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Mathematics Teacher Educator

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Mathematics Teacher Educator (*MTE*) (ISSN 2167-9789) is a joint venture between the National Council of Teachers of Mathematics and the Association of Mathematics Teacher Educators. *MTE* is published online only in September and in March by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502.

Mission and Goals of *MTE:* The *Mathematics Teacher Educator* will contribute to building a professional knowledge base for mathematics teacher educators that stems from, develops, and strengthens practitioner knowledge. The journal will provide a means for practitioner knowledge related to the preparation and support of teachers of mathematics to be not only public, shared, and stored, but also verified and improved over time (Hiebert, Gallimore, and Stigler 2002).

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EDITORIAL

Mathematics Teacher Educator: An Opportunity to Share, Verify, and Improve Practitioner Knowledge

Margaret S. Smith
Editor, Mathematics Teacher Educator

Mathematics Teacher Educator is the first journal dedicated specifically to issues in mathematics teacher education, providing a much-needed forum for supporting and improving the practice of educating teachers of mathematics. As the Editorial Panel articulated in the call for manuscripts (http://www.amte.net/publications/mte), the mission of *Mathematics Teacher Educator* is "to contribute to building a professional knowledge base for mathematics teacher educators that stems from, develops, and strengthens practitioner knowledge. The journal provides a means for practitioner knowledge related to the preparation and support of teachers of mathematics to be not only shared but also verified and improved over time. The journal is a tool to build the personal knowledge that mathematics educators gain from their practice into a trustworthy knowledge base that can be shared with the profession."

Building a trustworthy knowledge base for mathematics teacher education requires that manuscripts convey more than stories of practice, however compelling. They must describe a problem or issue in mathematics teacher education with which readers can identify; the methods/interventions/tools that were used to explore or address the problem or issue; and the means by which these methods/interventions/tools and their results were studied and documented. In addition, manuscripts must be connected to the existing knowledge base in mathematics teacher education, grounded in theory and/or previously published articles, provide evidence of the effectiveness of the intervention being described beyond anecdotal claims, make explicit the specific new contribution to our knowledge base, and provide sufficient detail to allow for verification, replication in other contexts, or modification by subsequent authors.

The articles in this first issue of *MTE* provide both rich illustrations of these criteria and solid examples of the types of problems and issues that are sure to be of interest to practitioners who contribute to the preparation and professional development of pre-K–grade 12 preservice and in-service teachers of mathematics (e.g., mathematics teacher educators, mathematicians, teacher leaders, school district mathematics specialists, professional developers). Although the articles share these general characteristics, they differ along several dimensions: the nature of the issue or problem being addressed, the context in which the intervention was enacted, and the level and experience of the teachers participating in the work. The descriptions that follow provide a brief summary of the articles, highlighting the three key dimensions along which they vary.

In the opening article, "Mathematics Preservice Teachers Learning About English Language Learners Through Task-Based Interviews and Noticing," Anthony Fernandes describes the use of task-based interviews in a content course focused on geometry and measurement to help preservice middle school teachers develop an awareness of the challenges that English language learner (ELL) students face and the resources on which they draw as they learn mathematics and communicate their thinking in English only classrooms. He provides evidence that in addition to developing awareness, preservice teachers also adopted strategies that were aligned with best practices for teaching ELLs outlined in the literature.

In "The Role of Writing Prompts in a Statistical Knowledge for Teaching Course," Randall E. Groth describes the use of writing prompts to help preservice elementary teachers (K–8) enrolled in a content course focused on statistics develop statistical knowledge for teaching (SKT). He provides evidence that preservice teachers developed SKT as well as knowledge of introductory college-level statistics.

In "Capitalizing on Productive Norms to Support Teacher Learning," Laura Van Zoest and Shari L. Stockero describe the results of a study they conducted with both preservice and in-service secondary mathematics teachers to determine the extent to which teachers' experiences and learning in an initial methods course had long-term effects on their professional practice. The authors argue that explicitly cultivating professional norms impacts teachers' knowledge and habits of practice. Specifically, cultivating professional norms improves teachers' own mathematical understanding, particularly the specialized content knowledge needed for teaching; supports teachers in learning to view and analyze classroom practice in productive ways; provides teachers an experiential basis for thinking about fostering productive norms in their classrooms; and helps teachers to develop professional dispositions that support continued learning from practice.

In “The Content-Focused Methods Course: A Model for Integrating Pedagogy and Mathematics Content,” Michael D. Steele and Amy F. Hillen posit the creation of hybrid courses that focus on developing specific mathematical content (in this case functions) in the context of a methods course with the intent of helping teachers developed more integrated knowledge of content and pedagogy. In the specific example provided, preservice and in-service teachers with elementary, secondary, and special education backgrounds collaboratively engage in a course that is designed around three key principles, which the authors argue are generalizable to a wide range of teacher education settings.

In the final article in this issue, “Using ‘Lack of Fidelity’ to Improve Teaching,” Anne K. Morris describes how variations in the implementation of lesson plans can serve as a source of information for improving curricula. She draws on her observations of two instructors of a content course for preservice elementary teachers (K–8) and identifies significant variations and positive adaptations in the lessons that lead to increasingly rich lesson plans that, she argues, can move toward building an accumulated knowledge base in teacher education.

Although these five articles represent diversity along several dimensions, as with any finite set of exemplars, they do not begin to exhaust the possibilities for articles that would be suitable for *MTE*. For example, none of the articles focuses on professional development for in-service teachers, delivery systems other than face-to-face meetings, or field experiences for preservice teachers. As the collection of *MTE* articles grows over time, it is expected that a wider range of issues, contexts, and populations will be addressed.

Because one of the goals of the journal is to build a knowledge base for the field, submissions that deliberately build on prior published work are encouraged. Careful descriptions of how previous methods/interventions/tools have been modified and the comparison/contrast to earlier reported results should be articulated.

Please consider contributing to the journal by writing an article or serving as a reviewer. As you can see by the authorship of the articles that appear herein, authors have a range of experience (assistant professors to full professors), are at different types of institutions, are housed in different departments, and have different areas of expertise. What they have in common is a passion for teacher education and the motivation to share their work with colleagues in order to improve the practice of teacher education.

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COMMENTARY

Mathematics Teacher Educator: A Milestone in the History of the Association of Mathematics Teacher Educators

Marilyn Strutchens
President, Association of Mathematics Teacher Educators

The Association of Mathematics Teacher Educators (AMTE) is excited to serve as a co-partner with the National Council of Teachers of Mathematics (NCTM) in publishing *Mathematics Teacher Educator* (*MTE*), a practitioner journal for mathematics teacher educators, which will serve as a milestone in the history of AMTE. The mission and goals of *MTE,* listed below, support our members and our organizational goals.

> ***Mathematics Teacher Educator*** **will contribute to building a professional knowledge base for mathematics teacher educators that stems from, develops, and strengthens practitioner knowledge. The journal will provide a means for practitioner knowledge related to the preparation and support of teachers of mathematics to be not only public, shared, and stored, but also verified and improved over time. (Hiebert, Gallimore, & Stigler, 2002). (www.nctm.org/publications/content.aspx?id=28143)**

Furthermore, mathematics teacher educators are the intended members of the audience for *Mathematics Teacher Educator*, with *practitioner* broadly defined as anyone who contributes to the preparation and professional development of pre-K–12 preservice and in-service teachers of mathematics. Mathematics teacher educators include mathematics educators, mathematicians, teacher leaders, school district mathematics experts, and others.

AMTE is the largest professional organization focused on mathematics teacher preparation and has approximately 900 members. The goals of AMTE are to promote (1) effective mathematics teacher education programs and practices; (2) communication and collaboration among those involved in mathematics teacher education; (3) research and other scholarly endeavors related to mathematics teacher education; (4) professional growth of mathematics teacher educators; (5) effective policies and practices related to mathematics teacher education at all levels; and (6) equitable practices in mathematics teacher education, including increasing the diversity of mathematics teachers and teacher educators.

The February 2012 annual meeting of the AMTE culminated a yearlong 20th anniversary celebration. One of the celebratory moments was the announcement that the first issue of *Mathematics Teacher Educator* would be published in 2012. *MTE* will help AMTE to address several of its goals. First, *MTE* will help AMTE to meet its goal of promoting *effective mathematics teacher education programs and practices* by publishing articles that showcase evidence-based programs and practices and describe how mathematics teacher educators and their partners developed them. By highlighting the voices of practitioners, the journal will enable them to share their personal struggles and how they overcame them to move the programs and practices forward, enabling others to gain insight as to what they may face in taking on similar endeavors.

AMTE's second goal of fostering *communication and collaboration among those involved in mathematics teacher education* can also be enhanced by *MTE*. Articles published in *MTE* can serve as catalysts for practitioners to discuss issues of practice, such as developing mathematics courses that help secondary preservice teachers develop mathematical knowledge for teaching (Conference Board of the Mathematical Sciences, 2012) or providing professional development for teachers to help them to effectively implement the Common Core State Standards for Mathematics (CCSSI, 2010).

Third, articles published in *MTE* will assist in the goal of supporting the *professional growth of mathematics teacher educators*. With the existing economic conditions, mathematics teacher educators are afforded fewer opportunities to travel to conferences. Thus, *MTE* will be an increasingly important venue for providing mathematics teacher educators with opportunities to learn about the experiences of their colleagues from around the country. They will gain perspectives from those colleagues about what they implemented, how, and the results.

As a past series editor for AMTE's monograph, I am excited about the potential that the journal has for our members and other constituents. Our monograph series was only available to members and initially focused on specific topics. In the last series, the call was more general, and a broader array of issues related to practice was discussed. However, we welcome the opportunities that will come with the publishing of *MTE*. The members of

AMTE and others will have an ongoing venue to submit practitioner articles for mathematics teacher educators, and they will also have the opportunity to build communities of practice around issues of importance to the field.

Moreover, *MTE* will eventually become a part of the JSTOR collection, which will make articles more accessible than they were in the monograph. The JSTOR collection is "a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content on a trusted platform of academic journals, primary sources, and books" (JSTOR, n.d.). Also, the electronic nature of the journal will make it possible to link articles that build on each other and to other resources. While we feel that the monograph series has largely served its purpose, we may still have some special focus monographs in the future.

As I stated earlier, AMTE is excited to be a co-partner with NCTM in publishing *MTE*. The leadership of both organizations felt that it was perfect timing in that both organizations were thinking about developing the same type of journal at the same time. Even though our initial needs may have been different, I think that we have created a journal that will meet the needs of both organizations. As we launch our first issue, I would like to personally thank the editors, members of the editorial panel, and the reviewers who have made this first issue of MTE a reality.

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COMMENTARY

Mathematics Teacher Educator: The Evolution of a New Journal

J. Michael Shaughnessy
Immediate Past President of the National Council of Teachers of Mathematics

In the past decade, there has been a growing interest in the role that practitioners play as stakeholders in and coproducers of professional knowledge and of research knowledge in mathematics education (Kieran, Krainer, & Shaughnessy, in press). Although the wellspring of professional knowledge and craft wisdom of teachers has been sparsely tapped in previous decades, there are now signs from all over the world that teachers are playing an increasingly important role in research on the teaching and learning of mathematics (Bednarz, 2004; Fernandez & Yoshida, 2004; Herbel-Eisenmann & Cirillo, 2009; Huang & Bao, 2006; Jaworski et al., 2007; Makar & O'Brien, in press). A recent conference brought together practitioners, teacher educators, and researchers in mathematics education to develop a research agenda that will provide closer links between research and practice (NCTM, 2010).

Concurrent with the growing interest in incorporating practitioner perspectives into research and development in mathematics education, there has been an increased awareness in the field of the need to systematically grow our research knowledge base on the professional development of mathematics teachers and on the education of prospective mathematics teachers. We need to better document and share what we learn from the education of prospective and practicing mathematics teachers. The impact of many mathematics education in-service projects and effective advances in preservice mathematics teacher programs have rarely been adequately shared with the field, in part because there have not been outlets to encourage the publication of such work. In the past, what is known about effective preservice and in-service mathematics teacher development has tended to have been gleaned from isolated studies rather than from a systematic program of research. To counter this trend, Weiss (1999) provided guidelines for the evaluation of professional development efforts in mathematics teacher education. Recent work has begun to identify and synthesize elements of effective professional development in a systematic way (e.g., Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; Desimone, 2009; Guskey & Yoon, 2009; Sztajn, Marrongelle, & Smith, 2011).

These new emphases in the work of mathematics educators—forming research partnerships with practitioners and investigating the education of preservice and practicing mathematics teachers in a more scientific and evidence-based manner—have had a growing influence on the thinking of both the National Council of Teachers of Mathematics (NCTM) and the Association of Mathematics Teacher Educators (AMTE).

In the fall of 2009, discussions about the possibility of creating a new journal focused on the practice of mathematics teacher education began to percolate in discussions at the National Council of Teachers of Mathematics. The Editorial Panel for the NCTM practitioner journal for secondary teachers, *Mathematics Teacher* (*MT*), had been receiving an increasing number of strong manuscripts dealing with issues around the preparation of preservice teachers and on the professional development of in-service teachers. However, many of these manuscripts did not fit the charge and purpose of *MT*, a journal of mathematics and mathematics teaching for secondary teachers. These articles were also different from the usual types of research articles in many mathematics education research journals. As discussions continued, it became clear to NCTM that there was a growing need in the mathematics education community for an entirely different kind of professional journal than the Council had in its portfolio.

Around the same time that NCTM was brainstorming the possibility of starting a new journal for teacher educators, the Association of Mathematics Teacher Educators formed a task force to study the efficacy of launching a new professional journal quite similar to the kind of journal that was being considered by the NCTM Board of Directors. In January 2010, representatives from the leadership of AMTE and NCTM met to discuss the possibility of a collaborating to create such a journal. A task force consisting of members of both AMTE and NCTM was formed to study the matter more deeply and to provide further information and recommendations on whether to pursue a joint journal effort, and if so, how to proceed. In the summer of 2010, the task force submitted a formal motion to the boards of directors of NCTM and AMTE to cosponsor the creation and publication of a new journal. The motion to create a new journal was approved by both the boards, an initial Editorial Panel was formed, and the first editor was appointed for a new journal, *Mathematics Teacher Educator (MTE).*

The *MTE* was created, among other reasons, to build a professional knowledge base in mathematics teacher education that stems from practitioner knowledge. The journal will provide a means for practitioner knowledge related to the preparation, support, and development of mathematics teachers to be made public, shared, stored, verified, and improved over time—necessary conditions for practitioner knowledge to provide a solid foundation for professional knowledge (Hiebert, Gallimore, & Stigler, 2002). Through MTE we hope to increase our professional and research knowledge about teacher education, through accounts of exemplary preservice and in-service mathematics teacher education programs; in reports of effective classroom pedagogical strategies; with studies of effective ways of developing the content knowledge and pedagogical content knowledge of preservice and in-service teachers; and eventually, through scholarly reviews of materials and resources for the mathematical education of teachers.

The National Council of Teachers of Mathematics is thrilled to cosponsor this new journal, *Mathematics Teacher Educator,* together with the Association of Mathematics Teacher Educators. We believe that *MTE* is uniquely positioned to encourage the publication of scholarly work that will enable the profession to systematically investigate critical issues around the preparation of preservice mathematics teachers as well as the professional development and leadership development of in-service mathematics teachers and teacher leaders. Over time we envision a growing body of work that will draw upon the best of what we can learn from practitioner knowledge to help inform and continually update best practices for preparing future mathematics teachers and for developing and strengthening existing mathematics teachers. With focused effort to build and extend these knowledge bases, we hope to be able to improve the recruitment, education, and retention of excellent mathematics teachers for many future generations of our students.

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Mathematics Preservice Teachers Learning About English Language Learners Through Task-Based Interviews and Noticing

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This article describes the 3rd cycle of an intervention in a mathematics content course that was designed to foster awareness among middle school mathematics preservice teachers (PSTs) of the challenges that English language learner (ELL) students face and the resources they draw on as they learn mathematics and communicate their thinking in English-only classrooms. Pairs of PSTs engaged 2 different ELL students in a videotaped task-based interview using 4 measurement tasks. Following each interview, the PSTs wrote a structured report guided by Mason's (2002) framework of *noticing.* The results of the intervention indicated that the PSTs went beyond awareness of ELLs' needs and challenges and also adopted strategies outlined in the literature that were aligned with best practices for teaching ELLs. The article also discusses the potential of the intervention and how it can be used by other mathematics educators.

Key words: Preservice teacher education, Task-based interviews, Noticing, English language learners

The U.S. population of English language learners (ELLs) is growing, and there is a great need to prepare *all* mathematics teachers to work with these students (Bunch, 2010; Lucas & Grinberg, 2008). Between 1980 and 2009, the ELL student population experienced a growth spurt, rising from 10% to 21% of students (National Center for Education Statistics, 2010). However, specific preparation of teachers to work with ELLs has not kept pace with this growth. The National Center for Education Statistics (NCES) reported that out of the 41% of teachers who had ELLs in their classrooms, only 13% of those teachers received ELL-specific training (NCES, 2002). In their study of 417 teacher preparation programs, Menken and Antunez (2001) found that less than 17% prepared preservice teachers (PSTs) to work with diverse students, a category that includes but is not limited to ELLs. This lack of teacher preparation persists despite the growth in ELL population (e.g., Christian, 2006; Gandára & Maxwell-Jolly, 2006; Márquez-López, 2005).

This article reports on the third cycle of an intervention that I, as the instructor of a mathematics content course, carried out to support PSTs' work with ELLs in their future classrooms. This intervention provided multiple experiences for the PSTs to participate in task-based interviews (Goldin, 2000) and develop their observation skills through a framework of *noticing* (Mason, 2002). PSTs became more aware of the challenges that ELL students face and the resources that they draw on as they learn mathematics in English-only classrooms. Further, through the interviews, the PSTs also developed concrete strategies for assisting ELL students, which will be useful in their future classrooms.

Language Demands for ELLs

De Jong and Harper (2005) pointed out that the pervasiveness of language in human activity leads to a tendency for teachers to look "through" language rather than "at" it. In the case of mathematics, there is a tendency to assume that it is universal and, as a consequence, that it involves minimal linguistic challenges for ELLs (Barwell, 2005; Walker, Ranney, & Fortune, 2005). Further, teachers may assume that "good teaching," with little or no modification, is enough to reach all students, including ELLs (de Jong & Harper, 2005). However, extensive research has illustrated the connection between language and mathematics, and the impact that language has on the teaching, learning, and assessment of mathematics (e.g., Bailey, 2007; Barwell, 2005; Clarkson, 2007; O'Halloran, 2005; Schleppegrell, 2004, 2007; Veel, 1999). I highlight a few aspects of the language demands that ELLs encounter as they learn mathematics in English-only classrooms. Note that even though non-ELLs face similar demands, the cognitive load is magnified for ELL students as they learn new content in a language they are still learning (Campbell, Adams, & Davis, 2007).

Cummins (2000) provides a useful distinction between the everyday *conversational language* that students encounter on a regular basis and the *academic language* that they encounter in school subjects such as mathematics. One part of the academic language consists of the *register*—the unique lexical and grammatical features that students can draw on in a content area to make meaning (Halliday, 1978). The *mathematics register* includes lexical aspects such as vocabulary that is unique to mathematics (e.g., words such as *coefficient* and *denominator*) and other everyday terms that have specialized meaning in mathematics (e.g., *rational* and *difference*; Bailey, 2007; Pimm, 1987). The latter can prove confusing to ELLs, who are learning English and the content at the same time (Bailey, 2007; Garrison & Mora, 1999; Lager, 2006).

Besides the challenge of lexical aspects, the mathematics register also includes unique grammatical features such as the use of the nominal group to pack information into a sentence (Veel, 1999). For example, "the volume of a rectangular prism with sides 8, 10, and 12 cm" (Veel, 1999, p. 197) consists of the elaboration of the noun *prism*. The prenumerative qualifier—the volume of—endows the prism with the mathematical attribute of volume; the classifying adjective—rectangular—subclassifies the prism into the existing taxonomies; and the qualifier—with sides 8, 10, and 12 cm—restricts the range of meaning of the prism. The use of complex nominal groups, like the one described, allows more information to fit into a sentence, thus increasing its lexical density (Eggins, 2004; Schleppegrell, 2004, 2007; Veel, 1999).

