with a summary of Rob's words. His summary, however, was essentially devoid of any mathematical content, suggesting that he may not have understood the ideas that Rob offered. It is not the case, however, that Gage did not know the mathematics needed to understand Rob's comment, since it was directly related to the same point that Gage attempted to make just a few minutes earlier. It is also not the case that Gage lacked the discussion management or pedagogical skills needed to elicit students' ideas. He asked several questions in an attempt to clarify Rob's point, and he engaged in significant mathematical discussions with students at other times in the course. We conclude that Gage did pay attention to Rob, but he was unable to understand Rob's point, and/or whatever he did understand, he did not consider sufficiently valuable to pursue.

Having eliminated other potential knowledge-related sources, we contend that there is evidence that Gage's failure to recognize the potential in Rob's contribution was rooted in an absence of some SCK relevant to the mathematical discussion at hand. To follow students' mathematical reasoning in real time requires the ability to recognize mathematical ideas in and/or infer those ideas from students' vaguely expressed or partially formed ideas (component practice 1). Postclass interviews revealed that Gage found following students' reasoning quite difficult at times—a situation that was exacerbated by the variety of instructional tasks for which he was responsible at any time:

Gage: I need to do too many things. I need to try to follow the train of thought carefully, and I need to try to figure out when is it a good place to do something. I have to look for certain clues that I am sensing are good ones to do something, and I— which is kind of a detached observation kind of thing. And the other is to be part of the thought process, to really follow it along, to try to direct it a little. There's a conflict there. I can't do both.

On the one hand, Gage needed to attend to the direction of the entire conversation of the class, deciding when would be "a good place to do something." On the other hand, following students' thinking carefully demanded that he "be part of the thought process," reflecting an essential component of analytical scaffolding. In combination, these roles of being simultaneously "detached" and "a part of the thought process" left him feeling conflicted, as well as overwhelmed by his "need to do too many things."[6] It is quite likely that Gage's knowledge of the mathematical domain and additional time to reflect on Rob's comments would have revealed to him their potential, but it is SCK that supports the recognition of valuable student contributions without having to do so much mathematical "work" in the moment while so many other issues are simultaneously demanding attention.

What specific SCK might have helped Gage under the circumstances? It is difficult to identify such knowledge by its absence, but here we analyze the mathematical work that a teacher would have needed to carry out to recognize the students' contributions as relevant and useful for moving the conversation toward the mathematical goals of the activity. Even though Gage likely possessed all the formal mathematical knowledge of the material under discussion, "unpacking" (Ball & Bass, 2000) such knowledge into underlying conceptual challenges related to students' informal and intuitive ideas was needed. Making sense of Rob's contribution required a series of inferences. Gage needed to infer meaning behind Rob's imprecise language: "that's a specific circumstance . . . where that happens to

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[6] In addition to the roles of knowledge examined here, Gage also held certain beliefs about mathematics, teaching, and learning that influenced his instructional practices. Although we do not present such an analysis here, Gage had beliefs about his role in large-group discussions that shaped decisions he made about when and how to direct the conversation. Those decisions were influenced by the PCK and SCK he had available, and so we chose to focus this analysis only on knowledge-related issues. This should not be taken to suggest that there are not other factors influencing the nature of the instruction Gage provided.

work." (Gage attempted to question Rob on just that point.) Then, even without necessarily being clear about what Rob perceived, Gage would have needed to recognize that $P(t) = 2e^t$ also "happens to work," because, in Rob's words, "just, by going back to the original equation [$dP/dt = P$], you can just substitute between P and e^t [$2e^t$], you can derive the equation that says P is equal to e^t [$2e^t$]." Finally, he would then have needed to judge how such a connection, once made with Rob and the rest of the class, might be used further to assist students in understanding the difference between the two differential equations at hand, as well as the reason that one rather than the other better modeled the population situation in question.

All of this required that Gage be able to take a student's unfamiliar and perhaps surprising way of perceiving the mathematical situation, understand the student's perspective, and simultaneously "translate" it into a more mathematically normative and useful idea. This analysis of the component practices underlying analytic scaffolding highlights the significant amount of real-time mathematical work required of Gage while feeling overwhelmed by his classroom roles.