There is a further expansion of linguistic demands in current reform (NCTM, 2000; NGA Center & CCSSO, 2011) classrooms, as students are expected to master discourse features such as making conjectures, justifying their solutions, building on other students' ideas, and presenting solutions as part of the classroom community (Bailey, 2007; Moschkovich, 2002).

The Use of Task-Based Interviews in Teacher Preparation

The discussion up to this point highlights the need for PSTs to be aware of the linguistic aspects that affect the teaching and learning of mathematics to ELLs (Fillmore & Snow, 2005). This awareness justifies adapting mathematics instruction to accommodate the needs of ELLs. For example, ELL students may benefit from explicit instruction and modeling of the discourse features in mathematics (Khisty & Chaval, 2002). Informal discussions with PSTs from the mathematics content courses that I taught revealed that they had minimal opportunities to interact with ELLs in prior educational experiences. Generally, the PSTs tended to view mathematics as being universal and minimally language intensive, and as involving symbols that could be transferred across languages (e.g., 1 + 1 = 2 was the same whether you spoke Spanish or English). However, they accepted that word problems could pose linguistic challenges for all students, not only ELLs. It is also important to note that mathematical notation and procedures may be different for recent immigrant students in their home country (Perkins & Flores, 2002). Research has shown that PSTs who are not aware of the role that language plays in the teaching and learning of mathematics are less likely to make linguistic modifications in their classrooms to accommodate ELLs (Lucas, Villegas, & Freedson-Gonzales, 2008). Based on my experiences with the PSTs from the content courses, and the needs engendered by the changing demographics in mathematics classrooms across the country, I wanted the PSTs to become aware of the resources ELL students draw on and understand the challenges that these students face as they learn to communicate mathematically in English-only classrooms.

There were two major factors that prompted the use of task-based interviews (Goldin, 2000). First, engaging in task-based interviews allowed the PSTs to go beyond the correct answers to problems to understand the students' thinking (Goldin, 2000). In the process of interviewing students, the PSTs were able to interact with students and observe the possible impact of language on students' mathematical performance and the resources students drew on to communicate their thinking. My own research with task-based interviews revealed the rich nature of ELL students' mathematical thinking when they were provided with the appropriate support and asked probing questions during the interview (Fernandes, Anhalt, & Civil, 2009). I conjectured that with appropriate support, the PSTs could replicate this experience, which in turn would ground their thinking about the influence language has on the teaching and learning of mathematics for ELLs.

Second, the research literature recommends that PSTs learn through direct experience. In multicultural education, direct experiences such as cross-cultural immersion and tutoring students from diverse backgrounds have had a positive influence on the beliefs that predominantly White PSTs hold about these students (Gay, 2002; Giroux, 1988; Grant & Secada, 1990; Nieto, 2000; Sleeter, 2001; Sowa, 2009; Waxman & Padrón, 2002; Zeichner & Hoeft, 1996). Griego-Jones (2002) found that PSTs who had tutored ELL students held beliefs that were in line with the research about second language learning. This idea has also been demonstrated in mathematics education. Opportunities to learn about children's mathematical thinking positively influenced PSTs' initial beliefs about mathematics teaching (Ambrose, 2004; D'Ambrosio & Campos, 1992; Vacc & Bright, 1999).

Noticing

In addition to providing PSTs with direct experiences, research on teacher development also recommends the incorporation of reflection (Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010; Mewborn, 1999). Without this component of reflection, the experiences could simply serve to reinforce deficit beliefs that PSTs have regarding diverse students (Grant, 1991; Grant, Hiebert, & Wearne, 1998). Based on two prior cycles of this intervention in previous semesters (this article reports on the third cycle), I observed that even though the PSTs reflected on their interactions with the ELL students, they focused on the strategies that the students used to solve the problem. Although this is consistent with the typical use of task-based interviews, in this intervention their focus was redirected toward the linguistic aspects of the interactions with the students. Mason's (2002) framework of *noticing* was used in the third cycle to focus PST attention on the complete mathematical communication within the interview. Mason pointed to *noticing* as being key to professional development and the first step toward action, stating that people learn through experience, and this causes people to react in habitual ways. These habitual ways of interacting with others influence people to classify others and react stereotypically to situations before they realize it. This appeared to be the case with the PSTs.

In previous cycles of the intervention, PSTs had a tendency to make a quick judgment and classify the student's strategy as correct or incorrect. This quick judgment of the student's attempt to solve the problem prevented PSTs from exploring the possible reasons why the student produced that solution. By slowing down their judgments about the ELL students' solutions, the PSTs could open up opportunities to notice possible linguistic challenges that the ELL students faced and the resources they used.

Mason suggests that professional noticing is about being sensitive and becoming systematic without acting automatically. In his book *Researching Your Own Practice: Discipline of Noticing* (2002), he outlines processes through which one could become more sensitive. For the purposes of this intervention, I focused on one process, the creation of *accounts*. He describes two forms of recording what we notice: *accounts-of* and *accounts-for. Accounts-of* refers to recording an event as it would be seen and felt by another observer, by paying careful attention not to involve emotion or judgments. Making a judgment could mean that we have labeled something too fast, and this could blind us to new interpretations. To *account-for* something means offering "interpretation, explanation, value-judgment, justification, or criticism" (p. 40). By writing *accounts-of,* the observer leaves things open. He or she and others can revisit the incidents at a later stage and make interpretations.

The intervention included the use of this process of accounts as a starting point to develop PSTs' sensitivity to noticing linguistic aspects during interviews. Additionally, the intervention included my feedback on the process; I reviewed the PSTs' accounts and provided them with alternative interpretations. These points will be elaborated further in some of the sections below.

The Intervention

The intervention consisted of a semester-long project in four phases (see Figure 1), which was integrated into content courses I taught for middle school mathematics PSTs. The intervention described in this article was the third cycle conducted in a geometry and measurement course. Topics in this course included perimeter and area of two-dimensional shapes, surface area and volume of three-dimensional objects, and proofs in Euclidean geometry, including parallel lines, triangle congruence, and properties of various quadrilaterals. There were 32 PSTs in total, 10 males and 22 females; however, 1 female student dropped the course after conducting the first interview. I did not consider her report as part of my analysis. There were 20 Caucasians, 7 African Americans, 3 Hispanics, and 1 Middle Eastern student. All the students were in the second or third year of the teacher preparation program, which contained a special mathematics strand for PSTs who expressed an interest in teaching the middle grades (Grades 6-8). Four out of the 31 students had also participated in the second iteration of the intervention in a previous course.

Phase one (Figure 1) consisted of one class period that was used to introduce the project, engage the PSTs in solving the four measurement tasks, watch two video clips of a researcher interviewing ELL students, and craft an interview script. The second phase involved pairs of PSTs interviewing individual ELL students from a group of fifth and sixth graders at a local intermediate school. A Flip video camera (Cisco) was given to each pair to record the interview. The recording was used to assist PSTs with the written report they submitted after each interview. The PSTs interviewed a second ELL student and submitted another report in the third phase. This interview was also recorded with a Flip video camera. Finally, the fourth phase involved the PSTs sharing what they learned from this interview in a class discussion. The sections below will outline the selection of tasks and the four phases of the intervention.

Selecting Tasks

The four NAEP measurement tasks (Figure 2) were chosen based on prior research and the potential they had to foreground various linguistic challenges for ELL students. Since NAEP does not report performance data about ELLs, I used NAEP data on Hispanic students to guide the selection of tasks. Though the data are not entirely aligned, this strategy seemed reasonable, as 79% of the students in the Hispanic category are ELLs (McKeon, 2005).

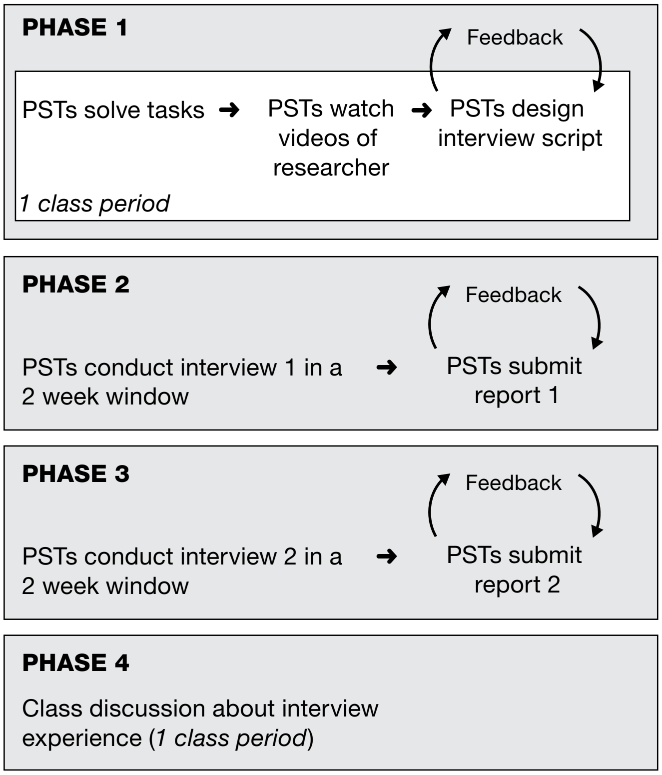
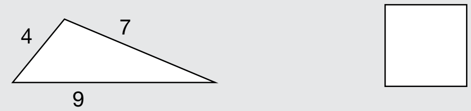


Figure 1. **Phases of the intervention.**

Lubienski (2003) pointed out that the biggest difference between Whites and Hispanics on the eighth-grade NAEP mathematics exam was in the content area of measurement, and this was the motivation to choose that topic for the interview tasks. Based on the Lubienski article, I assumed that tasks (shown in Figure 2) for which there were "big" differences between the performance of Whites and Hispanics (as shown in Table 1) could possibly reveal interesting linguistic challenges for ELLs with proper probing. Interviews I had previously conducted with ELL students (Fernandes, Anhalt, & Civil, 2009) revealed linguistic

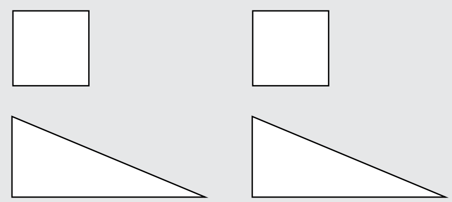
Task 1: The Triangle and Square problem



If both the square and the triangle above have the same perimeter, what is the length of the side of the square?

Task 2: The Area Comparison problem

[*The following cutouts of N (square) and P (triangle) are provided with the problem. Note that the height of P is the same as the side N and the base of P is twice the side.*]



Bob, Carmen, and Tyler were comparing the areas of *N* and *P*. They each conclude the following:

(a) Bob: *N* and *P* have the same area

(b) Carmen: The area of *N* is larger

(c) Tyler: The area of *P* is larger

Task 3: The String problem

Brett needs to cut a piece of string into 4 equal pieces without using a ruler or other measuring instrument. Write directions to tell Brett how to do this.

Take 4: The Tile problem

How many square tiles, 5 inches on a side, does it take to cover a rectangular area that is 50 inches wide and 100 inches long?

Figure 2. **The interview tasks.**

Table 1

NAEP Performance Data on the Four Interview Tasks

Task	Year	Grade level	Difficulty (easy, medium, hard)	Percentage correct: White vs. Hispanic
The Triangle and Square problem	1996	4	Hard	29, 14
The Area Comparison problem	1996	8	Hard	34, 15
The String problem	1996	4	Hard	6, 2
The Tile problem	2009	8	Hard	19, 9

challenges. For example, for the Triangle and Square problem, when an interviewed student read the "if-then" conditional clause, she was not able to solve the problem because of her focus on the word "if"; she claimed that it was possible that the triangle and square did not have the same perimeter.

The second factor for choosing the tasks was to ensure that there was a blend of problems that used different modes of presentation, challenged ELL students on various linguistic facets, and also allowed them to use diverse resources to explain their mathematical thinking. For example, even though the Area Comparison task was considered a "hard" eighth grade problem in NAEP (task 2 in Table 1), it included cutouts that the students could manipulate to explain themselves orally. A written explanation could prove more challenging. The Area Comparison problem would provide opportunities for the PSTs to contrast students' verbal explanations with their written solution. The String problem and Tile problem (tasks 3 and 4 in Figure 2) could pose linguistic challenges because they contain a complex clause (e.g., "into four equal pieces without using a ruler or other measuring instrument") and an embedded clause, (e.g., "square tiles, 5 inches on a side"), which would have to be unpacked by the students to successfully solve the problems. In the case of the String problem, similar to the Area Comparison problem, the students could use concrete materials (i.e., an actual string) to display their thinking, which would again allow the PSTs to contrast the students' oral solution with their written work.

Phase 1: Developing the Interview Script and Pre-interview Preparation

The PSTs were introduced to the project during the first week of the semester. I outlined the goal of the project, which was for PSTs to develop an awareness of the challenges that ELL students faced when learning mathematics in English-only classrooms and the resources that these students used to communicate mathematically. During the same class period, the PSTs solved the interview tasks on their own, and there was an in-class discussion about possible challenges that ELL students could encounter when they solved the same problems. In these initial discussions, the PSTs pointed to possible mathematical challenges that the students could face, such as not knowing how to find the area or perimeter of a shape. In terms of linguistic challenges, the PSTs pointed mostly to vocabulary (e.g., students not knowing the meaning of "measuring instrument"). Because the PSTs had never interviewed students, I presented examples of a researcher interviewing two ELL students about the Triangle and Square problem. One of the clips highlighted the challenge that an ELL student had with the "if-then" conditional clause and the probing questions that the researcher asked to clarify the student's thinking. I also discussed my own experience with interviewing students and additional challenges, such as confusion between area and perimeter.

After our discussion, the PSTs brainstormed in their groups and developed an interview script for the four problems that encompassed possible scenarios that could play out during the interview. In the feedback that I provided, I emphasized that the purpose of the interview was not only to determine if the students could get the correct answer but also to understand their thinking and, if necessary, to provide them with appropriate scaffolding so that they could eventually solve the problem. In keeping with Moschkovich's (2002) ideas of viewing the resources that students bring to the classroom as assets rather than liabilities, I encouraged the PSTs to also accept gestures and drawings as an integral part of the students' explanation of their thinking process.

Phase 2: The First Interview and Report

The PSTs completed the first interview in a two-week window. They visited the intermediate school (fifth and sixth grade) and interviewed ELL students selected by the English as a Second Language (ESL) teachers. Each interview was conducted by a pair of PSTs, one acting as the interviewer and the other responsible for setting up the camera and taking notes. The latter PST could also ask questions if he or she felt the need to do so. For those PSTs that did not have a partner, I provided filming support. The PSTs began by introducing themselves and the project to the ELL student; they were encouraged to have an informal discussion with the ELL student to make him or her feel comfortable during the process. The PSTs provided the student with the first task and allowed some time for the student to solve the problem independently. Once the student indicated that he or she had finished, the PSTs engaged him or her in an interaction to understand the student's solution and probe him or her further. In some cases, the PSTs began this interaction earlier, if the student asked a question about the task that he or she was reading. Because the school placed time constraints on the activity, the PSTs engaged the students for 40-45 minutes and in some cases skipped the fourth task (the Tile problem).

After the interviews, the PSTs were required to submit a detailed report with guiding questions (see Figure 3) based on Mason's (2002) constructs of providing *accounts-of* and *accounts-for.* The guiding questions were designed to spur the PSTs to notice aspects of language that may have influenced the mathematical performance of the student. The *accounts-of* questions related to detailed descriptions of what the student did on his or

Accounts-of

1. What did the student do on his or her own? Provide details.
2. What support did you provide, if any, after the student worked on the problem independently? Provide details about the scaffolding process that you may have used.

Accounts-for

3. In your opinion, what did the student find challenging about these questions? Provide evidence from your descriptions for each task and consider both the mathematics and language.
4. In your opinion, what strengths and resources did the student bring to the problems? Provide evidence from the descriptions and consider both the mathematics and language.
5. Note any other comments about the student's thinking or language or your interaction with the student.
6. Comment on the presence of concrete materials (cutouts, graph paper, string, etc.) and drawings in the problems. Did they help or hinder the student? What role do you see concrete materials and drawings playing in ELLs' learning of mathematics? Why? Provide details.
7. Comment on the student's writing for questions that required written responses. Provide details.

Other

8. In your opinion, what sort of support would this student need in the classroom to understand and do well in math? Explain with examples.
9. Overall, what did you learn about ELL students' mathematical thinking and teaching mathematics to ELL students? Elaborate at least three points in detail.
10. What was your biggest surprise in the interview?

Figure 3. **Guiding questions for the reports.**

her own and what he or she did with assistance (questions 1 and 2). Once the PSTs answered these for each of the four tasks, the *accounts-for* questions (questions 3–7) required them to take a more holistic view and go back over their descriptions of the four (or three) completed tasks and notice patterns in aspects that were challenging to the student, the resources that the student employed, and the student's use of concrete materials, communication, and writing. Further, the "other" questions required PSTs to make inferences about the support that ELL students would need in the classroom and what they learned about the teaching and learning of mathematics to ELL students.

For some questions (e.g., 3 and 4), where there was a chance that the PSTs could overlook the linguistic aspects of the student's responses, I explicitly asked them to consider the language in addition to the mathematics. I provided the guiding questions to the PSTs before the interview to help them prepare probing questions ahead of time. The goal of working within this structured framework was to maximize the PSTs' opportunities to focus on linguistic aspects that arose during their interactions with the students.

Note that the guiding questions themselves would not elicit *accounts-of* or *accounts-for*; it was through the process of instructor feedback and PSTs reworking their written reports that the descriptions and evaluations would come to resemble *accounts-of* and *accounts-for* as described by Mason (2002). The guiding questions are useful to the instructor to assist the PSTs in moving their writing in this direction by emphasizing descriptions for the first set of questions and emphasizing evaluations and judgment for the second set.

Phase 2: Feedback on the First Report

The PSTs submitted their reports electronically for feedback and grading. The reports were graded based on four criteria: detailed descriptions, quality scaffolding, insightful reflections, and depth of language issues covered. These criteria were shared with the PSTs before they conducted their first interview. The PSTs were required to provide details of how the interview unfolded so that another person, if he or she was present, could confirm the details. Thus the PSTs were to avoid making judgments about the student's statements and were instead instructed to report on what happened and what was said in detail. The quality of scaffolding criterion examined whether the PSTs' questions were leading rather than getting the student to grapple with the problem. Insightful reflections referred to the quality of the responses for questions 3–10. More weight was given to claims that were backed up in the descriptions. Finally, I examined the linguistic issues that the PSTs discussed in their answers to questions 3–10.

I first provided the PSTs with feedback on their reports and asked most of them to add more detail or to justify a statement with an example. In some cases, I watched part of the videotape together with the PSTs, and we jointly discussed areas where they could provide more detail and talked about possible linguistic issues that they might consider for further analysis. I later graded the reports after they had a chance to reflect and incorporate my feedback.

The reports allowed me to focus on their descriptions, assist with their interpretations, and provide suggestions on improving their second interview. This was a key part of the intervention. For example, if a PST mentioned that a student did not understand the concept of area, I asked the PST to think about other ways that the student could express his or her understanding of area besides the use of a formula, such as pointing to the area of the table or floor, shading area in a figure, or using graph paper. By accepting a broad range of student approaches, some of which may not seem "mathematical" (according to the PSTs) at the outset, the PSTs could appreciate the students' thinking and understand linguistic challenges and how the students were using resources in conjunction with speech to make meaning and partake in mathematical practices (Moschkovich, 2002).

Phases 3 and 4: Second Interview and Report and Experience Sharing

Because most of the PSTs were interviewing students for the first time, the second iteration allowed them to have richer interactions and improve their probing of the student based on what they learned from the experiences in the first interview. Based on the feedback from the first report, the PSTs refined their interview script and interviewed a different ELL student toward the end of the semester. Once again they submitted a report that I graded, and in some cases I asked them to revise their reports. As a conclusion to the project, the PSTs shared something new that they had learned about the teaching and learning of mathematics to ELL students during an in-class discussion. The next section discusses the impact of the intervention.

Impact of the Intervention

The major goal of the intervention was to build awareness among the PSTs of the challenges that ELL students face and the resources that ELL students draw on to communicate their mathematical thinking. To document the impact of the intervention with respect to this goal, I initially focused on the PSTs' responses to questions 3, 4 and 9 (see Figure 3). I created a separate document that compiled each of the 31 PSTs' responses from both reports for these three questions and used this as the starting point for examining the impact of the intervention. I specifically looked at the linguistic challenges that the PSTs described and the resources the PSTs mentioned that the ELL students used in connection to these challenges. I triangulated these points with their responses to other questions, particularly the descriptions they provided in response to questions 1 and 2. I also examined portions of the videotape where they were interacting with the ELL students to ensure that their interpretation was grounded in their interactions. Further, I had close interactions with all the PSTs during the project, and during the feedback process I clarified my interpretation of their statements.

The following sections will describe the challenges (understanding the questions and writing) and the resources (using concrete materials to assist with communication) reported by the PSTs. Further sections will discuss what the PSTs reported on learning through the task-based interviews and the few cases where prior deficit beliefs about ELLs were reinforced.

Linguistic Challenges

All 31 PSTs brought up the linguistic challenges that the ELL students faced during the interviews. In particular, these challenges arose in students' understanding of the questions and explaining their thinking in writing.

Understanding the Question

By allowing the ELL students to initially work independently on the task, the PSTs noticed challenges students faced in understanding the question. Some ELL students read the problem multiple times, others asked for the meaning of words that were unclear, and some guessed at what the question was asking by using portions of the problem that they understood. In some cases, the PSTs helped the students understand the question by getting them to read and explain the different parts back to them. By doing so, the PSTs were able to isolate parts of the question that were challenging to the students and assist them with the language. In some cases, especially for the Triangle and Square problem, the ELL students were able to solve the task with assistance, and this convinced the PSTs that the ELLs were challenged with the language in the question. One PST wrote,

> **I learned that ELL students' difficulty with language does affect their math [performance], but it does not affect their mathematical thinking. The student I worked with had difficulty understanding the language of the question. . . . But, once the student understood the question she was able to mathematically think correctly and figure out the answer to the question.**

The PST observed that assistance with the language in a question could make a difference in whether the student used an appropriate procedure to solve the problem. In the String problem, a number of PSTs observed that the ELLs were not using the whole string to form four equal pieces. On further probing, they linked the linguistic challenge to the phrase "a piece of string," which the ELLs assumed to mean a part of the string that was provided. In these cases, the students were able to rectify their solution method based on the assistance they got from the PSTs. One PST asked the ELL student to think of the string as Twizzlers© (a type of candy) that had to be divided among four friends. This scaffolding from the PST helped the ELL student understand the problem and then solve it.

The PSTs noticed the challenge for the students in the Tile problem lay in the phrase "five inches on a side," which they tended to ignore or misinterpret in their solution. For example, one ELL student ignored this phrase and counted all the squares on the graph paper that was offered. The PSTs also pointed to the numerous pieces of information that the students had to coordinate to solve the problem. For example, one PST wrote, "He had to work with inches, tiles, a small square, a big rectangle. He also had to figure out how all of them were connected in order to find the final answer." In the Tile problem, where the students were required to integrate the information and determine the mathematical approach they would take for a solution, most of the PSTs reported the challenges facing the student as both linguistic and mathematical. After observing how ELL students grappled with understanding the questions, some PSTs suggested that the questions could be modified with simpler language to ensure that the ELL students understood them.

Writing

The PSTs noticed that the ELLs were challenged by explaining their thinking in writing, and some preferred just an oral explanation for their solution strategy. In most cases, the PSTs mentioned that students' written work was difficult to understand. Besides commenting on the incorrect spelling and grammar, the PSTs noted that the ELL students tended to write the way they spoke: "…and cut like two of the pieces…." This is common because students are familiar with spoken communication and draw on this resource for their writing if they have not been introduced to various genres of writing and ways of presenting their ideas (Gibbons, 2002).

Some PSTs commented on the structure of the sentences that the students used and reported that these were "run-on sentences". This referred to sentences which made use of conjunctions to chain their ideas: "Well, first take each end of the string and connect them, then take the other end that the string made and connect it to the two ends of the string, you then would cut the pieces of each end." Again, the use of chained clauses are characteristic of early writers who need explicit instruction to develop academic writing using more condensed clause structures (Schleppegrell, 2004).

In the case of the String problem, many PSTs were successful in getting the student to rethink their written explanation to achieve clarity by using the string to work through the steps and illustrate to the ELL students that their oral solution did not match their written instructions. This prompted the students to correctly modify their writing to match the sequence of steps that they used to cut the string. Further discussion of the use of concrete materials is described in the next section.

Resources

In their discussion about the resources that ELLs used during the interview, concrete materials featured prominently in solving the problem and communicating their solution. The use of concrete materials, such as the string and the cutouts, were especially useful for the students for whom providing a coherent written solution was challenging. These students could use the materials, along with informal language, to demonstrate their solution. One PST says,

> **I can't stress enough how helpful the string and the cutouts were for [student name]. She used the cutouts to solve the area problem. Not only did they help her solve it, but they were a big factor in her communicating how she did it. . . . Where her writing was a little confusing, she was able to demonstrate using the string very clearly. . . . I think the availability of concrete materials to aid in understanding and communicating are vital for these [ELL] students and should be used extensively in the classroom.**

Note that, even though the PSTs thought that the use of concrete materials would be beneficial in work with ELL students, there were some who noticed that just providing the concrete materials was not enough and some support also was required. For example, in the Tile problem, a PST noticed that the ELL student assumed the square on the graph paper represented a tile with unit dimensions instead of 5 x 5, the dimensions specified in the problem. The PST had to help the student use the graph paper to appropriately represent and solve the problem. Overall, most of the PSTs reported that the concrete materials were a resource that ELL students employed to understand and communicate their thinking. The concrete materials also opened opportunities for the PSTs to understand the ELLs students' thinking and in some cases, such as the String problem, got them to modify or revise their solution.

What PSTs Learned From the Interviews

Based on their interview experience, most PSTs concluded that language could prove to be a challenge for ELL students. In the words of one PST,

> **[The ELL] is learning a new language and learning a new concept (math) and that is a lot for a child to do together. It's like double the work.**

Here the PST seems to understand that ELL students who are learning the content and the language at the same time face an added cognitive load (Campbell, Adams, & Davis, 2007). Most PSTs discussed adjustments that they would make to their mathematics class to account for the extra cognitive load that the language posed for the ELL students. For example, the PSTs reported that they would allow ELL students more time to process information, slow down their speech, and integrate strategies that would help the students with reading and writing the content. For example, one PST recommends that "reading, writing, and math are all covered in [the math] class," and goes further in stating that teachers should provide opportunities for students to integrate aspects of the language as they learn the mathematics content. Such opportunities could take the form of having students read the mathematics problem, interact with peers as they solve the problem, and provide a written explanation of their thinking. Thus the PSTs went beyond being aware of ELLs' needs and challenges to learning specific strategies that aligned with the research on best practices for working with ELLs.

The PSTs also reported that there was a lot of diversity among the ELL students that they interviewed and thus mentioned that they would avoid making "sweeping generalizations" in their future encounters with this group of students. For example, some PSTs mentioned that they would be careful not to automatically conclude that ELL students struggled with mathematics. After conducting the interviews and interacting with the ELL students, many PSTs were surprised that the students could speak English, as they assumed that the students would have difficulty speaking. However, in some cases, the PSTs assumed that students' conversational proficiency meant that these ELL students were no different from non-ELL students: "I don't know if you could really call these kids ELL students because it seems like they already know the language fluently." These PSTs seemed to assume that fluency in *conversational language* automatically meant proficiency in *academic language*.

Reinforcing Beliefs

The interviews, in a few cases, seemed to reinforce prior beliefs that PSTs had about mathematics being universal. This was usually the case when ELL students successfully solved the problems with minimal assistance with the mathematical concepts. One PST expressed this idea as "two plus two is four no matter what language or dialect you speak." This particular PST had experience teaching algebra in eighth grade and did not consider the linguistic assistance he provided the ELL student to be *linguistic* assistance. Rather, he considered it to be mathematical assistance that he would provide to ELL and non-ELL students alike. For example, in the Tile problem, when the student struggled to understand the phrase "five inches on a side," he used the cutout from the Area Comparison problem to demonstrate the dimensions of the tile. Later he used the cutout to illustrate how the tile would cover the rectangular area, which prompted the student to successfully work out the number of tiles that covered the space. In our interactions, he explained that he provided such assistance to non-ELL students as well; thus, according to him, this illustrated a mathematical challenge rather than a linguistic challenge. As such, he reiterated that mathematics was universal and that the same issues that challenged ELL students also challenged non-ELL students. Having such a belief ignores the fact that ELL students face an additional cognitive load because of the language (Campbell, Adams, & Davis, 2007).

Deficit beliefs about ELLs, such as "ELL students typically haven't had proper schooling before arriving here and generally do not receive proper help at home," were expressed by a few PSTs who interviewed students who needed a lot of prompting to solve the problem. However, in these cases, I also observed that the PSTs expected the students to express their mathematical knowledge in very narrow ways that fit with how they themselves would solve the problem. For example, for the Tile problem, one PST expected the student to use division to find the number of tiles. Initially the PST provided graph paper that the student used to work out the total number of tiles by simply multiplying the number of actual squares along the length and width of the sheet and ignoring the dimensions of the tiles in the problem. Instead of attempting to build on the student's approach, the PST tried to funnel the student toward the use of division. When the student struggled to do so, it seemed to reinforce the PST's deficit beliefs about ELL students.

Deficit beliefs about the use of native language were reported by two PSTs, who assumed that the students were taking a long time to work out the problems due to having to translate between English and Spanish. However, there was no overt evidence of this in the videotapes of the interviews. One of these PSTs concluded that translating back and forth would be "extremely taxing" on the student. In essence, these two PSTs' comments in the report seemed to view the native language as a hindrance for the student's mathematical performance rather than an asset that could be used in the classroom. The PSTs statements imply that "taking longer" indicates a lack of understanding—again expressing a narrow view of what it means to know and do mathematics. Research has established that ELLs may take a little longer in calculations; however, this does not reflect their level of mathematical understanding (Moschkovich, 2010).

Others Using the Intervention

Although the intervention took place in a content course for middle school PSTs, it is flexible enough to be integrated into various mathematics content and methods courses and can be used to help PSTs notice the linguistic issues that arise in the mathematics curriculum and how ELL students negotiate them. The interview tasks can consist of NAEP questions that relate to the topics being discussed and share some of the characteristics with the tasks that were used in this study. The instructor can pilot some of the tasks in interviews with ELL students to determine which ones have the potential to benefit the PSTs in their interviews.

Further, the instructor will need some assistance from the schools. In this intervention, the ESL teachers at the school obtained parental permission on my behalf (for videotaping), identified the ELL students to be interviewed, and coordinated the PSTs' visits during the two-week window for each interview. The ESL teachers went even further and outlined their program and how ELLs were classified and answered specific questions that the PST may have had.

Because most PSTs are new to conducting task-based interviews, a significant amount of time is invested at the beginning of the project helping the PSTs write detailed descriptions and notice linguistic aspects in the videotape. In my case, I spent time reading the reports, viewing the videotapes, providing the PSTs with appropriate feedback on their reports, and in some cases also viewing sections of the videotape together with the pairs. I found that the PSTs also learned from informal interactions among themselves as they shared experiences of what worked and what did not work with each other. For example, one PST shared how he pretended not to understand the questions and thus encouraged the ELL student to elaborate and explain the questions and the mathematical thinking to him in great detail. In the future, I plan to incorporate these interactions into the structure of the intervention by building in more discussion time during class. Knowledge of the basics of systemic functional linguistics (e.g., Eggins, 2004) is also essential in understanding the linguistic complexity in the formulation of problems and how this may impact ELL students' communication of their mathematical thinking.

PSTs tend to need more assistance in probing students during the interview and providing detailed descriptions in their reflections at the beginning of the course; as they gain experience over time, they get better. The four PSTs who participated in the second and third cycles of the intervention showed improvement in their probing. For example, one of these PSTs was able to reframe the String problem using a scarf that the student was wearing. The PST first complemented the student on the attractive scarf and then asked her to imagine how she would divide it equally with three other friends who wanted to have the same scarf but could not purchase the same one at the mall. By reframing the problem this way, the PST could get at the student's understanding. I noticed this flexibility in the PSTs' probing as they gained more experience with the interviews. The level of detail that the PSTs provided in their descriptions were more aligned with Mason's notion of *accounts-of* as they made fewer statements that were evaluative or could not be verified by another observer (if one was present).

Discussion

Overall, the interview experience along with the composition of accounts and feedback from the instructor have the potential for helping PSTs notice the linguistic challenges that ELL students face and resources that they use to communicate mathematically. The guiding questions, based on *accounts-of* and *accounts-for*, serve to focus PSTs on the linguistic aspects of students' responses. The potential for noticing is maximized initially when the instructor uses the PSTs' descriptions to provoke further thinking about the possibilities in the student work. This allows the PSTs to look beyond the familiar methods to solve the problems; probe students appropriately; and notice the challenges of understanding the questions and the resources, including gestures, drawings and concrete materials, that ELL students use to build meaning that goes beyond speech. Videotaping the interview allows the instructor and the PST to recall incidents and interpret them in new ways. The videotape is also useful in bringing to the fore incidents, especially those involving linguistic issues, that may not be captured in the initial descriptions as the PSTs may not consider them important. The continued informal interactions with the instructor over the course of the project also add to the PSTs' overall learning.

The interview experience goes beyond fostering PSTs' awareness to developing concrete strategies that assist ELL students and are aligned with best practices advocated in the research. Some of these strategies include isolating linguistic challenges in the wording of a question, using concrete materials and drawings to help the students understand the problem and communicate their thinking, adapting speech, providing more time for the students to work on the problem and communicate their thinking, and analyzing and critiquing the students' written work—all skills that will be useful in PSTs' future classrooms for teaching all students.

In his review of research on how to prepare mainstream secondary content-area teachers to work with ELLs, Bunch (2010) emphasized the need for integrating the focus on language and content so that the teachers have the "opportunity to understand the language demands in their own lessons" and can "capitalize on the linguistic resources that ELLs already bring to the classroom, and create instructional settings that expand students' access to content learning and development of language and literacy" (p. 374). The task-based interviews, along with a framework of *noticing* can provide the needed integration of the content and the language so that PSTs can notice the linguistic challenges that ELLs face and the resources that they draw on to communicate their thinking. The ultimate aim of teacher preparation is not to prepare expert teachers, but to prepare teachers who can continually learn from their teaching (Hiebert, Morris, Berk, & Jansen, 2007). Developing their skills of interviewing and *noticing* can help teachers continually learn from *all* their students.

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Author Note

The author would like to thank Leslie Kahn, Tamsin Meaney, Josh Chesler, Vic Cifarelli, Laura McLeman, Mary Capraro, and the anonymous reviewers for valuable feedback at various stages of the preparation of this manuscript.

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The Role of Writing Prompts in a Statistical Knowledge for Teaching Course

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Teachers of grades Pre-K-8 are charged with the responsibility of developing children's statistical thinking. Hence, strategies are needed to foster statistical knowledge for teaching (SKT). This report describes how writing prompts were used as an integral part of a semester-long undergraduate course focused on building SKT. Writing prompts were designed to help assess and develop the subject matter knowledge and pedagogical content knowledge of prospective teachers. The methods used to design the prompts are described. Responses to a sample prompt are provided to illustrate how the writing prompts served as tools for formative assessment. Pretests and posttests indicated that prospective teachers developed both SKT and knowledge of introductory college-level statistics during the course. It is suggested that teacher educators employ and refine the prompts in their own courses, as the method used for writing and assessing the prompts is applicable to a broad range of statistics and mathematics courses for teachers.

Key words: Statistical knowledge for teaching, Mathematical knowledge for teaching, SOLO Taxonomy, Writing prompts, Formative assessment

Current scholarship in teacher education reveals the complexity of the knowledge needed by teachers. Subject matter knowledge alone is not sufficient. The Learning Mathematics for Teaching (LMT) project characterized mathematical knowledge for teaching (MKT) as consisting of two primary elements: subject matter knowledge and pedagogical content knowledge (Hill, Ball, & Schilling, 2008). Pedagogical content knowledge helps teachers make subject matter comprehensible to students. It has been described as a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own form of professional understanding" (Shulman, 1987, p. 8). Components of subject matter knowledge and pedagogical content knowledge hypothesized by the LMT project are shown in Figure 1.

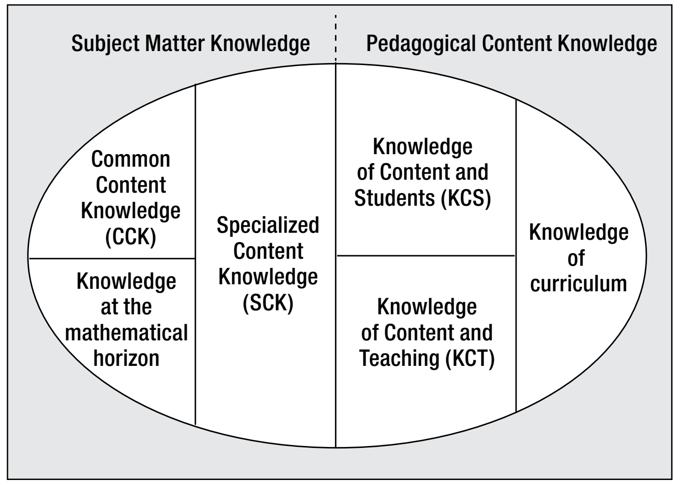

***Figure 1.* Hypothesized components of subject matter knowledge and pedagogical content knowledge (Hill, Ball, & Schilling, 2008, p. 377).**

This report describes how I used writing prompts in a semester-long undergraduate course devoted to building prospective Pre-K-8 teachers' subject matter knowledge and pedagogical content knowledge for teaching statistics. In this article, I use the term "statistical knowledge for teaching" (SKT) rather than MKT to acknowledge statistics and mathematics as distinct disciplines (Groth, 2007; Moore, 1988). For example, many statistical activities, such as study design, survey question design, and measurement, have substantial nonmathematical components (Rossman, Chance, & Medina, 2006). Although the LMT model explicitly focuses on MKT, researchers have found it to be of use in describing SKT as well (Burgess, 2011). The degree of overlap between the two disciplines makes it feasible to ground the discussion of SKT in a theory of MKT (Groth, 2007), and the methods I describe for designing and assessing writing prompts are not restricted to use in statistics courses. Some specific examples of subject matter knowledge and pedagogical content knowledge for statistics will be discussed next.

Subject Matter Knowledge

Subject matter knowledge includes common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon (Hill, Ball, & Schilling, 2008). *Common content knowledge* is that which is required in teaching as well as in other professions. Examples include knowing how to compute and interpret frequently used measures of center and spread, understanding the idea of random sampling, and recognizing variability as a central object of study in statistics (Groth, 2007).

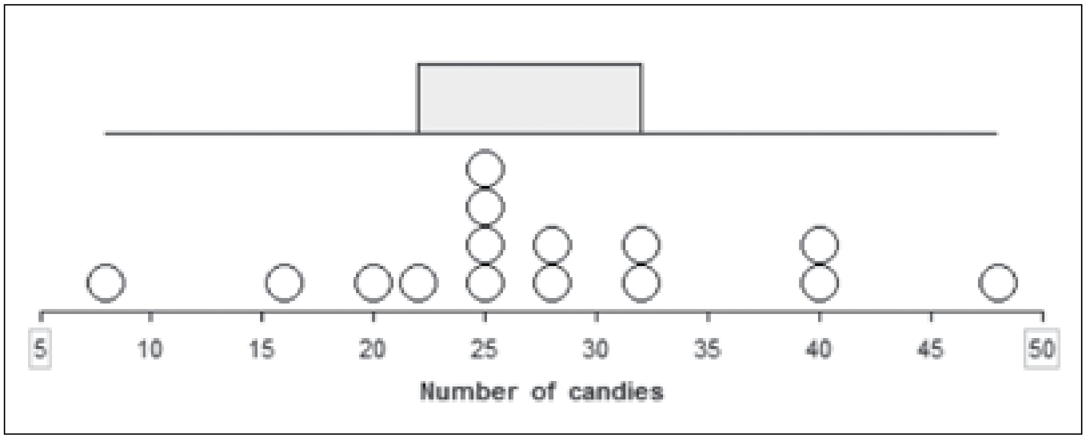

***Figure 2.* Representing a data set for number of candies per student with a dot plot and hat plot.**

Specialized content knowledge is that which is unique to teaching. It allows teachers to select representations to make subject matter comprehensible. For instance, hat plots produced using dynamic statistics software (Konold & Miller, 2005; Figure 2) are not conventional data displays, but they provide a useful intermediate step between reading dot plots and box plots (Watson, 2008). Children often have difficulty interpreting box plots because they condense the data to display summary statistics rather than showing individual values (Bakker, Biehler, & Konold, 2005). Hat plots address this difficulty because they are generally displayed above plots showing individual values, as in Figure 2. By examining both representations simultaneously, students can begin to understand how individual values contribute to a more condensed display. The median is initially not included in a hat plot to avoid another intuitive difficulty: in a box plot, the median is usually closer to one of the quartiles than the other, even though the same number of data points resides within each quarter of a box plot (Watson, Fitzallen, Wilson, & Creed, 2008). Hat plots allow students to initially focus on the more intuitive idea of "modal clump" (e.g., the middle 50% of the data is highlighted in Figure 2), which connects to children's tendencies to partition data into low, middle, and high categories (Konold et al., 2002). Once students understand how a display can condense data and partition it into groups, adding the median to a hat plot showing the middle 50% of the data can complete the transition from dot plots to box plots. The hat plot representation, therefore, can be considered an element of specialized knowledge because it helps make box plots comprehensible, and it was invented to serve this purpose rather than to be widely used among those outside the teaching profession.