Episodes 1 and 2 both reveal how it is possible for an instructor to be challenged in providing analytic scaffolding in class discussions, even when students raise good mathematical questions and propose sensible mathematical contributions. Frequently, students' questions and contributions do not emerge in clear, precise mathematical language. It takes a skillful teacher to recognize and reveal the good mathematics that lies beneath even students' best attempts at articulating their thoughts. This analysis demonstrates that even one skilled in the mathematical domain being taught may require particular forms of SCK to provide analytical scaffolding in the demanding atmosphere of inquiry-oriented classroom discussions.

CASE 2: WHEN STUDENTS POSE NOT-SO-GOOD IDEAS

In the previous episodes, we saw how challenging it can be for an instructor to follow students' thinking and recognize how students' mathematically correct suggestions can contribute to the goals for the activity. It is also possible, however, that mathematically *incorrect* contributions can be used by an instructor to move a discussion in a mathematically productive direction. In the following episode, we provide examples from Gage's classes of the challenges that teachers may encounter in their efforts to provide analytic scaffolding when students' contributions are not mathematically correct.

The episode comes in two parts. In the first part, a student offered a suggestion that was not productive for the problem under consideration. Gage was unable, in the moment, to follow the student's reasoning, so he could not determine whether there were learning-related reasons for discussing the student's idea. He reacted by making a move to keep the conversation going, but students continued in an unproductive direction until Gage politely cut off the discussion without resolution. In the second part, several students offered mathematically incorrect, but potentially pedagogically useful, suggestions. Gage recognized the suggestions as incorrect, but he did not see the mathematically productive ideas contained in them. This time,

Gage focused on the incorrect aspects of the students' ideas and directed the discussion away from the mathematical goals for the problem. In both cases, we propose that Gage lacked the SCK to follow the students' ideas and the PCK to envision how their ideas, even though they were incorrect, could have been used productively. Had he possessed such knowledge, he might have been better prepared to provide the analytic scaffolding that would have better served the mathematical goals of the class.

Episodes 3A and 3B

   Context. The class had been working on solution strategies for various forms of differential equations. They had solved problems by separation of variables and discussed that approach. The students then began a problem for which the solution could not be obtained using the technique of separation of variables. The problem was designed so students would have the opportunity to develop the "integrating factor" technique or, as it was called in this class, the "reverse product rule" technique for solving particular kinds of differential equations. In traditional curricula, this is often presented to students in ways that make both the origin of the approach and the reasons why it works quite mysterious. The goal here was for students to develop the technique themselves in a way that highlighted its connection to things they already knew about techniques of differentiation. In particular, the objective was for students to recognize in the differential equation an expression that resulted from differentiation using the product rule, and to develop an approach that would enable them to integrate by "reversing" the product rule. This would permit an explanation and use of integrating factors that was directly connected to students' prior calculus knowledge.

   The students' previous work on a problem led them to develop Equation A, which Gage had rewritten as Equation B, intended to suggest the product rule:

$$\text{A: } \frac{dP}{dt} = 2 - \frac{P}{10+t} \qquad\qquad \text{B: } (10+t)\frac{dP}{dt} + P = 2(10+t)$$

   The goal was for students to identify the left side of B as the derivative with respect to t of $(10 + t)P$.

   Narrative and transcript, 3A. Students discussed the equation for a few minutes at their tables, and then Gage called the class back together. He asked one student to explain what he was thinking. The student said that the approach his group had tried did not work. Gage commented briefly on the utility of pursuing things that do not seem to work and then reiterated that the task at hand was to try to integrate. Another student, Matt, entered the discussion and provided an abstract description of his approach. His idea involved rearranging the terms in the equation so one could make a substitution that would simplify the equation, resulting in an equation

that would be solvable using techniques with which they were already familiar:

Gage: But somehow we're trying to integrate, right? Everybody used that word all the time earlier, that we're just trying to integrate. Well, of course, it's not that easy. We have all seen, so far, you can't just integrate.

Matt: Ah, I get it!

Gage: Matt, what did your "Ah" . . . ? Just, not the whole thing, just what, what thing hit you?

Matt: Well, if you can fiddle with two parts and make them equal, we can remove some stuff. Like if you get P of t equal to some thing, you also know P of t equals the integral of dP/dt, so you could get P of t equal to something, and then that something equals the integral of dP/dt, which you would, and then you could fiddle, slide the parts around, OK, so they cancel out, maybe?