Horizon knowledge helps teachers understand how activities done during a given lesson foreshadow more advanced ideas to be studied later on. For example, elementary school students work on describing populations (e.g., students in their classroom) using descriptive statistics and graphs, but a transition to using samples to make inferences about larger populations must eventually be made. Teachers who have horizon knowledge of formal statistical inference can pose questions that prompt students to think about the extent to which data from one class may generalize to a larger population (e.g., all students in school or all students in the country). They can also look for opportunities to emphasize ideas that comprise the foundation for formal inference, such as sample size, randomness, sampling variability, and bias (Ben-Zvi, Gil, & Apel, 2007). As they do so, teachers can bring students progressively closer to techniques of formal inference by gradually formalizing their early statistical investigations.

Pedagogical Content Knowledge

Pedagogical content knowledge includes knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. Each component marks out a type of knowledge needed specifically for tasks related to teaching.

Knowledge of content and students can be described as "content knowledge intertwined with knowledge of how students think about, know, or learn . . . content" (Hill, Ball, & Schilling, 2008, p. 375). This type of knowledge allows teachers to anticipate difficulties students will have in learning a subject and address them when planning and implementing lessons. Research provides a fair amount of information about developmental levels through which Pre-K–8 students are likely to pass as they learn statistics and probability. Jones et al. (2000) mapped levels of thinking one can expect from elementary school students in regard to describing, organizing, representing, and analyzing data. Mooney (2002) did the same for middle school students. One of the key insights from these studies is that students often use their own idiosyncratic strategies to handle data before progressing to conventional methods. Knowing about common student strategies can help teachers anticipate potential inroads and obstacles in developing students' statistical thinking.

Knowledge of content and teaching entails having a repertoire of content-specific strategies for teaching concepts. For instance, it is beneficial for teachers to know how to help students understand the arithmetic mean as a fair share and as a balance point. To portray the mean as a fair share, students can be given snap cubes to represent the numbers of a given object belonging to each person in a group. The cubes can then be piled together and redistributed so that everyone has the same amount (if the total number of cubes is not a multiple of group size, distributing fractional amounts of a cube can be discussed). The amount each person receives is the mean. To help students understand mean as a balance point, teachers can give students data on the number of pets each student in a class has. Sticky notes can be used to construct a dot plot of the data set, and students can move all of them to the mean value of the data set. To preserve the original mean, they can experiment with ways to rearrange the sticky notes so the balance point remains at the original mean. In doing so, they produce a number of data sets that all have the same mean (Franklin et al., 2007).

Knowledge of curriculum suggests knowing the structural characteristics of curricula. One part of developing curriculum knowledge can be coming to understand different philosophies underpinning Pre-K–8 curricular materials for teaching statistics. Several reform-oriented curriculum series funded by the National Science Foundation (NSF) are based on principles of inquiry-oriented instruction that may be unfamiliar to prospective teachers whose school experiences were more traditional in nature (Senk & Thompson, 2003). Lloyd and Behm (2005) found that when presented with reform-oriented curriculum materials, prospective teachers tended to gravitate toward traditional-looking elements of the texts rather than inquiry-oriented ones. Understanding the purpose and benefits of inquiry-oriented instruction enables teachers to implement innovative curricula with fidelity to the intentions of the curriculum designers.

The Statistical Knowledge for Teaching (SKT) Course

I used the Learning Mathematics for Teaching (LMT) conceptualizations of subject matter knowledge and pedagogical content knowledge as starting points in designing a one-semester SKT-focused course for prospective Pre-K–8 teachers. The use of the LMT framework prompted me to go beyond just common content knowledge goals for the course. Although common content knowledge is important, it is ideally developed in tandem with other types of subject matter knowledge and pedagogical content knowledge. An immediate implication of the LMT framework is that it is not adequate to ask prospective teachers simply to solve mathematics and statistics problems in such a course, even if those problems focus on conceptual understanding (Hiebert & Lefevre, 1986) and have high levels of cognitive demand (Smith & Stein, 1998). Therefore, I set out to design course experiences that required engagement with all aspects of the LMT framework.

The core teaching strategies used in the course and their connections to the LMT framework are shown in Figure 3. Inquiry-oriented statistics activities relevant to

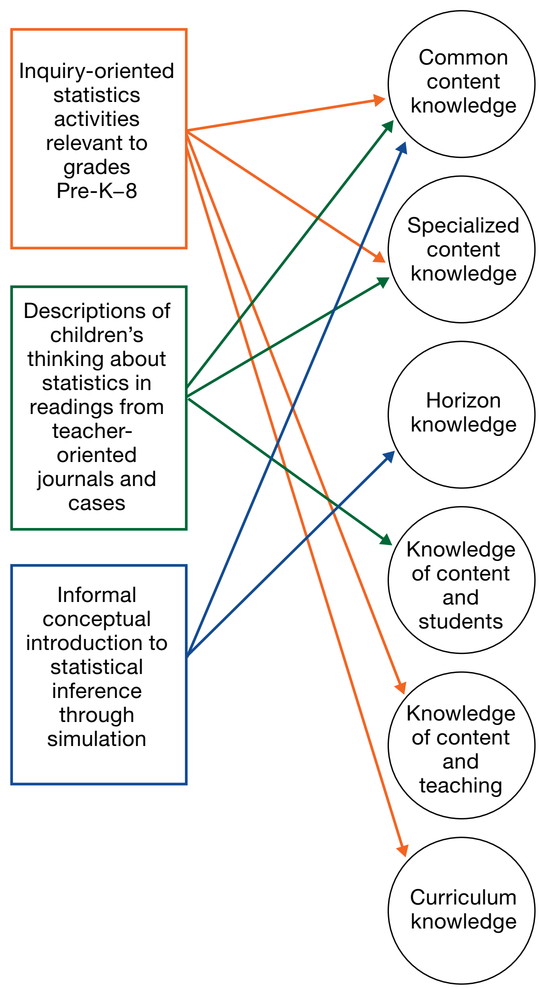

Figure 3. **Connections between core instructional strategies and the LMT framework.**

grades Pre-K–8 were selected from the required course textbook (Perkowski & Perkowski, 2007) and other sources (e.g., Burns, 2000; Rossman & Chance, 2008; Scheaffer, Gnanadesikan, Watkins, & Witmer, 1996). The required textbook integrated activities from NSF-funded curricula (Senk & Thompson, 2003). These activities were intended to build conceptual understanding of statistics (common knowledge) while also providing ideas for teaching specific content (knowledge of content and teaching) and an introduction to inquiry-oriented curricula (curriculum knowledge). To supplement and extend the inquiry-oriented activities done in class, I selected readings from *Teaching Children Mathematics* and *Mathematics Teaching in the Middle School*. These readings, as well as two classroom cases I selected from *Discovering Mathematical Ideas* (DMI) (Russell, Schifter, Bastable, Konold, & Higgins, 2002), helped build knowledge of content and students by providing descriptions of children's thinking about statistics. They also helped build specialized knowledge by introducing representations suitable for making content understandable to children (e.g., hat plots) and common knowledge by prompting readers to think conceptually about core statistics content (e.g., choosing between mean and median to describe data). Near the end of the semester, a unit introducing formal inference was included. The unit used materials described by Garfield and Ben-Zvi (2008) to provide an intuitive foundation on the meaning of sampling distributions, hypothesis testing, and confidence intervals through simulation. It was included in the course to foster horizon knowledge by providing a sense of statistical content studied beyond the Pre-K–8 curriculum and to build common content knowledge of inference ordinarily included in college-level statistics courses.

This report will focus in-depth on writing prompts used in conjunction with the course readings from *Teaching Children Mathematics* and *Mathematics Teaching in the Middle School*. The relationships among statistics content, the selected readings, and supporting course activities are shown in Table 1. Although the writing prompts were used in conjunction with most of the statistical topics included in the course, they were not used for all. Specifically, units on bivariate data and inference were not accompanied by articles and writing prompts. However, writing prompt sets did comprise more than 60% of the homework assignments for the course.

Writing Prompts in the SKT Course

I chose writing as a means to help prospective teachers analyze the teacher-oriented journal articles because it encourages learners to place organizational structures on their thinking (Vygotsky, 1987). As a self-reflective activity, writing supports learners' metacognition, enabling them to select and employ appropriate problem-solving strategies (Pugalee, 2004). Writing about a text can also help support generative reading of it. Generative reading involves applying background knowledge to interpretation of a text, thinking about relationships among ideas within a text and across texts, and identifying important concepts (Borasi, Siegel, Fonzi, & Smith, 1998).

Figure 4 provides an overview of the design and use of writing prompts in the SKT course. I wrote five prompts for each article. (The full set of writing prompts is available. See "Supplement: Assignments." Prompts addressed both subject matter knowledge and pedagogical content knowledge. Some of the questions in the prompts required literal reading, and others required generative reading. As I read responses to the prompts, I had both summative and formative assessment purposes in mind. I used a rubric to assign summative scores to each set of writing prompts, and I examined responses to selected prompts in more depth to gain insight about adjustments to the course to help advance prospective teachers' learning. It should be noted that although the process outlined in Figure 4 was developed in a statistics course for prospective elementary school teachers, it is not necessarily restricted to a single subject area or grade band. Details about the process are provided in the remainder of this section.

Each of the writing prompts was designed to address one or more of the six components of the LMT framework described earlier: common knowledge, specialized knowledge, horizon knowledge, knowledge of content and students, knowledge of content and teaching, and curriculum knowledge (Hill, Ball, & Schilling, 2008). Six types of questions described by Day and Park (2005) were used within the prompts to encourage active reading: literal comprehension, reorganization, inference, prediction, evaluation, and personal response. Examples of each type of prompt and their alignment with the LMT framework are provided in Table 2. Literal comprehension questions are those that can be answered directly from a portion of the text. The remaining five types of questions require generative reading. Reorganization questions require piecing together information from various parts of the text. Inference questions go beyond literal reading of the text to prompt students to draw on background knowledge and experiences while reading to formulate a response. Prediction questions involve extending a text by drawing on knowledge obtained from reading it. Evaluation questions prompt readers to express reasons for agreement or disagreement with a portion of the text. Personal response questions prompt readers to express their feelings about the text and its content. Some writing prompts contained more than one of the six types of comprehension questions, and some also were

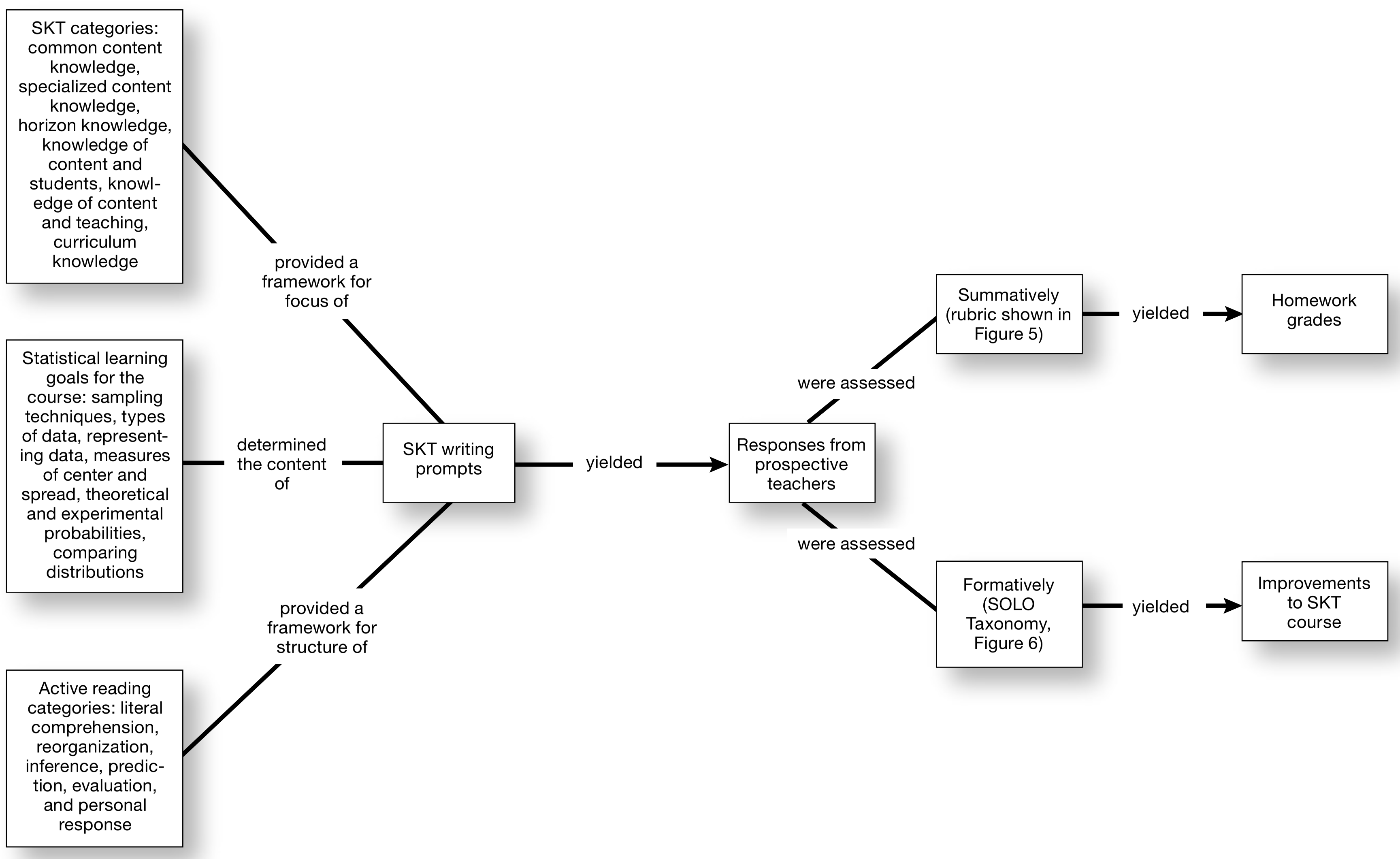

Figure 4. **Summary of design and use of writing prompts in SKT course.**

Table 1
Relationships Among Statistical Content for SKT Course, Readings, and Supporting Course Activities

Statistics content	Selected readings	Sample supporting course activities
Sampling techniques	"How do students think about statistical sampling before instruction?" (Jacobs, 1999) "Capture and recapture your students' interest in statistics" (Morita, 1999)	Completing "Random Rectangles" activity (Scheaffer, Gnanadesikan, Watkins, & Witmer, 1996) to establish the distinction between random and subjective sampling and to compare distributions of statistics produced by them.
Types of data	"Statistics in the elementary grades: Exploring distributions of data" (Franklin & Mewborn, 2008) "It's a fird! Can you compute a median of categorical data?" (Leavy, Friel, & Mamer, 2009)	Discussing classroom cases from *Developing Mathematical Ideas* series (Russell, Schifter, Bastable, Konold, & Higgins, 2002) to become familiar with how children tend to organize both quantitative and categorical data.
Representing data	"Reflecting on students' understanding of data" (McClain, 1999) "Students' interpretations of misleading graphs" (Harper, 2004) "The representational value of hats" (Watson, Fitzallen, Wilson, & Creed, 2008)	Analyzing hat plots and box plots with TinkerPlots (Konold & Miller, 2005) and Fathom (Finzer, 2002) to understand how a hat plot serves as a transitional representation between a dot plot and a box plot and to analyze how changes in a data set correspond to changes in accompanying box plot and hat plot graphical representations.
Measuring center and spread	"Mean and median: Are they really so easy?" (Zawojewski & Shaughnessy, 2000) "Statistics in the middle grades: Understanding center and spread" (Kader & Mamer, 2008)	Solving problems from *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2004) to understand the mean conceptually and to become familiar with the structure and goals of reform-oriented mathematics curricula.
Theoretical and experimental probabilities	"Predictions and probability" (McMillen, 2008) "Providing opportunities to learn probability concepts" (Tarr, 2002) "Enriching students' mathematical intuitions with probability games and tree diagrams" (Aspinwall & Shaw, 2000)	Completing "Numbers on a Line" activity (Burns, 2000) to compare theoretical and experimental probabilities for rolls of a pair of dice using a physical simulation. The physical simulation set the stage for computer-based simulations that took advantage of technology to run a large number of trials and analyze the resulting distributions.
Analyzing distributions	"Proportional reasoning: Lessons from research in data and chance" (Watson & Shaughnessy, 2004)	Describing shape, center, and spread of distributions of numbers of orange candies in a sample in an activity from *Workshop Statistics* (Rossman & Chance, 2008)

Table 2
Sample SKT Writing Prompts

Sample writing prompt	Article	Reading comprehension question types	Relevant SKT components
In your own words, explain how snap cubes can be used to determine the arithmetic mean of a set of quantitative data.	Franklin and Mewborn (2008)	Literal comprehension	Knowledge of content and teaching
How are hat plots similar to box-and-whisker plots? How are they different?	Watson, Fitzallen, Wilson, and Creed (2008)	Reorganization	Specialized content knowledge
On p. 438, the authors commented in regard to item 3, "This type of item assesses students' conceptual understanding of mean." Explain what the authors may mean by "conceptual understanding." How is it different from other types of understanding?	Zawojewski and Shaughnessy (2000)	Inference	Curriculum knowledge

Table 2—Continued

Sample writing prompt	Article	Reading comprehension question types	Relevant SKT components
Why did Eric, Paloma, and Kenji each have different estimates for the number of fish in the population of Lake Amanda? How much variability in student estimates do you think you would have if you had 25 students in your class? Why?	Morita (1999)	Reorganization, prediction	Common content knowledge
On p. 417, the author claimed, "They (the students) have formulated on their own this fundamental idea in statistical inference: larger samples tend to yield less sampling variability and therefore more accuracy." Do you agree with this claim? Why or why not? What evidence is provided in the article to support the claim?	Morita (1999)	Evaluation	Horizon knowledge, knowledge of content and students
In your own words, describe the different methods students used in combining their individual samples during the capture-recapture activity. Which method do you find the most appealing? Why?	Morita (1999)	Personal response	Knowledge of content and students

designed to assess more than one of the six components of SKT, as illustrated in Table 2.

The writing prompts served as summative assessments in that they were assigned homework grades. I graded the sets of writing prompts along five dimensions: inclusion of required information, clarity and organization, conciseness, depth of thought, and evidence of understanding. The rubric used to grade each set is shown in Figure 5. Prospective teachers were shown the rubric before completing the assignments. Along with giving them a sense of how the assignments would be graded, the rubric allowed me to provide feedback and assign grades in an efficient manner.

Grading with the rubric was only the first stage in my analysis of responses. I analyzed responses to prompts that elicited a wide range of thinking in more depth by using the Structure of the Observed Learning Outcome (SOLO) taxonomy (Biggs & Collis, 1982; Biggs & Tang, 2007), which has been gainfully employed in several studies of statistical thinking (e.g., Groth & Bergner, 2006; Jones et al., 2000; Mooney, 2002; Watson & Moritz, 2000). Figure 6 shows a diagram summarizing the SOLO levels of response observed for a prompt. It is based on a visual model devised by Biggs and Collis (1982). Figure 6 shows that *prestructural* responses draw on information not directly relevant to a task. *Unistructural* responses draw on a single relevant aspect, and *multistructural* responses draw on more than one aspect. *Relational*-level responses include connections among relevant aspects, and *extended abstract* responses include aspects beyond those required for a successful response. It should be noted that using the SOLO taxonomy does not limit one to identifying only five levels of response. Pegg and Davey (1998) theorized that the middle three SOLO levels form a repeating cycle that can be extended indefinitely. The depth of analysis provided by using SOLO was valuable for the purpose of formative assessment, as the levels of response to writing prompts indicated potentially fruitful adjustments to the course.

The Design and Assessment of a Sample SKT Writing Prompt

In order to illustrate the design and assessment of writing prompts in the SKT course, a sample prompt is described next. This extended example details the design of the prompt, a SOLO analysis of the responses, and use of the formative assessment information gained from the SOLO analysis to inform instruction.

Design of Prompt

A writing prompt to support the SKT learning goal of distinguishing between experimental and theoretical probability was assigned with an article that provided an overview of activities intended to facilitate the implementation of NCTM's (2000) data analysis and probability standards for grades Pre-K–8 (Tarr, 2002). One of the

	Levels of achievement		
Criteria	**Needs improvement**	**Meets expectations**	**Exceptional**
Inclusion of required information	**0 Points** Most components requested in the assignment description are missing.	**1 Point** Most components requested in the assignment description are present.	**2 Points** All components requested in the assignment description are present.
Clarity and organization	**0 Points** Problems with grammar, spelling, mechanics, or writing style obscure meaning.	**1 Point** Few problems with grammar, spelling, punctuation, mechanics, and writing style.	**2 Points** No problems with grammar, spelling, punctuation, mechanics, and writing style.
Conciseness	**0 Points** Writing is too brief to convey necessary points or the writing is long and rambling.	**1 Point** The main points requested in the assignment description are addressed with some degree of efficiency and eloquence.	**2 Points** The main points requested in the assignment description are addressed with a high degree of efficiency and eloquence.
Depth of thought	**0 Points** No unique insights related to the project components are provided.	**1 Point** The writer provides unique insight related to some of the components in the assignment description.	**2 Points** The writer provides unique insights related to all of the components in the assignment description.
Evidence of understanding	**0 Points** There is little evidence that the writer understood any of the main concepts related to the components in the assignment description.	**1 Point** Evidence that the writer understood the main concepts related to most components of the assignment description is provided.	**2 Points** Evidence that the writer understood the main concepts related to each component of the assignment description is provided.

Figure 5. **Rubric used to assign grades to sets of writing prompts.**

article recommendations was to use probability simulations to develop students' intuitions about random phenomena. Tarr provided the example of having children predict how many times a coin would come up "heads" when flipped a given number of times. After making a prediction, children were to perform coin flips and gather data. They were then asked to revisit their initial predictions in light of the data and revise their thinking as necessary.

To engage prospective teachers in reading Tarr's article generatively, one of the writing prompts I posed was, "Explain how simulations of random phenomena can help students develop correct intuitions about probability." In terms of the Day and Park (2005) reading comprehension question categories, this was an "inference" item because it prompted prospective teachers to draw upon examples presented in the article as well as similar activities they had experienced during class to formulate responses. In terms of the LMT framework, the prompt was intended to elicit knowledge of content and teaching because it assessed understanding of a content-specific teaching strategy. It was also intended to elicit knowledge of curriculum because probability simulation is not just a teaching strategy to be used for a single lesson, but rather recurs throughout the study of statistics.

SOLO Assessment of Prompt

Prospective teachers providing prestructural responses to

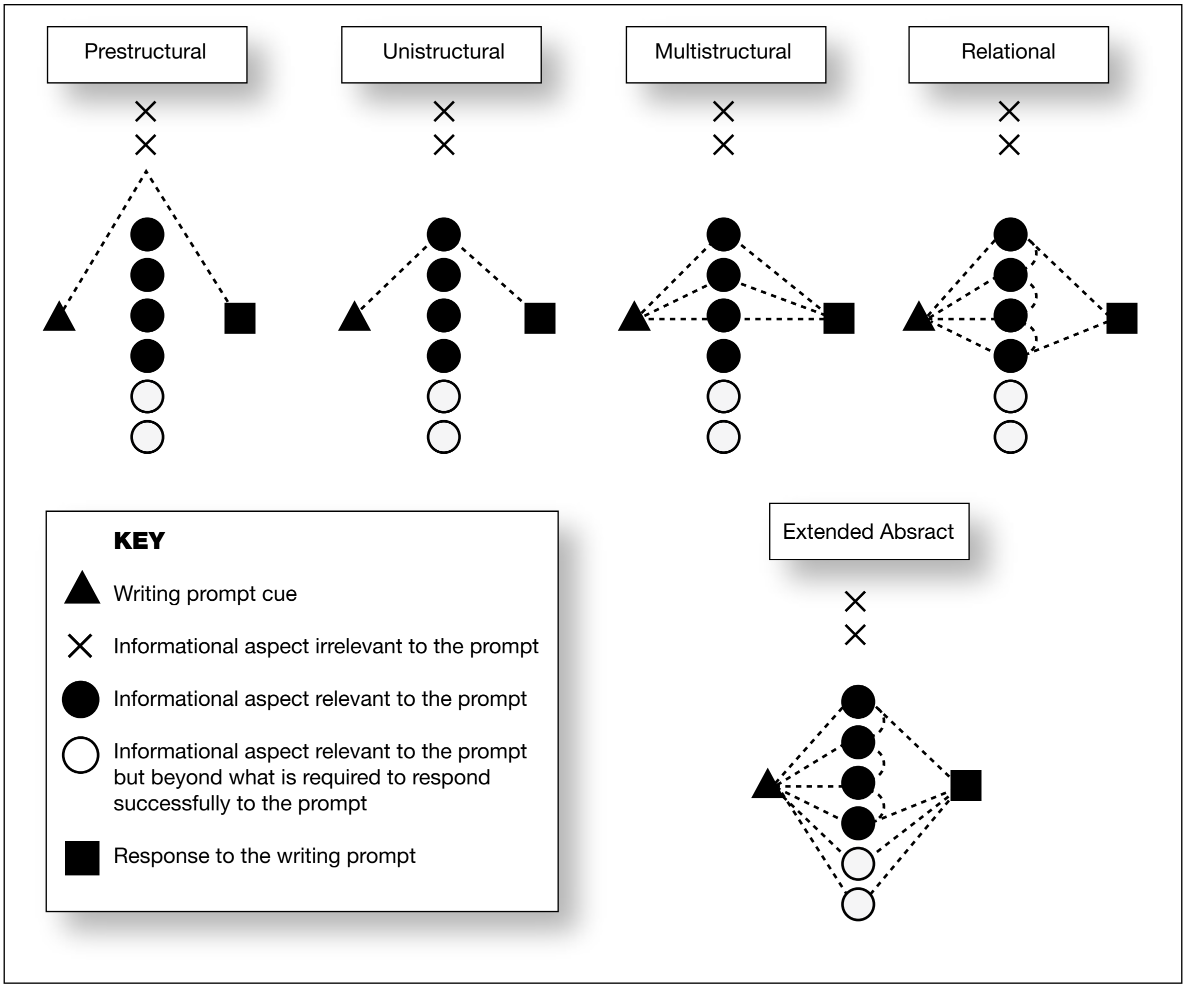

Figure 6. Diagrammatic representation for mapping SOLO levels exhibited in response to writing prompts.

the prompt exhibited no evidence of progress toward understanding the role of probability simulation in instruction. Steph, for example, wrote, "Random phenomena throws children off because it disproves what they have always thought was correct. It shows something that is extremely unlikely to happen and is hard to explain." Although portions of the response were true, it did nothing to explain how probability simulations may be helpful to students. It started to list potential difficulties in thinking students may have, but not how the recommended strategy may help remedy those difficulties.

Unistructural responses showed a degree of progress toward explaining how probability simulations may help students. However, the responses did not go beyond the single aspect of stating relevant terminology. In some cases, statistical terminology was included, but not explained, as in Sonya's response: "Simulations of random phenomena can help students develop correct intuitions about probability because it informally supports the idea of randomness and variability." Although all of the statistical terms in her response were relevant to responding to the writing prompt, she did not offer an explanation of *how* simulations would support children's understanding of randomness and variability. In other unistructural responses, pedagogical terminology was used, but not explained, as in Karen's response,

> **Using random situations to explain probability to students is a very successful way because it involves realistic situations that can be hands-on or they can relate to, to explain the material that is trying to be covered. When students are able to relate or do a project hands-on they are able to grasp the material better from my understanding.**

Karen used pedagogical terms like "hands-on" and "realistic situations" in lieu of giving content-specific insight about what probability simulations may contribute to children's thinking.

Multistructural responses included the aspect of relevant terminology, but also included the relevant aspect of what may happen statistically as students carry out simulations. Christine, for example, wrote, "Random phenomena can help students develop correct intuitions about probability because students can do trials to determine how often an event will happen. Doing trials can give them a range of different numbers, and then they can find the average." The response provided examples of activities that may occur during a probability simulation along with relevant terminology. The sample activities described, however, were not organized around a coherent theme (e.g., in Christine's response, it is not clear why students would want to "find the average"). Nonetheless, such responses suggested a greater amount of understanding of the nature of teaching strategies that incorporate probability simulation than unistructural responses.

Relational-level responses went beyond multistructural ones by describing how the elements of a probability simulation become useful pedagogically when the unifying theme of encouraging children's metacognition is placed at the forefront. Ken, for instance, wrote,

> **The simulations can help students because you can have them take an experiment and predict what they think will happen then run the simulation and see what the actual number would be. This allows the students to see what predictions they made actually were realistic and probable and what predictions were a little unreasonable. This process gives them a better understanding of probability.**

In relational-level responses, metacognition provided a means for linking the activities that occur during a probability simulation to their pedagogical purpose and value. Not only were the activities that occur during a simulation described, but the manner in which teachers can support children's reflection on their thinking was used to explain how simulation could be used as part of an overall teaching strategy.

Connecting the above SOLO analysis to the diagrammatic scheme in Figure 6, in the sample writing prompt, the cue (▲) was to explain how probability simulations can help develop children's thinking. Prestructural responses (■) simply stated that children have difficulty with random phenomena. Although true, this observation is irrelevant (×) to explaining how simulations might remedy the difficulties. Unistructural responses (■) touched on the relevant aspect of terminology (●), but did not mention aspects that would help illustrate its meaning. Multistructural responses (■) mentioned relevant aspects (●●●) of terminology and events that occur during a probability simulation. Relational responses (■) used the idea of metacognitive activity as an umbrella to explain how the various events that occur during a simulation can help develop children's thinking. This helped the responses progress beyond the multistructural level by explaining how relevant aspects (●●●) in multistructural responses complemented one another. No extended abstract responses (■) to the prompt were observed, but these might involve explaining how other pedagogical ideas, not specified explicitly in the writing prompt (○), might fit together with probability simulation to help form a coherent curricular approach to remedying children's difficulties with random phenomena. For instance, an extended abstract response might describe the advantages and disadvantages of online applets or dynamic statistics software for carrying out simulations.

Use of Formative Assessment Information from SOLO Analysis

The SOLO analysis for the sample writing prompt informed my approach to probability simulations with the class. To help more of the prospective teachers understand the importance of metacognition within the context of using probability simulations during instruction, I began to more consistently ask them to predict the results of probability simulations in class before running them. Once the simulations had been run, they were encouraged to compare the results to their original predictions and discuss reasons for discrepancies or agreement between the two. Additionally, to help them begin to reason about how technology can be used in conjunction with probability simulations, I introduced applets from the National Library of Virtual Manipulatives (http://nlvm.usu.edu/) and the freeware program Sampling Sim (http://www.tc.umn.edu/~delma001/stat_tools/). For example, when we began to study the behavior of sampling distributions, I asked the class to predict the shape, center, and spread for sampling distributions as the sample size varied. They then tested their predictions using Sampling Sim. As they did so, some began to predict that larger sample sizes lead to sampling distributions that are more tightly clustered around the population parameter. They then tested their conjectures with Sampling Sim, which allowed them to quickly simulate the gathering of various-sized random samples and then reconcile the results with their original predictions.

Effects on Prospective Teachers' Learning

Using writing prompts in place of solely subject matter-based homework problems was a substantive departure from conventional practices for undergraduate statistics courses. Although some of the prompts contained problems to develop subject matter knowledge, many were also designed to build elements of pedagogical content knowledge. Curious to examine the extent of prospective teachers' learning in a course where the primary homework tasks were writing prompts, I administered two assessments at the beginning and at the end of the course. The first was a statistics test developed by the LMT project (G. Phelps, personal communication, June 11, 2010). I used it to gain a sense of prospective teachers' SKT development during the course. This was an early draft of the test, and did not have equated forms, but it did align very closely with the course learning goals. The second assessment was the Comprehensive Assessment of Outcomes in a First Statistics Course (CAOS) test (delMas, Garfield, Ooms, & Chance, 2007). As the name implies, CAOS assesses the extent to which students develop conceptual understanding of ideas generally encountered in introductory college-level statistics courses.

Although writing prompts were assessed using the SOLO taxonomy, the LMT and CAOS examinations provided better assessments of learning gains from the beginning to the end of the course. The SOLO analyses provided valuable snapshots of prospective teachers' thinking at various points in time, but it was not feasible to track changes in SOLO levels across tasks because the sets of tasks all dealt with different statistical content. Hence, any changes in the level of response seem just as easily attributable to the difficulty of the content as they would be to general cognitive gains in SKT. SOLO analyses could, however, be used to track learning gains if similar sets of tasks were administered periodically throughout a course.

Results from the LMT and CAOS tests both indicated that prospective teachers made notable progress toward learning goals for the SKT course. On the LMT test, the mean difference between pretest and posttest scores was statistically significant ($N = 22$, $M = 0.64$, $SD = 0.52$), $t(21) = 5.79$, $p < .0001$, 95% CI [0.41, 0.87]. The mean change from pre- to posttest of 0.64 IRT units indicated that participants on average improved their SKT scores by 0.64 standard deviations between pretest and posttest administrations. On the CAOS pre-test, the mean percent correct was 36.54%, and on the posttest it was 51.54%. The mean difference between CAOS pretest and posttest scores was statistically significant ($N = 21$, $M = 15$, $SD = 12.01$), $t(20) = 5.72$, $p < .0001$, 95% CI [9.53, 20.47]. In comparison, the typical score on the CAOS pre-test for a national sample of students from undergraduate introductory statistics courses was 44.9%, and the typical posttest score was 54% (delMas, Garfield, Ooms, & Chance, 2007). Although the mean percent correct for the SKT class was slightly below 54%, the gain from pre-to-post was slightly greater.

The CAOS test results indicated that students in the SKT course left with approximately the same degree of conceptual understanding of statistics subject matter as students enrolled in conventional introductory college statistics courses. This was an important finding, since the SKT course had replaced a general education statistics course for the prospective teachers involved. Additionally, the LMT test results indicated that they gained statistical knowledge specifically required for teaching, which was not targeted in the general education course that used to be required. The observed gains in conceptual understanding of introductory college statistics and SKT helped justify continuing to steer prospective teachers into the SKT course. While it is not possible to attribute the observed learning gains directly to the writing prompts, the scores provide evidence that the writing prompts can play a prominent role in courses that build both subject matter knowledge and pedagogical content knowledge.

Conclusion

The ideas for designing and assessing writing prompts that have been discussed in this article can be used in a variety of content courses for teachers. Although statistical knowledge for teaching was the focus of this article, the ideas offered can be applied more broadly. Specifically, as teacher educators find articles that address elements of subject matter knowledge and pedagogical content knowledge, they can design prompts using the reading comprehension question types (Day & Park, 2005) that have been described and use the SOLO framework (Biggs & Collis, 1982; Biggs & Tang, 2007) to assess the levels of responses they elicit. The sample prompts described in this article, and included in the online supplement, provide examples of the types of items that may ultimately be designed and used. Readers are encouraged to experiment with the sample prompts and to design their own to elicit various aspects of SKT and MKT. Continuous design, trial, revision, and dissemination of prompts can contribute to a collective set of items to be used by those in the mathematics teacher education community for the purpose of supporting content courses for teachers. The development of tools to support such courses is particularly vital in light of calls to intertwine the development of subject matter knowledge and pedagogical content knowledge in the mathematical preparation of teachers (Kilpatrick, Swafford, & Findell, 2001; Conference Board of the Mathematical Sciences, in press).

I hope that this manuscript will contribute to teacher educators' discourse about SKT and MKT. Specifically, I hope it will spark discussions about the roles that writing prompts, generative reading, and the SOLO taxonomy can play in the process of developing and assessing prospective teachers' subject matter knowledge and pedagogical content knowledge. In addition to contributing to the practice of teacher education, such discussions can help refine theories of the components of SKT and MKT and how they may be assessed. The LMT framework and SOLO are useful tools to inform teacher educators' discussions, but they, like all models, can always be improved. The writing prompts discussed above and the method for producing them provide catalysts for further refinement of SKT and MKT theory and methods for assessment. Relevant questions for further discussion include: What other components of SKT and MKT might exist? How might the components interact with one another? What percentage of writing prompt responses fit well with one of the categories of the SOLO taxonomy? What other formative and summative assessment techniques might profitably be used in conjunction with SKT and MKT writing prompts? By examining such questions, we can develop increasingly effective approaches for fostering SKT and MKT and assessing their development.

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SUPPLEMENT

The Role of Writing Prompts in a Statistical Knowledge for Teaching Course

Randall E. Groth

Assignment 1

Article to read: Jacobs, V. R. (1999). How do students think about statistical sampling before instruction? *Mathematics Teaching in the Middle School, 5,* 240–246, 263.

Questions:

1. Write three of your own original scenarios about sampling. The first should involve random sampling, the second should involve restricted sampling, and the third should involve self-selected sampling. Explain why each scenario fits each category.

2. On p. 244, the author stated, "Students seemed to focus on the possibility of extreme outcomes without realizing that the probability of their occurrence was low." What does this mean? Provide your own example of a situation where a student may exhibit this behavior.

3. What does "fairness" mean, in the statistical sense? What is a common student conception of "fairness" that differs from the statistical sense? Give an example of a situation a young student with a nonstatistical notion of fairness may consider to be unfair.

4. Why do some students believe that a survey is not useful if survey respondents do not all answer the same way?

5. In 200–250 words, describe a general strategy you would use for teaching young students about survey sampling and how you would assess their understanding. Then provide a rationale for your general strategy.

Assignment 2

Article to read: Franklin, C. A., & Mewborn, D. S. (2008). Statistics in the elementary grades: Exploring distributions of data. *Teaching Children Mathematics, 15,* 10–16.

Questions:

1. Explain the difference between categorical and quantitative data. Give your own example of a statistical question that young students could investigate involving categorical data. Also give your own example of a statistical question that young students could investigate involving quantitative data.

2. In the second column on p. 12, the authors provided a bulleted list of five extension questions for the shoe activity. Write two of your own extension questions, and explain why they should be added to the list.

3. On p. 13, the authors stated, "it is inappropriate to ask children to determine the mean of a set of categorical data." Why is it inappropriate? In your own words, explain how Snap Cubes® can be used to determine the arithmetic mean of a set of quantitative data.

4. On pp. 14–15, a bulleted list of questions is given that teachers can ask students when interpreting the results of the soccer investigation. Write two of your own questions to add to the list, and explain why they should be added.

Assignment 3

Article to read: Leavy, A. M., Friel, S. N., & Mamer, J. D. (2009). It's a fird! Can you compute a median of categorical data? *Mathematics Teaching in the Middle School, 14,* 344–351.

Questions:

1. Provide your own example of a data set for which the median cannot be determined. Then provide your own example of a data set for which the median can be determined. Explain your thinking.

2. How can the median of a data set be determined using a paper strip marked with square grids? Give two of your own examples of data sets that would help illustrate the approach for young students, and show how the model applies to your examples.

3. See problem 1.4 in Figure 3. Identify one question that cannot be answered by using the data from the graphs and tables the students created. Explain why the question cannot be answered and tell what additional information you would need to answer the question.

4. Describe two types of student errors that occur when working with nominal categorical data: computing a numerical value for the median and computing a categorical median. Illustrate how the two errors can occur using a data set of your own.

5. In 200–250 words, describe a general strategy you would use for teaching young students that you cannot find the median of categorical data and how you would assess their understanding. Then provide a rationale for your general strategy.