Matt's description was somewhat vague and did not include specific steps to take with the equation. During the next part of the exchange, Gage said he was not able to follow the suggestion and asked to "hear one more" from the class. Another student volunteered, but instead of offering a different idea, he offered to try to explain what Matt had proposed:

Gage: […] I can't quite get my hands on, on your thought. Um, I think . . . the problem is that you, of course, don't know P-prime of t. But, ah, maybe, I would, I would like to, let's maybe, let's hear one more, Craig?

Craig: Well, I was just gonna, gonna try to explain what I thought Matt was saying . . .

Gage: OK.

After some discussion between Gage and Craig, the particular approach was still unclear to Gage. We omit most of the transcript of this exchange for reasons of space, but we note that Gage expressed his inability to figure out the approach in the midst of the discussion by saying things such as, "I couldn't quite see it." At this point, Matt's idea had not been significantly clarified, and Gage suggested that the students pursue it further outside of class:

Gage: You may want to pursue this further, OK? So that maybe it becomes a little clearer. I can't— . . . Can I try to point out something else? Maybe we'll try to do something else. But, I'm not rejecting this. [Points to the expression that Craig had suggested] I'm just trying to kind of redirect, because there's something that—. I'm going to bring that over to the side so I don't have to erase it. But please, can you [looking at Craig and Matt, implying they should pursue this on their own] . . . outside . . . then you . . . pursue this a little further and see, and see what happens.

Interpretation and analysis, 3A. Recall that the goal of this part of the activity was for students to notice that the equation with which they were working looked similar to what one might obtain from using the product rule to find a derivative. The solution strategy that Matt had in mind did not appear to make use of

recognizing the equation as being in such a form. What he described is perhaps based on some idea he had connected to recursive forms of equations and/or some other techniques he had seen in the past for simplifying equations. In any case, what Matt said did not contribute anything new to the discussion that appeared to be related to the form of the equation and the product rule.

Gage said that he did not follow the idea Matt described ("I can't quite get my hands on, on your thought") and stated one potential problem with Matt's proposed solution ("you, of course, don't know P-prime of t"). Gage kept the discussion moving (asking for an additional idea by saying "let's hear one more") and called on Craig. This accomplished the goal of keeping the conversation going, but the open-ended nature of the request did not specifically indicate that Matt's suggestion was not a productive approach toward the mathematical goal. Absent direction from Gage, the second student responded by attempting to clarify and elaborate on Matt's suggestion. The next several exchanges involved further discussion of Matt's approach to the problem. Gage invited additional conversation about the idea but also expressed his continued lack of clarity about the approach ("Let's just take a look at it, because, I couldn't quite see it."). Gage did not provide much guidance during his discussion with Craig and no especially useful ideas surfaced. After several turns of talk, Gage brought the discussion to an end.

During the postclass interview with the author–researcher (AR), Gage confirmed that he was genuinely unable to follow Matt's ideas (and was not, for example, feigning confusion as an instructional device):

AR: Matt was, you're trying to get them to look at the left-hand side of the equation and Matt comes giving suggestions that involve rearranging the equation all over and trying to solve for P or something.

Gage: I have no clue what he was saying.

The way Gage provided closure to the discussion suggests that his SCK and the additional time spent thinking about and discussing Matt's idea did not enable Gage to figure out his reasoning or determine the relevance of the mathematics contained in it (component practices 1 and 2). He encouraged the students to pursue the ideas on their own time and seemed to suggest that their approach was potentially productive ("I'm not rejecting this"). The discussion of Matt's approach did not bring out any ideas connected to the "reverse product rule" approach that Gage wanted the class to pursue. The way Gage ended the discussion, however, left the students without a clear sense of whether or not the type of approach they had been discussing was mathematically appropriate. On the one hand, the discussion of the approach did not seem to have helped them make progress on the problem, but at the same time, Gage encouraged the students to pursue the approach later on their own.

In the interview that followed this class, Gage expressed his dissatisfaction with the way the discussion had progressed and his frustration with his inability to see the mathematical ideas in what the students said:

Gage: I'm not happy with what happened, what actually. I mean, it was another example of not really moving along in ways that I foresee actually before I go into the classroom. The thing is I can't handle all these strange things that come up. They just come up and I'm like, "What the [expletive]? Where does that come from? Where does that lead?" [He went on to say] I just don't understand and haven't thought enough about differential equations as a subject to be taught so that I feel any flexibility at all. I mean I just feel like no matter what anybody says, I just don't know, "Well, should I stop them because I don't know where this is going?" Well then I have to stop everybody essentially and I have to go back to just telling them what I'm thinking. I just can't do it.