Assignment 4

Article to read: McClain, K. (1999). Reflecting on students' understanding of data. *Mathematics Teaching in the Middle School, 4,* 374–380.

Questions:

1. On p. 374, the author asked, "Do students first need to know how to construct various types of graphs before they can engage in an analysis of data, or can they learn how to construct various types of graphs by engaging in data analysis?" Write a response to the author's question. Explain how your response compares to the position taken by the author of the article.

2. On p. 375, the author stated, "My assessment of their (the students') performance would not be based solely on whether they made a histogram and made it correctly but would focus more on how they reasoned about organizing and representing the data." Do you agree with this decision? Why or why not?

3. On p. 377, the author stated, "I was not clear whether the students were making a modified histogram or simply grouping the data points into categories that they named with numeric intervals." What is the difference between the two activities?

4. Examine the student graphs shown in Figures 2, 3, and 4c. Discuss the strengths and weaknesses of each one.

5. On p. 380, the author stated, "As we deliberated, we decided to find situations in which the two data sets had very similar means even though the individual data points in one of the sets varied greatly." Invent two data sets that are very different but have similar means. Use a context for the data that would be engaging for young students (similar to the battery life example on p. 380).

Assignment 5

Article to read: Harper, S. R. (2004). Students' interpretations of misleading graphs. *Mathematics Teaching in the Middle School, 9,* 340–343.

Questions:

1. Respond to the NAEP test items shown in Figure 1 in your own words.

2. Respond to the NAEP test items shown in Figure 2 in your own words.

3. Respond to the NAEP test items shown in Figure 3 in your own words.

4. Invent a set of data that would be interesting for young students to analyze. Construct two correct graphs for the data. One of the graphs should be misleading. Explain why one graph is misleading and the other is not.

5. Drawing upon the sample student responses reported at the end of the article, describe three major types of difficulties students may have with interpreting misleading graphs.

Assignment 6

Article to read: Zawojewski, J. S., & Shaughnessy, J. M. (2000). Mean and median: Are they really so easy? *Mathematics Teaching in the Middle School, 5,* 436–440.

Questions:

1. Write a response to item 1 in Figure 1. Explain your reasoning completely.

2. Write a response to item 2 in Figure 1. Explain your reasoning completely.

3. Write a response to item 3 in Figure 1. Explain your reasoning completely.

4. On p. 438, the authors commented in regard to item 3, "This type of item assesses students' conceptual understanding of mean." Explain what the authors may mean by "conceptual understanding." How is it different from other types of understanding?

5. Explain why some students believe the mean is always a better indicator of typical value than the median. How might you convince these students that the median is more appropriate in some cases?

Assignment 7

Article to read: Kader, G., & Mamer, J. (2008). Statistics in the middle grades: Understanding center and spread. *Mathematics Teaching in the Middle School, 14,* 38–43.

Questions:

1. In your own words, and drawing upon the ideas in the article, explain why histograms and box plots are more challenging to use and interpret than line plots, dot plots, and picture graphs.
2. Construct two different sets of data that have the same mean. The data sets should have different numbers of values. Compute the SAD and MAD for each set of data. Show your work. Explain what the SAD and MAD tell you about the sets of data.
3. How is the MAD similar to the standard deviation? How is it different? How might understanding the MAD help students prepare to study the standard deviation?
4. Write your own responses to each of the questions shown in Table 1 on p. 41. Explain your reasoning.
5. Write your own responses to each of the questions shown in Table 2 on p. 42. Explain your reasoning.

Assignment 8

Article to read: Watson, J. M. (2008). The representational value of hats. *Mathematics Teaching in the Middle School, 14,* 4–10.

Questions:

1. Beyond generating hat plots, how can the software program TinkerPlots® help students learn statistics?
2. How are hat plots similar to box-and-whisker plots? How are they different?
3. Why is it desirable to have students work with hat plots before working with box-and-whisker plots?
4. How can hat plots help students make the transition from focusing on individual data values to focusing on group characteristics?
5. Invent a data set that would be interesting for young students to analyze. Construct a dot plot, a hat plot, and a box-and-whisker plot for the data. Describe the conclusions one can draw about the data from each representation.

Assignment 9

Article to read: McMillen, S. (2008). Predictions and probability. *Teaching Children Mathematics, 14,* 454–463.

Questions:

1. Explain the difference between experimental and theoretical probability in your own words.
2. Explain why experimental probabilities do not always match the theoretical probabilities.
3. Which cards in Figure 3 (p. 459) involve theoretical probability? Which cards in Figure 3 involve experimental probability? Justify your answers.
4. Explain how technology can be useful when teaching the distinction between theoretical and experimental probability.
5. Examine the worksheets at the end of the article for activities 1 and 2. Describe at least one modification you would make to the worksheets in order to help improve students' learning experience. Explain why you made the modification.

Assignment 10

Article to read: Tarr, J. (2002). Providing opportunities to learn probability concepts. *Teaching Children Mathematics, 8,* 482–487.

Questions:

1. What is the difference between estimating the relative likelihood of events and quantifying likelihood numerically? Which of the two should elementary school children do first? Why?
2. Write a problem or scenario you would share with young students to help them understand the idea that the sum of the probabilities of all sample space outcomes is 1 (or 100%). Explain how the problem or scenario would help them understand this idea.
3. Explain how simulations of random phenomena can help students develop correct intuitions about probability.
4. Describe an activity that could help students understand the idea, "that, for a given event, the experimental probability (through repeated trials) is more likely to approximate the theoretical (actual) probability as the number of trials increases" (p. 486).

5. Is the beanbag game described in the “Probability and Area” section of the article (p. 486 and Figure 6 on p. 487) fair? Justify your response. Is taking an equal number of turns an essential requirement for the game to be fair? Why or why not?

Assignment 11

Article to read: Aspinwall, L., & Shaw, K. (2000). Enriching students’ mathematical intuitions with probability games and tree diagrams. *Mathematics Teaching in the Middle School, 6,* 214–220.

Questions:

1. Explain how the tree diagram in Figure 2 shows that “odd it out” is a fair game.
2. Describe how Bill, Clara, Denise, and Ahmed differed in their intuitions about activity 3 (in Figure 3) before doing the activity.
3. Explain why activity 6 (in Figure 8) is not a fair game.
4. Explain how tree diagrams can help students refine their intuitive ideas about probabilistic situations. Provide at least one specific example from the article to support your explanation.
5. Provide your own example of a probabilistic situation that can be analyzed by using tree diagrams. Explain how you would use the situation in a classroom setting to teach students.

Assignment 12

Article to read: Watson, J. M., & Shaughnessy, J. M. (2004). Proportional reasoning: Lessons from research in data and chance. *Mathematics Teaching in the Middle School, 10,* 104–109.

Questions:

1. Write your own responses to each of the four tasks shown in Figure 1 (and described on p. 105). Explain your reasoning.
2. Describe at least two different reasoning patterns students may make when comparing unequal-size groups. Discuss the strengths and weaknesses of each reasoning pattern.
3. Provide a response to each part of the task shown in Figure 2.
4. Describe three types of strategies you can expect students to use in answering the sampling task shown in Figure 2.
5. Add one of your own follow-up questions to the list in the first column of p. 109. Explain how your follow-up question would help enhance students’ learning.

Assignment 13

Article to read: Morita, J. G. (1999). Capture and recapture your students’ interest in statistics. *Mathematics Teaching in the Middle School, 4,* 412–418.

Questions:

1. In your own words, explain how the “capture-recapture” method of sampling works. Are samples produced using this method likely to provide reasonable estimates? Why or why not?
2. Why did Eric, Paloma, and Kenji each have different estimates for the number of fish in the population of Lake Amanda? How much variability in student estimates do you think you would have if you had 25 students in your class? Why?
3. In your own words, describe the different methods students used in combining their individual samples during the capture-recapture activity. Which method do you find the most appealing? Why?
4. Describe a method for helping students get a feel for sampling variability within the context of the sampling-resampling activity. Explain why the method is likely to help students understand the nature of sampling variability.
5. On p. 417, the author claimed, “They (the students) have formulated on their own this fundamental idea in statistical inference: larger samples tend to yield less sampling variability and therefore more accuracy.” Do you agree with this claim? Why or why not? What evidence is provided in the article to support the claim?
6. At the end of the article, the author stated, “Now what or who else can we tag? The possibilities are endless.” Write your own example of a situation where you could lead students to use the capture-recapture method to estimate the size of a population.

Capitalizing on Productive Norms to Support Teacher Learning

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We draw on research into the durability of sociomathematical and professional norms to make a case for attending to productive norms in teacher education experiences. We illustrate that productive norms have the potential to support teacher learning by (a) improving teachers' own mathematical understanding, particularly of specialized content knowledge; (b) supporting teachers to productively view and analyze classroom practice; (c) providing teachers an experiential basis for thinking about fostering productive norms in their classrooms; and (d) helping teachers to develop professional dispositions that support continued learning from practice. This work points to the importance of intentionally considering the norms cultivated in teacher education experiences, assessing their productivity, and strategically focusing on those that provide the best support for teacher learning.

Key words: Norms; Sociomathematical norms; Professional norms; Teacher learning; Teacher education

No teacher education experience, no matter how well designed or thorough, will be sufficient to prepare teachers for all that they will face in their future classrooms (Feiman-Nemser, 2001; Hiebert, Morris, Berk, & Jansen, 2007). This makes it critical that the limited time teacher educators have with teachers—particularly in methods classes—be used to lay a foundation that can be built on as they engage in the practice of teaching. One way to help do this is to intentionally cultivate patterns of behavior that support both short- and long-term teacher learning.

Knowing that the nature of a classroom's norms has been shown to significantly affect the learning that takes place within the classroom (e.g., Cobb, Wood, Yackel, & McNeal, 1992; Kazemi & Stipek, 2001), many mathematics teacher educators intentionally cultivate norms that create the kind of environment they feel will support teacher learning in their classrooms. Often, however, these norms focus on engaging teachers in the learning rather than on supporting the learning itself. An example of this would be focusing on the norm of having teachers explain their mathematical thinking about a given task, without being intentional about developing norms for using that thinking to support understanding of the mathematical concept(s) underlying the task. The result is a high level of participation that meets an important process goal, but may fall short of meeting important content goals (see, for example, Stockero & Van Zoest, 2011). In this sense, norms are often an underutilized teacher education tool.

Our work suggests that some norms have the potential to support teacher learning beyond that which takes place in a particular course or even an entire teacher education program (Van Zoest, Stockero, & Taylor, 2011). We draw on our research into the durability of professional and sociomathematical norms intentionally fostered in an initial mathematics methods course to make a case for the long-term benefits of attending to productive norms in teacher education experiences. In doing so, we highlight four ways in which productive norms have the potential to support teacher learning. We conclude with implications for teacher education and questions for future work.

Defining Norms

In classrooms, *norms* are regular patterns of behavior that affect the nature of the learning that occurs within them. In some cases, teachers (in our work, teacher educators) may intentionally foster specific patterns of behavior, but norms exist regardless of whether the teachers and students are aware of them (Bauersfeld, Krummheuer, & Voigt, 1988; Voigt, 1998).

Yackel and Cobb (1996) made a key distinction between *social* and *sociomathematical* norms. *Social* norms are regular patterns of behavior that can apply to any subject area and, thus, are not unique to mathematics classrooms, while *sociomathematical* norms are specific to mathematical activity. Seago, Mumme, and Branca (2004) introduced the term *professional norms* to indicate standard patterns of behavior unique to learning about teaching.

These different types of norms are often related. For example, the social norm of *supporting one's answer with an explanation* creates the need for the sociomathematical norm of *what counts as a mathematical explanation* and is related to the professional norm of *backing up* *claims about teaching and learning*. *Supporting one's answer with an explanation* is a social norm because it is not unique to a mathematics classroom; it could also be a norm for interacting in an English, science, or history class. *What counts as a mathematical explanation* is unique to mathematics, although the norm could look different in different classrooms. For example, in one classroom, saying what one did might suffice, while in another, the explanation might require providing mathematical justification for what one did.

The majority of work with sociomathematical norms has been in the context of learning what Ball, Thames, and Phelps (2008) described in their *Domains of Mathematical Knowledge for Teaching* as *common content knowledge*: "mathematical knowledge and skill used in settings other than teaching" (p. 399). Our work with teachers, however, also focuses on the development of *specialized content knowledge*: "mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400). Even though the level of the activity is different, we have found that the sociomathematical norms themselves are similar. For example, while the students' focus would be on providing a mathematical explanation for their solution, the teacher's focus might also include determining whether a student's explanation is sufficient and mathematically accurate.

Backing up claims about teaching and learning is a professional norm because it is specialized to the work of learning about teaching. Similar to the sociomathematical norm *what counts as a mathematical explanation,* this professional norm also varies across learning contexts. In one teacher learning setting, it might include initial impressions and simple reflections, while in another, teachers might substantiate claims about teaching and learning using classroom-based evidence, including student work, dialog, and other artifacts of practice.

Research on norms in mathematics education has at its core the intent to develop inquiry-based classrooms that engage learners in worthwhile mathematics (e.g., National Council of Teachers of Mathematics [NCTM], 2000). Thus, research has focused on how existing norms provide obstacles to this goal, what norms might support meeting the goal, and how these supportive norms can be developed in classrooms. In general, it has been established that intentionally fostering productive norms, particularly productive sociomathematical norms, can improve mathematics learning at any level—for example, elementary (Mottier Lopez & Allah, 2007), secondary (McClain, 2009), university (Stylianou & Blanton, 2002), teacher preparation (McNeal & Simon, 2000), and professional development (Clark, Moore, & Carlson, 2008). Of particular relevance to teacher education is the finding that an investment in developing these productive norms in methods courses can support teachers' future learning (Van Zoest et al., 2011). Drawing on this growing body of research on norms, we use the adjective *productive* to distinguish norms that support student learning from other norms that may have no effect on learning (e.g., the students always write in pencil) or may actually undermine it (e.g., the teacher does all the thinking during lessons).

In this article, we provide more detailed examples of two productive norms—one sociomathematical and one professional—that we use to illustrate the ideas in the remainder of the paper. The examples are drawn from a study investigating the extent to which prospective teachers' experiences and learning in an initial secondary school mathematics methods course have long-term effects on their professional practice (e.g., Van Zoest et al., 2011). Before continuing, we give an overview of both the course and the study.

The Course

The initial methods course was the first of three courses devoted to the teaching of secondary school mathematics in an NCTM (2000) *Standards*-based teacher preparation program that focused on teaching mathematics for student understanding. The first course focused on teaching at the middle school level, with an emphasis on analyzing and understanding student thinking and implementing instructional practices with small groups of students. The second course focused on using technology to support mathematics instruction, and the third focused on teaching at the high school level, with an emphasis on unit planning and whole-class instruction.

We approached both the development of the initial mathematics methods course and the research from a situated perspective (e.g., Borko et al., 2000). That is, we generated learning situations that were similar to those in which we intended the learning to be used, and we studied the way in which participants interacted in them. In the context of the initial methods course, we used the professional development curriculum *Learning and Teaching Linear Functions (LTLF): Video Cases for Mathematics Professional Development, 6-10* (Seago et al., 2004) to help prospective teachers learn to analyze student thinking and teacher decisions during classroom interactions, as well as the relationship between them. Each of the eight LTLF video modules began with the prospective teachers individually solving a mathematics problem, after which they shared and discussed their solution strategies as a group. The prospective teachers then viewed video clips of school students sharing their thinking about the same problem, and analyzed and discussed the student thinking and teacher actions seen in the video. This is similar to the type of ongoing analysis in which teachers need to engage in order to make sense of and build on student thinking during instruction. In addition, the prospective teachers had an opportunity to “try out” the ideas they were learning with small groups of middle school students. They did so by planning for and implementing tasks from the LTLF modules, after which they reflected on students’ thinking and ways in which they as the teacher either supported or inhibited that thinking. More details about the structure and content of the course can be found in Van Zoest and Stockero (2008a, 2008b, 2009) and Van Zoest, Stockero, and Edson (2010).

In the discussions of the LTLF video cases and of the prospective teachers’ work with middle school students in the initial course, the instructors focused on cultivating professional and sociomathematical norms embedded in the LTLF curriculum (see Table 1). These norms were intended to support the development of professional skills and dispositions necessary for teachers to productively study practice with their colleagues. Although we were intentional about cultivating these norms, at the time of the study we used what Bernstein (2004) called an *invisible pedagogy* in that neither the norms themselves, nor the moves we made to cultivate them, were made explicit to the teachers.[1] When the teachers shared their mathematical thinking, for example, we pushed them to provide a mathematical justification, rather than just report the procedure they had used, but did not explicitly discuss that we were cultivating justification as a desired pattern of behavior. Research on the learning outcomes of the course before and after incorporating the LTLF curriculum (Stockero, 2008a; 2008b) documented, among other things, evidence of prospective teachers engaging in the norms embedded in the LTLF curriculum—norms that had not been evident among prospective teachers in the course prior to incorporating the curriculum.

The Study

The study looked at the long-term effects of teacher experiences in the previously described initial methods course on their professional practice. The participants were 11 prospective secondary school mathematics teachers (PTs) enrolled in the third methods course, and 16 beginning secondary school mathematics teachers (BTs) who were graduates of our program with fewer than four years of teaching experience. The PTs had been enrolled in the initial methods course in four different semesters, with 1 to 4 enrolled in the course during any given semester; the BTs had been enrolled in five different semesters, with 2 to 4 concurrently enrolled. Both authors taught and designed the course, but approximately half of each participant group had taken it from other instructors. The other instructors were mentored by the first author, used the same curriculum, and cultivated the same norms. Including both the PTs and BTs in the study enabled us to look at the extent to which documented learning outcomes persisted at different points in time.

To understand how the initial methods course activities may have supported long-term teacher learning, we separately engaged the PT and the BT groups in activities centered on the Counting Cubes Problem[2] in Figure 1.

Table 1
Sociomathematical and Professional Norms in the LTLF Curriculum (*Seago et al. 2004*)

Sociomathematical norms	Professional norms
Naming, labeling, distinguishing, and comparing mathematical ideas [*naming and comparing*]	Listening to and making sense of or building on others’ ideas [*listening*]
Using mathematical explanations that consist of a mathematical argument, not simply a procedural description or summary [*mathematical argument*]	Adopting a tentative stance toward practice—wondering versus certainty [*tentative stance*]
Raising questions that are related to the mathematics and push on understanding of one another’s mathematical reasoning [*pushing understanding*]	Backing up claims with evidence and providing reasoning [*evidence*]
	Talking with respect yet engaging in critical analysis of teachers and students portrayed on the video [*critical yet respectful*]

1 As a result of what we have learned from our research program, we now make more explicit to teachers the specific norms that we are cultivating in our work with them. This allows the productive norms that they are experiencing to become a topic of discussion, adding another layer of potential learning. By doing this, we are able to more fully take advantage of the spectrum of ways that norms can support teacher learning.

2 The Counting Cubes Problem and the accompanying video are from the Turning to the Evidence project (see Seago & Goldsmith, 2005).

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The *n*th building?

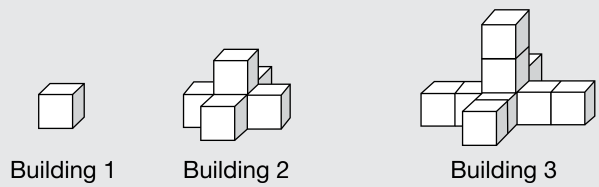

Figure 1. **Counting Cubes Problem solved by the teachers in the study and the students in the video they watched.**

These activities were similar to those they had participated in during the initial methods course. The PTs engaged in the activities during one 80-minute class session in their third (and final) methods course of the program; the BTs engaged in the activities during a 1-day professional development session held as part of the study. Neither of the two other mathematics teacher education courses in the program used video analysis as an instructional tool. Beyond the expectation in the second course that written responses to mathematics problems were to include detailed explanations of their thinking and the expectation in the third methods course that prospective teachers listen to each other as they discuss ideas about teaching, there was no evidence to suggest that the norms cultivated through the LTLF curriculum (see Table 1) had been specifically addressed in the remainder of the mathematics teacher education courses. None of the BTs' professional development experiences since graduation (as indicated in an online survey) had used video case analysis as an instructional tool or been focused specifically on mathematics instruction. Thus, there was no evidence to suggest that any of the participants had engaged in discussions grounded in representations of practice where norms such as those in Table 1 were intentionally cultivated since they had taken their initial methods course.