We find significant Gage's suggestion that he had not "thought enough about differential equations *as a subject to be taught* [italics added]" that would allow him greater "flexibility." Gage appeared to recognize a distinction between his general knowledge of differential equations and his knowledge of it as a subject to be taught—the very distinction that the construct of SCK is intended to capture. Had Gage's SCK enabled him to figure out the mathematical ideas the students were suggesting and determine that they were not connected to the mathematical goal of the activity, he might have been able to provide different kinds of guidance for the students, or, to use his language, be more flexible in his responses. Had Gage understood Matt's suggestion more thoroughly and recognized it as mathematically unproductive for the goals of the problem, he might have been able either to direct the discussion away from it or to pursue the ideas deliberately so students could come to see that the approach he offered was not going to be successful. Instead, Gage's request for more ideas, and the way he eventually brought the conversation to an end, meant students had fewer opportunities to deepen their understanding of the issues at hand. With additional guidance, students might have had, for example, some additional insights into the nature of the equation; those insights might have led them to more productive solution strategies in the future.

In this particular case, however, it is possible that Gage's choices were also limited because of his inexperience in handling situations in which students posed ideas he could not evaluate on the spot. That is, he may not have had sufficient pedagogical tools at his disposal for handling the situation in a less abrupt way. On the one hand, this supports our interpretation that he had reached the limits of his PCK and SCK and did not know what else to do. On the other hand, it also suggests potential for growth of his skills in leading discussions. All teachers will, at one time or another, find themselves in situations in which their existing practices and knowledge cannot sufficiently support them. Seymour and Lehrer (2006) highlight such situations as opportunities for the coemergence of new practices and PCK (and, we add, SCK).

Narrative and transcript, 3B. In the second part of the episode, after closing down the discussion of Matt's idea, Gage stopped asking for additional ideas and instead began writing an equation on the board. The author–researcher (AR) videotaping

the class recognized this as the start of a method for formally deriving the integrating factor. Had Gage continued, he would have provided the students with a method for solving the problem but in a way that diminished the opportunity for students to connect it to ideas they had studied in the past. In a previous conversation, Gage had invited the author–researcher to take part in class discussions or make suggestions at any time. In practice, this occurred only a few times during the entire semester, and this was one such occasion. Hoping that the students might still solve the problem on their own, the author–researcher interrupted Gage and drew the students' attention to the fact that they had seen equations in this form many times before:

AR: Can I interrupt?
Gage: Yes.
AR: Before you do that?
Gage: Yeah.
AR: I'm still trying to see if, if something was clear to everybody, and I was going to phrase it this way, if I would. What's up there on the left-hand side of the equation [AR refers to

$$(10+t)\frac{dP}{dt}+P]$$

          is something that all of you have seen, things like hundreds and hundreds of times over your calculus careers.
Gage: Hint. Hint.
Ss: [Laughter]
AR: How about going there first? Where have you seen this before?

In the flurry of discussion that followed, several students offered ideas that were not Gage's desired answer of "product rule:"

Tony: I was thinking initially, like, to me it looked like a chain rule kind of thing.
Gage: Chain rule? . . .
Tony: But I couldn't get anywhere with that, though.
Ron: Yeah. . . .
Dan: Differentiation by parts?

   Gage followed up on both the suggestion of "chain rule" and "differentiation by parts," directing the students' attention to the features of those answers that did not match the features of the equation with which they were working:

Gage: Differentiation by parts? What is that?
Dan: I don't remember, because . . .
Craig: Oh!

Ss: [Laughter]

Gage: Craig, you said, "Oh."

Craig: Well, yeah, somebody said the chain rule, and that, isn't that like, 'cuz we got a derivative of the . . .

Gage: The chain works with an inner and outer function, right?

Craig: Yeah.

Gage: A function inside of another function. I don't know. Do we have a function inside of another function here?

Craig: Well . . .

Tim: Looks like it.

Gage: Looks like it?

Craig: The *P* of *t* over 10 plus *t* seems like it might be . . .

Tim: Yeah.

Ron: With the chain rule, don't you multiply . . . if I remember correctly?

Gage: The chain rule looks like what?

Ron: You multiply.

Gage: The chain rule . . . well, where do I go? Maybe here? If I become neutral and start using neutral letters, you know what? It's kind of like, if you have *f* of *g* of *x*, then if you differentiate you get *f*-prime evaluated at *g* of *x* times *g*-prime of *x*.