To gain insight into the participants' individual thinking and their interactions in the group, data for the study included both participants' written work and recordings of the group discussions. The individual work included solutions to the mathematical task, predictions about potential student solutions, and reflections on the video cases and on the session overall. The writing prompts and those given by the authors in their role as session facilitators were carefully worded and intentionally left open-ended to avoid directing the participants' thinking or prompting them to consider norms. For example, participants were asked, "What did you notice in this segment about students' thinking?" and "What did you notice about the teacher's questions, contributions, actions, or role in instruction?"

Transcripts of the recordings and the written work were coded independently by at least two researchers for examples and counterexamples of each targeted norm. Counterexamples were important to document because they allowed us to determine whether a behavior that violated a targeted norm was recognized and addressed by other group members. The research group met throughout the process to verify that the coding was consistent and to resolve any differences.

The researchers then looked across the coding to determine what behaviors were normative for the group. This analysis involved developing multiple charts that cross-referenced examples and counterexamples for each targeted behavior by participant and data source. These charts were used to determine the number of participants who engaged in each target behavior and the number of behaviors in which each participant engaged. This allowed the researchers to draw conclusions about whether each behavior was normative for the group. Note that classifying a behavior as a group norm did not mean that everyone engaged in it all the time, but rather that it appeared to be the standard pattern of behavior to which the group aspired. Thus, a behavior was classified as normative if most participants engaged in the behavior when appropriate to do so, and when they did not, the behavior was corrected or addressed by another member of the group.

For more details on the study methodology and results, including individual and group analyses of the PTs and BTs, see Van Zoest et al. (2011). Henceforth, the PTs and BTs collectively will be referred to as "the teachers." In the following section, we provide examples of two productive norms that will be used to illustrate the ideas in the remainder of the article.

Examples of Productive Norms

Mathematical Argument

The sociomathematical norm of *using mathematical explanations that consist of a mathematical argument, not simply a procedural description or summary* [referred to as *mathematical argument*] (Seago et al., 2004) has been found to create rich opportunities for students to engage as mathematical thinkers (see Yackel, 2002, for an analysis of argumentation across grade levels). Because proof and justification are central to the discipline of mathematics, this norm is particularly important to mathematics instruction that focuses on sense-making and developing a deep understanding of mathematical ideas—qualities advocated by the NCTM *Standards* (e.g., 2000) and the *Common Core State Standards for Mathematics* (CCSSI, 2010). We turn now to examples from our study to explore what counts as a mathematical argument.

We begin by looking at some attempts to provide mathematical arguments for the Counting Cubes Problem (Figure 1) that were identified as counterexamples to the norm because they lacked adequate mathematical justification. For example, in response to the prompt "show how you arrived at your solution," one teacher wrote about his expression, $5x - 4$, "I counted the differences, noticed that the pattern increased by 5 each time, so I chose $5x$. Then I used mathematical thinking to decide on what to add or subtract." Similarly, another teacher explained the constant term negative four as follows: "[B]uilding One started with one. That means five less would have been negative four. Building Zero would have been negative four cubes. And that's where the negative four comes from." Both of these cases were identified as counterexamples to the *mathematical argument* norm because the responses simply summarized the process used to arrive at a final expression, rather than justifying why the final expression made sense mathematically.

The following two responses exhibit the norm, even though the explanations left room for improvement. To justify her expression, $5n - 4$, a teacher wrote, "My solution accommodates my visualization of 5 blocks adding every [time] to the original cube: one cube spreading out at its arms." While this teacher justified the first term of the expression, $5n$, she made no attempt to explain the negative four, rendering her argument incomplete. Another teacher provided a stronger explanation of his expression for the total number of cubes in the nth figure, $n + 4(n - 1)$: "The solution relates to the picture by the single n as the center [column] growth, the 4 is the number of [horizontal] legs and the $(n - 1)$ is because each leg contains 1 less block than the figure number." Although his language was not precise (i.e., he identified n as the center column growth, instead of the number of blocks in the center column), he justified each part of the expression in relation to the diagram provided with the task. It is this justification based on mathematical ideas that is the intent of the *mathematical argument* norm.

The above examples were in the realm of *common content knowledge* (Ball et al., 2008) because they involved the teachers solving a basic algebra task. We turn now to an example that draws on *specialized content knowledge*. In this example teachers were asked to predict how students might think about the Counting Cubes Problem, drawing on specialized content knowledge because predicting others' thinking is unique to teaching. In this context, argumentation was used when teachers went beyond predicting correct or incorrect expressions that students might produce, to thinking about how students might visualize and make sense of the task. One teacher, for example, engaged in the norm of *mathematical argument* when she described one way that students might think about the task that would result in an expression of $5n - 4$:

> **So one of the ways that I thought of [how] a student might think of [the] arm length, if you think about the arm length as being the same as the building number, then [in the five arms] you would know you counted the middle block four times too many. So you could multiply the building number by five, but then subtract four.**

In this case, the explicit language and description that the teacher used in sharing her prediction of student thinking went beyond a procedural account of what students might do, to a justification of why the students' thinking would mathematically make sense.

Evidence

The professional norm of *backing up claims with evidence and providing reasoning* [referred to as *evidence*] (Seago et al., 2004) supports teachers in making sense of classroom events and drawing conclusions that will help them improve their practice. Rather than responding to events based on emotions or initial reactions that may not accurately reflect the underlying issues, this norm helps teachers learn to use classroom-based evidence to make decisions that support the development of students' mathematical understanding. In our study, this behavior was exhibited in two different ways: (a) when participants quoted the video transcript verbatim (or nearly so), and (b) when participants referenced specific line numbers from the transcript to support an argument. The transcript excerpt in Figure 2 illustrates these two ways.

During the professional development session, the facilitator prompted the following discussion by noting a teacher's observation that the students in the video were making sense of several different expressions and asking whether the participants had any observations regarding the connections being made among these expressions.

Teacher 1: Well, I think the teacher probably kicked it off when he said, "Are they the same or are they different?" . . . It's like the line, "Could someone think they can show that they're the same or different?" and Zach raises his hand. So, Zach is kind of prompted to go up to the board and say, "Hey, these are, you just have to use this distributive property thing."

Teacher 2: [The teacher] also asked, on line 32, um, to Cassie, "How is yours different or the same as what Arden and Yoshio did?" And that was one of the things I think [another teacher in her small group] pointed out, for me, maybe that Cassie didn't quite understand it. [Cassie] said, "The only thing that was different was that we subtracted and he added.'' And that really didn't—

Teacher 1: That doesn't make a lot of sense.

Teacher 2: It doesn't make a lot of sense. I mean, it makes it, maybe visually it makes sense, okay they have an adding sign and we have a subtracting sign, but it didn't get to really the root of what's different about it.

Teacher 3: For me, I thought 46 through 49 was like a big moment, where [Zach's] like, "I think what Arden is trying to do" and he nailed it, he said, "Arden's calling it, they're just renaming their variables."

***Figure* 2. Excerpt from discussion during the professional development session.**

Teacher 1 quoted the transcript verbatim ("It's like the line, 'Could someone think they can show that they're the same or different?'") to support her idea that the teacher's goal was to help the class realize that two of the expressions, $5n - 4$ and $1 + 5(n - 1)$, were the same, just written in a different way. Teacher 2 referenced a line number to further support the claim that the classroom teacher was trying to get the students to compare different mathematical expressions. Teacher 3 used line numbers to provide a rationale for his thought that Zach was the one who articulated what each group's expressions were representing [one group used zero as their first building number and the other group used one, resulting in different expressions]. In this excerpt, the teachers were spontaneously engaging with the *evidence* norm by using quotes from the transcript and line numbers to support their thinking. Rather than making unfounded claims or providing an emotional reaction to an idea under discussion, the teachers were engaged in analyzing and making sense of what was actually being said by the students and teacher in the video and what it meant in relation to student understanding of the mathematics. It is this emphasis on attending to aspects of classroom interactions that can be used to learn from teaching that makes the *evidence* norm productive.

In the following section, we use these two examples of productive norms—*mathematical argument* and *evidence*—to illustrate our findings about how cultivating productive norms in methods courses can support teacher learning (e.g., Van Zoest et al., 2011).

Reasons for Cultivating Productive Norms

Many teacher educators are aware of norms and take steps to cultivate specific norms in their teacher education contexts, yet fall short of taking full advantage of the different types of learning that norms might support. We have found that *productive norms* have the potential to support teacher learning by (a) improving teachers' own mathematical understanding, particularly the specialized content knowledge needed for teaching; (b) supporting teachers in learning to view and analyze classroom practice in productive ways; (c) providing teachers an experiential basis for thinking about fostering productive norms in their classrooms; and (d) helping teachers to develop professional dispositions that support continued learning from practice.

In the following sections, we describe, and draw on our work to illustrate, each of these reasons for cultivating productive norms in teacher education. Although we discuss the reasons separately to highlight the contributions each makes, we see them as interacting with one another in supportive ways to achieve the goal of improved classroom practice.

Improving Teachers' Mathematical Understanding

The reason for cultivating productive norms most commonly discussed in the literature (e.g., Grant, Lo, & Flowers, 2007; McNeal & Simon, 2000) is to help teachers improve their own mathematical understanding. Through cultivating specific sociomathematical norms, such as *mathematical argument*, learners are pushed to make sense of mathematical ideas they may previously have only superficially understood.

The examples in the *mathematical argument* section illustrate how cultivating this norm helps teachers develop a deeper understanding of mathematics. When teachers engage in this norm, they go beyond knowing how to get an answer, to understanding why the answer makes sense mathematically and what mathematical ideas underlie the solution process. Consider, for example, the subtle difference between the statements, "I counted the differences, noticed that the pattern increased by 5 each time, so I chose $5x$" and "My solution [$5n – 4$] accommodates my visualization of 5 blocks adding every [time] to the original cube: one cube spreading out at its arms." The first statement asserts *that* the number of blocks increases by 5 each time, while the second explains *why* this is the case. The second, we argue, is more productive in that the ability to provide this kind of justification is an important component of the *common content knowledge* teachers are being asked to help their students develop, knowledge that goes beyond learning procedures to making sense of mathematics (e.g., CCSSI, 2010; NCTM, 2000).

In the methods course, we specifically engaged teachers in doing mathematics and providing justification to prepare them to engage with the LTLF videos. We have found, however, that cultivating the *mathematical argument* norm also supports teachers in developing *specialized content knowledge*, as it helps them learn to recognize what student explanations might count as a mathematical argument. We see this in the following excerpt, in which a teacher discussed how students in the video were able to justify a part of a mathematical expression that the teachers themselves were unable to justify in their own discussion.

> **I couldn't figure out how to describe where you take away the four. 'Cause I did it like [another teacher] did it, with the four—well, I did it in a table, but then I also saw the $4(n – 1) + n$. I was like, "Oh, well, that's how you get your minus four." But I like how this [student explanation] actually shows this is how you take away the four.**

In this excerpt, the teacher provides some indication that hearing the student's mathematical argument helped her better understand the mathematics in the task. If the *mathematical argument* norm had not been established, it is quite possible that this teacher would not have been uncomfortable with her own inability to provide an argument, and thus, would not have noted the significance of the argument the student provided. Thus, cultivating the *mathematical argument* norm appears to have supported this teacher's own mathematical learning, as well as her ability to productively analyze practice—a second way that norms can support teacher learning.

Viewing and Analyzing Classroom Practice

The cultivation of productive sociomathematical and professional norms, such as *mathematical argument* and *evidence*, also supports teachers in learning to view and analyze classroom practice in productive ways, including making sense of student ideas, becoming more tentative about initial analyses, and seeking evidence to support conclusions about student learning (Stockero, 2008a, 2008b).

The sociomathematical norm of *mathematical argument* prepares teachers to both recognize when students provide a sound mathematical argument (as seen in the previous excerpt) and notice when a student's explanation may indicate an incomplete understanding of the mathematics. For instance, a teacher noted that two students in the video "had the slope figured out by their reasoning of the picture and found the intercept by fitting their line into their data. They didn't have conceptual reasoning based on the picture [for] why you should subtract 4." In this case, the teacher recognized that the student seemed to have a sound understanding of slope, but may not have fully understood the meaning of the intercept in this problem context. This analysis of practice is markedly different from that in which teachers make judgments about students' understanding based on whether or not their answer is correct.

Cultivating the *mathematical argument* norm in their initial methods course also supported the teachers in our study in noticing whether the norm seemed to be in place in the classroom they analyzed in the video. For instance, one teacher noted:

> **[I]t's very important that students were expected to explain their work to their peers. This verbal explanation—added onto their written work—makes misconceptions more obvious and also lets other students hear explanations [of] classmates. Also, [it] shows if they really understand what they did.**

Here, the teacher noticed that the norm of *mathematical argument* was in place and articulated the value of this norm for its ability to support teaching and learning. This type of noticing has the potential to support teachers in continuing to learn from practice, as it helps them to make sense of how mathematical understanding can be supported in a classroom.

The professional norm of *evidence* also supported the teachers in productively viewing and analyzing practice. Recall that this norm was exhibited when teachers used video transcript line numbers or quotes to support their analysis of practice. This use of evidence can be seen throughout the excerpt in Figure 2. The resulting dialogue is very different from that which occurs when analyses of practice are based on recollection and emotion—a common occurrence in teacher education settings. When the *evidence* norm is in place, teachers are able to engage in grounded analysis and reflection in which they learn to make sense of what is actually being said by the students or teacher. This helps teachers develop listening skills that are critical to student-centered instruction and learn to focus on key aspects of the interactions that matter to student learning—professional habits that lay a foundation for continued learning from practice. In addition, despite differences in the reflection time and type of evidence available, there is some indication that dispositions developed through teacher education experiences focused on analyzing artifacts of practice transfer to classroom instruction (Sherin & van Es, 2009). Thus, cultivating the disposition of using evidence to ground analyses of practice holds promise for supporting teachers in making evidence-based in-the-moment decisions during instruction.

Fostering Productive Classroom Norms

Since many teachers have not learned mathematics in student-centered classrooms where ideas were shared and discussed, a third reason for cultivating norms is to provide teachers with an experiential basis for thinking about fostering productive norms in their own classrooms. Teachers' ability to engage in and recognize the importance of productive norms for supporting mathematical learning is an important first step in cultivating these norms in their own mathematics classrooms.

Examples in previous sections illustrated how cultivating the sociomathematical norm of *mathematical argument* helped teachers consider what a sound mathematical argument might look like in a given instructional situation. However, even when the kind of argument a teacher might push for is clear, orchestrating productive discussions in which students justify and make connections among their mathematical ideas is still challenging (e.g., Smith & Stein, 2011). The professional norm of *evidence* helps teachers analyze specific teacher moves that might foster norms that support productive mathematical discussion and argumentation in their own classroom.

One teacher, for instance, noticed that "[the teacher] did not tell students, he asked students questions that focused them to specific aspects of the work (lines 32, 35, and 61)." Although this teacher did not list specific questions, an analysis of the transcript reveals that he was noticing that the teacher in the video asked questions that included: "How is yours different or the same as what Arden and Yoshio did?" (line 32), "Does that make it different? Is it the same, or what?" (line 35), and "Is your expression the same as any of the other ones? Because they all look different somehow. They have different numbers in them. Are any of them like equivalent or the same?" (line 61). In each case, the teacher noticed specific teacher moves that focused students on listening to and making sense of one another's ideas and on comparing and making connections among them—all productive norms in a mathematics classroom focused on using student thinking to develop mathematical understanding. Analyzing how other teachers cultivate productive norms provides teachers a foundation for developing ideas about cultivating such norms in their own classrooms.

Developing a Professional Disposition

A fourth reason for cultivating productive norms is to help teachers to develop professional dispositions that support continued learning from practice. This may be the most powerful way to think about taking full advantage of norms in teacher education, as it has the potential to promote learning that will lead to what Franke, Carpenter, Fennema, Ansell, and Behrend (1998) called *self-sustaining generative change*—change that will provide a basis for continued growth long past the end of the teacher education experience.

The discussion in the previous sections provides evidence of ways that norms might support this continued teacher learning. The examples illustrate how cultivating productive norms helped the teachers in our study develop a disposition of: (a) making sense of mathematics and expecting students to do the same; (b) carefully listening to and making sense of student ideas; (c) engaging in grounded analysis of practice; and (d) considering teacher moves that might allow them to cultivate productive norms in their own classrooms. These dispositions will allow them to continue to learn from practice, as together they form the foundation of a reflective practitioner—one who has the ability and propensity to engage in critical analysis and reflection, consider alternatives, and make connections between theory and practice.

We have some evidence that the teachers in our study who had classrooms of their own were, in fact, building on the dispositions developed in the methods course to support their instruction. For example, one teacher compared the mathematical arguments his own students might give to those given by the students in the video:

> **[In my classroom] I always like to hear somebody explain how to do it verbally, which I think was what really happened really well on the clip, because definitely being able to explain your reasoning and even teach somebody else how to do it is on a level of Bloom's Taxonomy that, you know, not only do they know it, but they can comprehend it and explain it as well.**

In this explanation, the teacher articulates the value of having students provide mathematical justifications for their solutions, rather than simply describing the procedures they used. This suggests that he was attempting to develop the *mathematical argument* norm in his own classroom. In general, the norm of *mathematical argument* supported the development of a professional disposition that led teachers to expect a mathematical justification for ideas. That is, they were not satisfied with students simply replicating what was said in a book or in a curriculum standard, but rather expected them to use reasoning and argumentation to help make sense of the mathematics being taught.

The use of evidence to support analyses of practice provides a means of connecting specific instances of practice with general theories about teaching and learning; these connections then serve as a basis for ongoing learning. One striking difference that we found between the PTs and BTs in our study was in whether the claims they used evidence to support were generalizations or specific claims. For example, the statement "[the teacher] did not tell students, he asked students questions that focused them to specific aspects of the work (lines 32, 35, and 61)" uses evidence to support a generalization about the teacher's actions, while the statement "I didn't really like how he funneled the question on line 85. It was a yes or no question" focuses only a specific instance.

In general, the PT teachers in our study were much more likely to invoke evidence to support specific observations, while the BTs' use of evidence was more balanced between supporting generalizations and supporting specific claims. We conjecture that the PTs may have been more cognizant of providing evidence since they were still in a university setting and not as far removed from the context in which this more academically oriented norm had been introduced, and thus did so more frequently in superficial ways. The fact that the BTs provided evidence in more meaningful ways suggests that the more significant aspect of this professional norm endures over time; that is, this norm supports teachers in using evidence to make sense of classroom events and draw conclusions that will help them to continue to improve their practice.

Implications for Teacher Education and Questions for Future Work

We have identified how productive norms can support teacher learning by (a) improving teachers' own mathematical understanding, particularly the specialized content knowledge needed for teaching; (b) supporting teachers in learning to view and analyze classroom practice in productive ways; (c) providing teachers an experiential basis for thinking about fostering productive norms in their classrooms; and (d) helping teachers to develop professional dispositions that support continued learning from practice. We highlighted the fact that although social norms, such as explaining one's thinking, are important, they fall short of supporting teacher learning unless they are coupled with sociomathematical and professional norms that support learning specific to mathematics teaching. As a result, mathematics teacher educators need to carefully consider the potential of focusing on a range of norms—social, sociomathematical, and professional—in terms of the many ways that such a focus supports both short- and long-term teacher learning.

Our work speaks to the importance of intentionally considering the norms cultivated in teacher education experiences. This includes identifying those that are productive—such as *mathematical argument* and *evidence*—and systematically integrating them into our curricula. In fact, this work has provided evidence that not only can productive norms be fostered and used to support teacher learning in a particular teacher education course (e.g., Stockero, 2008b), they can also support longer-term learning (Van Zoest et al., 2011). The finding that intentionally developing productive sociomathematical and professional norms early in a teacher education program can contribute to teachers' continued learning from practice is particularly encouraging given the benefits of *self-sustaining generative change* (Franke et al., 1998) to ongoing teacher development.

Fully capitalizing on the potential of productive norms to support teacher learning requires further work. First, we need to know what norms support meeting our teacher education learning goals. The norms discussed here—*mathematical argument* and *evidence*—have been shown to be productive and can be cultivated in teacher education experiences with confidence. In our work, we have found other norms—such as the sociomathematical norm of *naming, labeling, distinguishing, and comparing mathematical ideas*, and the professional norm of *listening to and making sense of and building on others' ideas*—to also be productive (Van Zoest et al., 2011). As other teacher educators systematically analyze the productivity of additional norms, we encourage them to share their findings with the mathematics teacher education community.