Craig: Isn't the—

Gage: So on both sides you really are looking at a chain, right? Something inside of some other function.

The next suggestion offered by a student was the desired one, and Gage affirmed the answer:

Craig: Is it the product rule?

Tim: I meant the product rule. That's what I was thinking.

Gage: Product rule!

The remainder of the discussion continued productively as Gage guided the students in discovering how to "reverse" the product rule and eventually generalize and formalize the method.

Interpretation and analysis, 3B. In contrast to the first part of the episode when Gage's contributions were primarily focused on keeping discussion going, here he provided analytic scaffolding. However, because he did not recognize the mathematically useful ideas contained in the students' suggestions (component practice 2), he directed the discussion away from a mathematically productive path instead of toward it.

Gage had anticipated that it would be difficult for students to see the product-rule pattern in the equation. In the interview following the class, the author–researcher said it "was, not surprisingly, a challenge of trying to come up noticing the product rule going on in there." Gage agreed, stating, "No, right, I wasn't surprised that that was challenging." It appears, however, that merely knowing it would be difficult

for the students (and knowing the mathematical content) did not prepare Gage to make sense of the mathematical ideas contained in the students' off-target answers and to connect those ideas to the desired answer.

When students offered "chain rule" and "differentiation by parts" as suggestions, Gage reacted to the latter of the two ideas. He responded, "Differentiation by parts? What is that?" Although his response came in the form of a request for clarification, he did not engage the student who made the suggestion, Dan, in a discussion of his idea or use it to help direct the conversation. Although Dan responded to Gage's question with "I don't remember," the microphone at Dan's table captured more of his comments, but they were inaudible to Gage because of the multiple conversations and student laughter occurring at the moment. Dan said, "like $u\ dv$ and $v\ du$ and that whole deal," suggesting that he had in mind key features of the "integration by parts" technique.

It is worth noting, however, that "differentiation by parts" actually provides a very good description of features of the equation, because the equation resembles one that could be integrated by recognizing it as having the form $v\ du + u\ dv$. This differential form is an abstract description of what results from taking the derivative of the product of two functions, v and u, and it generates the formula for "integration by parts." In suggesting "differentiation by parts," Dan had accurately identified the specific feature of the equation that Gage wanted the class to generate. What the students were supposed to do, however, was identify what rule for differentiating had been used to obtain the equation; what was actually a near miss of the correct answer was dismissed by Gage as incorrect and not pedagogically useful.

A similar story can be told about the pedagogical utility of the students' suggestion of "chain rule." Although, like "differentiation by parts," this answer is not precisely correct, it suggests that students were paying close attention to the form of the equation and thinking about techniques of differentiation that might lead to the equation. Gage did pursue the "chain rule" answer, but only by writing it on the board and explaining why it was incorrect.

During the postclass interview, Gage expressed his frustration with the "chain rule" situation, indicating that he did not understand why the suggestion might have been raised. It appeared to him to limit further progress:

Gage: … but the thing is, when they started talking about the chain rule, that just put a bullet through my head. I was like, whoa, where does *that* come from? And what am I gonna do with that?"

Unable to make use of the suggestions being offered to him, Gage interpreted the problem as a breakdown in communication:

Gage: It was like, how do we get them to see that this is the product rule? It's like 20 people in there and 18 of them say "chain rule" when they mean "product rule." And I'm like, well, then we can't communicate.

Although Gage was unable to recognize "differentiation by parts" or "chain rule" as potentially mathematically productive contributions to the conversation, either

of these suggestions could have been used to scaffold the discussion toward the key idea of seeing something in the form of the equation that can be undone by integrating. In the moment, however, he was unable to do the mathematical work of examining the entailments of the students' two answers and comparing them to the features of "product rule" in ways that made the mathematical connections apparent to him. Instead of being able to deploy the SCK necessary to see the potentially productive mathematical ideas embedded in the students' solutions, he made use of his knowledge of formal mathematical language and procedures to help students see what was mathematically incorrect about their answers.

Finally, we note again that Gage's actions in this episode do not suggest that he lacked the knowledge of any particular pedagogical practices that may have been of help to him. He made the moves that he did because he focused on the incorrectness of the students' ideas and chose to correct them, not because he did not know what to do. Had he recognized the potential of students' ideas for productive discussion, he could have relied on questioning practices with which he was quite familiar.

CONCLUSIONS AND IMPLICATIONS

The literature on mathematics education reform is replete with examples of valuable learning opportunities created for students by teachers who are masterful at orchestrating discussions. Over the past several decades, however, the research community has amassed examples of how well-intentioned, otherwise capable, teachers can sometimes lead students in discussions that fail to provide ample opportunities to learn the intended mathematical ideas. Although the factors contributing to these challenges are many, it appears that valuable learning opportunities in inquiry- and discussion-driven classes can be created through teachers' use of analytic scaffolding. In an effort to better understand factors that influence teachers' capacities to create rich learning opportunities via whole-class discussion, we conjecture that there are component practices of analytic scaffolding and that the analysis of knowledge needed to enact these practices provides detailed and useful insights.

Our observation of Gage's practices and his self-assessment/analysis of the effectiveness of those practices for achieving his mathematical goals led us to examine his struggles with three of the four component practices. In particular, there were instances in which he did not follow or fully understand the ideas that students expressed (component practice 1). This limited the amount of "raw material" he had available to use to help students construct solutions to the mathematical tasks or to direct their attention to productive (or unproductive) solution paths. In addition, Gage's success in providing direction and analytic support to his students was hampered by the difficulties he had in recognizing or figuring out how ideas that students contributed to the discussion were relevant to the mathematical goals of the lesson and to the learning opportunities for the students (component practices 2 and 3). These three component practices draw on aspects of a teacher's knowledge

such as knowledge of typical ways student think (correctly and incorrectly) about the task or content in question, knowledge of the curriculum in use, and knowledge to support the specialized type of mathematical work teachers do when dissecting and analyzing students' expressions of their ideas.

Gage sometimes encountered difficulties when students provided mathematically correct and pedagogically useful ideas, as well as at times when students offered incorrect and/or nonproductive ideas. In Episode 1, students struggled with several ideas, and Gage appeared to understand the ideas they were raising in the discussion, but because he did not know the role that these particular ideas play in the development of students' understanding of the more general topic, he did not utilize the students' contributions. Had he seen the relevance of the discussion to advancing his students' conceptual understanding, he might have pointed the students' attention in that direction. Lacking the PCK to see such potential in the discussion, however, he was unable to use the students' contributions as an opportunity for analytic scaffolding that would have helped the discussion progress toward the mathematical goal.

Similarly, in Episode 2, students made mathematically correct contributions to the discussion, but this time Gage was unable to recognize those contributions as mathematically useful and relevant to the discussion at hand. As a result, he missed an opportunity to use the contributions to provide direction to the discussion. Episodes 3A and 3B presented potentially even more complex tasks for Gage. In these episodes, he was faced with student contributions that were mathematically imprecise and incorrect. In the first instance, because Gage was unable to follow a student's suggestion, he did not see that it was unproductive for the task at hand. As a result, instead of using the student's contribution to illustrate why an apparently reasonable suggestion was unhelpful (or making use of it in some other way), he permitted the discussion of the idea to continue until eventually, still unsure about the specifics of the suggestion, he encouraged the students who made the suggestion to continue to pursue their approach outside of class. In the second instance, he responded directly to the incorrect aspects of students' contributions by explaining why they were wrong, but he did not recognize or figure out how the mathematics in the students' incorrect answers of "differentiation by parts" and "chain rule" could potentially be linked to the mathematical goals for the discussion.

From our perspective, with the luxury of time and multiple viewings of the tapes, we could follow ideas that were communicated in imprecise or unconventional ways, and we could see the potential that those ideas had for moving the agenda forward. We do not claim that Gage or teachers in general should be able to do all these things all the time. We do believe, however, that teaching expertise in reform-oriented practices of this sort is enhanced as teachers develop the types of knowledge considered here. Analyses such as ours reveal with greater precision the knowledge that teachers need to expand their capacity to carry out such practices more fluently and in the moment. Without this capacity, Gage's analytic scaffolding in these episodes was met with only limited success, despite his strong under-

standing of the mathematical content, clear vision of the learning goals for the lesson, and commendable ability to elicit contributions from students.

Our analysis reveals at a relatively fine-grained level of detail just how complex analytic scaffolding during classroom discussions can be, and why it can be so difficult for a teacher to carry out. This difficulty is only heightened as teachers must simultaneously provide social scaffolding as well. To provide social scaffolding, teachers must attend to students' contributions in order to monitor discussions and ensure that conversation takes place according to established classroom norms. Teachers providing analytic scaffolding must monitor the discussion for these same things, but they must simultaneously monitor and analyze the mathematical train of thought that emerges (or not) from students' contributions, both individually and collectively. Providing both kinds of scaffolding is demanding, and so it is not surprising that Gage and other teachers report a sense of being overwhelmed or feeling a lack of control over the discussion (e.g., Leikin & Dinur, 2007). This adds credence to the claim that substantial knowledge of mathematical content is not all that is needed to turn contributions from students into the building blocks of a productive discussion.

From a theoretical perspective, our work suggests how types of knowledge for teaching that are currently receiving significant attention among educational researchers can be connected to the fine-grained components of scaffolding practices considered to be valuable in reform classrooms. Observation of teachers such as Gage, a professional mathematician whose CCK was sound, reveals evidence that the mathematical work entailed by classroom teaching requires additional types of mathematical knowledge identified as SCK.

By attending to the distinction between an instructor's ability to recognize the significance of a student's contribution and the instructor's need to figure out its significance in the moment, we suggest practical criteria for distinguishing between the roles of PCK and SCK in teaching practice and note the possibility of a generative relationship that SCK may have with PCK. If, for example, a teacher is familiar with specific difficulties that students encounter when learning particular ideas, then when she sees/hears that difficulty, she can tap into the various things she knows that relate to that difficulty. If, on the other hand, she is not familiar with the particular difficulty, she may need to think through the mathematical ideas, unpacking for herself how various aspects of the idea might be (mis)understood by students in ways that lead to the difficulty she has observed. To respond to the student may also require some amount of work to figure out how to help students clarify their thinking by connecting the mathematical ideas they have raised to the desired ways of thinking. Doing this type of mathematical thinking may take time and effort. The next time the teacher sees or hears students expressing this difficulty, it may take less mathematical work to figure out what is going on and perhaps eventually the teacher will quickly recognize the student thinking and be able to call up the ideas she has generated in similar situations as a "chunk" of knowledge. In this way, SCK can be thought of as both supporting teachers as they do the

mathematical work specific to teaching and enabling teachers to learn through such work. The product of that learning then has the potential to become knowledge that can serve as PCK.

These hypotheses, along with our original hypotheses of the component practices of analytic scaffolding, suggest future research directions. Although our focus here has been limited to considerations of Gage's knowledge and use of the first three component practices, future analysis of Gage's or other teachers' teaching may shed light on similar issues surrounding teachers' knowledge and other component practices. As we have noted previously, we by no means wish to suggest that the whole of Gage's teaching could or should be characterized by the episodes and analysis presented here; rather, we have chosen these episodes in order to characterize the more prominent challenges he faced in his experience with the new curriculum. A more comprehensive view of Gage's teaching and of the relationship among PCK, SCK, and the component practices of analytical scaffolding require attention to his more successful experiences of leading classroom discussions, as well as to the development of his knowledge and practices over time. We anticipate future work in precisely this direction. Similarly, because we have had occasion to gather similar data on other instructors' teaching, we believe that much could also be learned by comparing Gage's experiences to parallel experiences of different instructors who are using the same new curriculum.

Insight into the knowledge needed to support specific instructional practices can help the mathematics education community enable more teachers to create rich learning opportunities through discussions like those that have been showcased in self-studies of exceptional teachers (e.g., Ball, 1993; Lampert, 2001). Researchers who examine and describe the practices and tools that successful teachers use to orchestrate discussions are making valuable contributions (e.g., Rasmussen & Marrongelle, 2006; Sherin, 2002a; Stein et al., 2008). The utility of those tools, however, comes from the ways in which teachers use them. In addition to identifying such tools, it seems important to understand what knowledge teachers need in order to use them well. Fine-grained analyses of teachers' practices and reasoning of the sort presented here provide one possible approach to enhancing our understanding of the connections between teachers' knowledge and the implementation of specific instructional practices.

We have delineated several component practices of analytic scaffolding and examined the PCK and SCK needed to enact those practices. The long-term goals of this line of research include understanding the experiences that enable teachers to develop these types of knowledge and practices. Determining the extent to which such formative experiences could be made available to teachers, perhaps through professional development opportunities, seems essential if research progress is to affect practice in a way that ultimately improves student achievement in mathematics.

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