Second, we need to know more about the sequencing of norms. Focusing on developing a large number of norms at the same time is not practical and risks diluting the benefits of the most productive norms. Knowing which norms are foundational and which ones are better introduced further into the program would be very helpful.

Finally, our experience suggests that additional learning can occur from discussions with teachers about why we are intentionally cultivating specific norms. As discussed previously, at the time of the study we were using an *invisible pedagogy* (Bernstein, 2004), in that the norms that we intended to establish were not made explicit to the teachers. After completing the study, however, we conjectured that it would have been beneficial to be explicit about the norms we were cultivating, the reasons we felt these norms would be productive, and the moves we were making to cultivate them. More work is needed to verify this conjecture and, if it is found to be true, to determine effective ways to make the use of productive norms more visible. Doing so may allow the cultivation of norms to affect teachers' learning in even more powerful ways.

Although there is more work to be done to take advantage of the opportunity that cultivating productive norms provide for meeting the challenging task of preparing mathematics teachers, there is enough information to get started now. As you think through your teacher education work, we encourage you to think about the norms that are currently in place, assess their productivity, and consider augmenting or replacing them with norms that have been demonstrated to be productive—such as *mathematical argument* and *evidence*. Doing so will lay a foundation that teachers can build on as they engage in the practice of teaching. Developing reflective teachers who can learn from their practice is essential for meeting the ambitious goals of mathematics teaching called for by NCTM (e.g., 2000).

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Author Note

The research drawn on for this article was supported in part by the National Science Foundation under grant no. ESI-0243558, awarded to Judy Mumme and Nanette Seago, WestEd. The opinions expressed do not necessarily reflect the views of the Foundation.

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The Content-Focused Methods Course: A Model for Integrating Pedagogy and Mathematics Content*

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In the majority of secondary mathematics teacher preparation programs, the work of learning mathematics and the work of learning to teach mathematics are separated, leaving open the question of when and how teachers integrate their knowledge of content and pedagogy. We present a model for a content-focused methods course, which systematically develops a slice of mathematics content in the context of typical methods course activities. Three design principles are posited that undergird the design of such a course, addressing the nature of the mathematics content, the sequencing and design of activities, and the ways in which the course addresses the needs of diverse learners. Data from an instantiation of one such course is presented to illustrate the ways in which the course design framed teachers' opportunities to learn about both content and pedagogy.

Key words: Teacher education; Mathematical knowledge for teaching; Mathematics methods.

As Linda Darling-Hammond (2010) points out, teacher education in the United States finds itself in a Dickensian conundrum. On one side, a great deal of political attention has been paid to improving the quality of teaching and learning, particularly in the fields of science, technology, engineering, and mathematics. On the other, pointed questions are being raised about the specific value of formal teacher education. In light of studies criticizing the mathematical training that US teachers receive (Schmidt et al., 2007; Schmidt, Houang, & Cogan, 2011), there has been a return to favoring mathematical preparation over education coursework for teacher certification in many states. This press has largely focused on the preparation of secondary teachers, with the notion that disciplinary specialists with some basic pedagogical instruction might be well equipped to teach. While the research community generally agrees that teachers certified through formal teacher preparation programs effect stronger learning outcomes in students (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2008; Darling-Hammond, 2006; Darling-Hammond & Bransford, 2005), little consensus exists regarding the features of mathematics teacher preparation that promote teacher and student learning. The call for common school mathematics standards has cascaded into calls for common mathematics teacher education standards (Simon, 2000; Wilson, 2011) consistent with student-centered instruction, and frameworks that support the development of such a knowledge base. Several researchers have taken up this call, describing the knowledge base for teaching mathematics (e.g., Ball, Thames, & Phelps, 2008; Stein, Engle, Smith, & Hughes, 2008), creating instruments for measuring teacher knowledge (e.g., Brown, Bush, & McGatha, 2006; Hill & Ball, 2004; Izsák, 2008), and linking those measures to student outcomes (Hill, Rowan, & Ball, 2005). While the field has made substantial progress in describing mathematical knowledge for teaching and in linking that knowledge to student outcomes, little work has been done to describe features of mathematics teacher education that support the development of this knowledge. In this paper, we describe design principles that undergird a model for a mathematics methods course for secondary teachers that systematically integrates mathematics content in ways that provide opportunities to learn mathematical knowledge for teaching.

Researchers have conceptualized the complex knowledge base for teaching in ways that incorporate content, pedagogy, and several conceptualizations of the intersections between the two across the K–16 spectrum (e.g., Ball, Thames & Phelps, 2008; Shulman, 1986; Speer & Wagner, 2009; Steele, 2005). A common thread across this work is that pedagogical knowledge is neither discrete nor conceptually separable from the knowledge of the mathematics content being taught. Knowledge of how to teach a particular slice of mathematics rests on one's knowledge of the mathematics in question; however, research that has investigated the development of mathematical knowledge for teaching has shown this process to be less additive (e.g., learn the content, then learn to teach it) and more

* In accordance with the policy of the *MTE* Editorial Panel regarding potential conflicts of interest involving the editor, the review of and publication decision for this manuscript were handled by Denise A. Spangler.

iterative. For example, Steele (2008) demonstrated the ways in which engaging in mathematical and pedagogical tasks can enhance different aspects of both knowledge bases; Speer and Wagner (2009) identified pedagogical dilemmas that arise during teaching that spark re-examination of the content and the further development of pedagogical capacity. Yet in both policy and practical circles, the work of learning mathematics content and learning to teach mathematics are bifurcated. Prospective teachers receive content and pedagogical instruction in different courses, often separated both temporally and organizationally within teacher education systems. While many elementary preparation programs feature mathematics for teachers courses that sometimes attempt to integrate these learning experiences, few such opportunities exist for secondary teacher candidates (for exceptions, see Hill, 2006; Senk, Keller, & Ferrini-Mundy, 2000).

One model integrating the study of content and methods for secondary teachers is what Markovits & Smith (2008) term a *content-focused methods course*. Content-focused methods courses (CFMC) situate the systematic development of mathematical knowledge for teaching in the context of the typical activities in a methods course (Markovits & Smith, 2008). Whereas a methods course might treat content opportunistically through isolated tasks or lesson plans that teachers prepare, and a content course might provide plausible connections to pedagogical practice, the content-focused methods course features discernible mathematical and pedagogical storylines that are tightly connected. In this article, we look back at a content-focused methods course intended to enhance teachers' mathematical knowledge for teaching and articulate a set of design principles common to the work. These principles can serve as a framework for the design of teacher education experiences that target mathematical knowledge for teaching across a wide range of mathematical content and in a variety of contexts: both preservice and practicing, both elementary and secondary teachers.

Content-Focused Methods Course Design Principles

1. Focuses on a narrow slice of mathematical content or process central to developing mathematical proficiency in secondary school.
2. Uses a guiding inquiry to frame and motivate the course and provide a unifying thread.
3. Organizes content and pedagogical activities into sequences that engage teachers across the continuum from learner to teacher.

Illustrating the Model With a Specific Example: A Content-Focused Methods Course on Function

A content-focused methods course centered on function (herein referred to as *the functions course* for simplicity) was designed using the three principles. We begin with a description of the course and context, followed by a discussion of the ways in which each of the principles influenced course design. We then describe in general the learning evident from the teachers who participated in the course and relate those data to the design principles.

Description of the Functions Course

The course was intended to enhance teachers' mathematical knowledge for teaching functions and to develop their capacity for enacting meaningful student-centered learning experiences around these ideas for secondary students. It was taught as a graduate-level course at a large urban university in the Midwestern United States. Course development and implementation were part of a larger research project whose goals were to design and study courses around case-based mathematics education materials. The goals of the course are shown in Table 1.

The course targeted preservice and practicing secondary teachers and was promoted as an "advanced methods" course. It was a required course for preservice secondary teachers at the end of a yearlong master of arts in teaching program and was offered as an elective for practicing teachers pursuing master's-level study. In addition, a number of elementary preservice teachers and in-service special educators with particular interests in mathematics took the course as an elective. (We reflect on the impact of the diverse teacher population later in this article.) The background of the 21 teachers enrolled in the course is shown in Table 2.

The principal investigator of the research project served as the lead designer and course instructor with support from a research team made up of teacher education researchers and graduate students. The authors of this article were graduate students on the research team and have subsequently refined and enacted the course as faculty members at other institutions. The research team (RT) began by selecting sets of mathematical tasks and narrative or video cases (featuring the same or a similar task) that represented rich learning opportunities related to functions, drawing primarily from Smith, Silver, and Stein's (2005a) set of algebra tasks and cases. The RT then created or adapted additional activities related to the mathematical tasks and assembled activity sequences

Table 1
Goals for the Functions Course

Mathematical goals	Pedagogical goals
Develop a mathematically accurate definition of function and use it to distinguish examples and nonexamples of function	Support the development of students' understanding of functions by encouraging and facilitating rich mathematical discussions
Distinguish linear and nonlinear and proportional and nonproportional functions	Identify and enact cognitively challenging mathematical tasks
Solve a variety of problems involving functions, using recursive or closed form terminology and notation	Identify factors that impact the maintenance and decline of cognitive demands during implementation
Create and make connections among multiple representations of functions	

Table 2
Demographic Data on Course Participants

	Preservice: postbacc MAT program	In-service: masters of education	Secondary education doctorate	TOTAL
Elementary (K–6, all subjects)	3	1		4
Secondary (7–12, mathematics)	10	5		15
Deaf Education		1	1	2
TOTAL	13	7	1	21

(called constellations) centered on a particular aspect of the mathematics of function. Figure 1 shows this collection of activities, with the colors representing the constellations, the shapes representing different activity types, and grey borders representing activities closely related to the guiding inquiry. (Figure 1 shows an enactment of the course during a 6-week summer term meeting 3 hours twice a week. The course has also been enacted during a typical 16-week semester.) Activities above the horizontal bar were enacted in class, with those below the bar representing homework assignments. We next describe the ways in which the design in Figure 1 reflected the three principles, and discuss how the team anticipated those principles and supported teachers' opportunities to learn.

Design Principle #1: A narrow focus: Algebra as the study of patterns and functions. The first principle for the content-focused methods course prescribes a narrow focus on an aspect of mathematical content central to the secondary mathematics curriculum. This principle establishes relevance for the mathematical content to be explored with respect to the work of teachers in their classrooms and affords an in-depth exploration of the content rather than a surface-level treatment of a variety of mathematical ideas. The content focus should cut across grade levels in some important way, be identified in standards documents as important to secondary mathematics, and be complex and challenging for both teachers and their students.

Therefore, the RT selected algebra as the study of patterns and functions as the focus for this course because it met the preceding criteria well. Function is an important cornerstone of secondary mathematics, which has become even more prominent with the rise of secondary mathematics curricula that explicitly use function as the grounding concept for the development of algebraic thinking (Alper, Fendel, Fraser, & Resek, 1997; Center for Mathematics Education, 2009; Cooney, 1996; Coxford et al., 1997). The Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010) underscore this importance by positioning functions as a high school content strand alongside algebra, geometry, modeling, and statistics and probability.

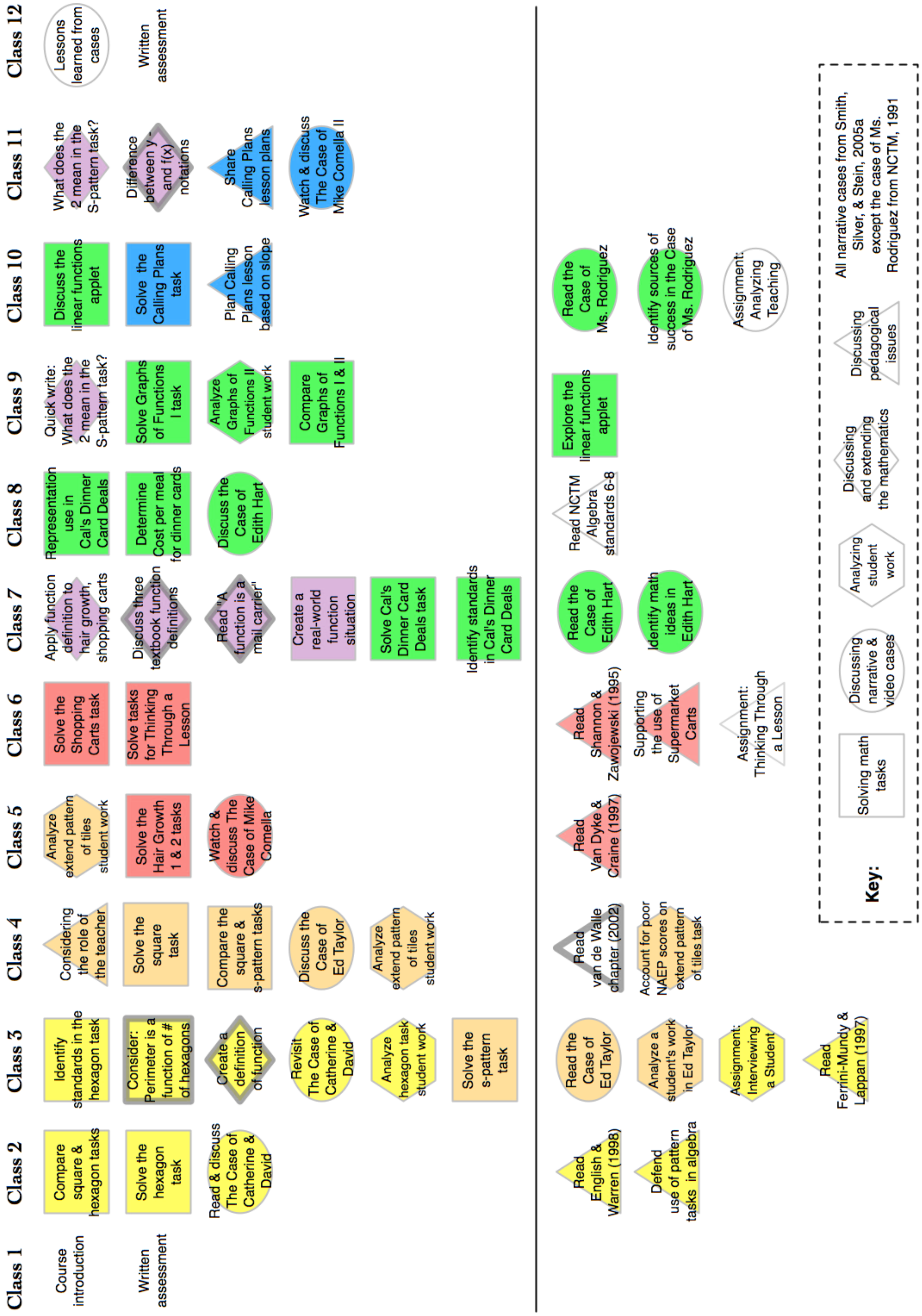


Figure 1. **Activities in the functions CFMC.**

Researchers investigating function have consistently found that students and teachers alike are frequently able to identify and generate examples at the heart of the function concept, such as continuous linear and quadratic functions, but struggle to identify more exotic functions, cannot always provide a mathematically complete definition of function, and are limited in their representational fluency (e.g., Even, 1998; Pitts, 2003; Sánchez & Llinares, 2003; Stein, Baxter, & Leinhardt, 1990). These characteristics together suggested that a content-focused methods course designed around functions would be a rich site for sustained mathematical inquiry.

Specifically, the RT conceived of the course as focusing on families of functional relationships that were familiar—largely linear and quadratic—to provide a diverse group of teachers with entry to the topic. At the same time, the team selected tasks that were mathematically extensible, allowing teachers to explore noncontinuous linear functions and rational functions built by transforming a simple linear function. The RT also recognized the importance of representational fluency—generating multiple representations of functions, moving flexibly between them, and describing the ways in which different representations are useful for noticing and analyzing specific features of a function. This set of ideas is important for developing both content knowledge and pedagogical practice.

Design Principle #2: A guiding inquiry: What is a function, and what are examples and nonexamples? A guiding inquiry is a question (similar to an essential question in K–12) designed to frame the course-long content focus. This inquiry establishes the importance of the in-depth study of a particular slice of content within mathematics and mathematics teaching. This sort of big-picture view is often lacking in curricula and standards documents and is an important aspect of teachers' curricular vision, which guides their decision making about what content is taught in the classroom (Drake & Sherin, 2008). The guiding inquiry also represents an opportunity to generalize from the set of particular mathematical tasks in the course to a larger mathematical structure and concept. The guiding inquiry should be about a topic for which most teachers will have initial ideas but one for which it is reasonable to believe teachers and students might have a limited understanding or misconceptions. In this spirit, the guiding inquiry should also be posed early in the course so as to reveal teachers' initial conceptions and revisited at key moments in the course to provide opportunities to refine and elaborate those initial understandings.

The RT selected the guiding inquiry of *What is a function, and what are examples and nonexamples of functions?* for the reasons outlined above: Teachers would likely have some fluency with functions, yet a rich understanding of the concept can elude teachers and students. A significant body of research has demonstrated that even if teachers and students can work with examples of functions, they may not have a clear definition of a function and the specific criteria that distinguish functions from nonfunctions (Pitts, 2003; Vinner & Dreyfus, 1989). Lack of a clear definition of function can lead to the over- or under-generalization of the function concept and can engender a limited view of function and obscure its mathematical utility. For example, teachers who conceive of a function as something that can always be represented graphically potentially miss important function examples such as the Dirichlet function or nonnumeric functions such as the relationship between letters and mailboxes (Sand, 1996). A reliance on a graphing requirement on the Cartesian plane also obscures geometrically-based functions such as transformations.

Motivating a course-long inquiry into a topic for which teachers may feel as if they already have a great deal of knowledge can be a challenge. To motivate deep consideration of the definition of function, the RT positioned the definition of function as something to be constructed and revisited over time rather than simply stated and taken as shared. The language of function was used in the discussion of the first mathematical task, and teachers were then asked to define function individually and in small groups. They were able to state their initial ideas about functions, providing the instructional team with a baseline gauge of what the teachers knew. The course instructor assembled a list that captured all publicly shared ideas, including incomplete or vague conceptions, and this list was posted for all subsequent class sessions. This list was then used both as a resource when considering future examples and as a living document to be modified over the course.[1] These recurring discussions helped to problematize the work on function.

Design Principle #3: Engage teachers across the continuum from learner to teacher. The notion of systematically developing content knowledge in a mathematics methods course is an important feature of the content-focused methods course model. In addition, the methods course must also develop pedagogical knowledge and link content and pedagogy in ways that are useful to the work of teaching. The use of authentic artifacts of practice (e.g., mathematical tasks, narrative and video cases of

1 While it did not happen in this particular instantiation, we have had teachers spontaneously request to edit an idea recorded on the list in subsequent iterations of the course.

teaching, student work, and lesson plans) is an important design consideration that supports the integration of the mathematical and pedagogical knowledge bases and connections to practice (Ball & Cohen, 1999). The content-focused methods course takes this connection a step further, using specific mathematical tasks as a grounding experience and starting point in an exploration of the mathematics as learner and teacher. Engaging in a mathematical task provides common ground to discuss the nuances of making sense of the mathematics. From this place, teachers can move back and forth between positions of learner and teacher, first in a protected way that may include analyzing third-party teaching artifacts such as narrative or video cases, sets of student work, or related mathematical tasks. As teachers develop deeper and more nuanced thinking about the content, they can move further on the continuum to consider the implications of taking different perspectives on the mathematics content on their teaching practice.

This principle led the RT to use particular activity structures in the content-focused methods course. For each mathematical task solved, teachers were asked to analyze the teaching of that task in some way (either through narrative or video cases), to consider students' thinking about that mathematics in some way, and to make connections to their classroom practice. Beginning with solving the mathematical task as a learner is a critical element; in grappling with the mathematics themselves, teachers are better positioned to analyze students' mathematical thinking and to consider how to support that thinking (Steele, 2008). The cases of teaching considered do not necessarily have to be exemplary cases but should raise important dilemmas about the teaching and learning of the content in question.[2]

Figure 2 shows the activities from the first constellation in the functions course placed along the learning-teaching continuum, with the numbering representing the order of activities. The constellation began with comparing the square and hexagon tasks and solving the hexagon task from a learner's perspective, followed by reading and discussing the teaching of the tasks in The Case of Catherine Evans & David Young (Smith, Silver, & Stein, 2005a). Activities 4 and 5, both homework, pushed teachers to consider the implications of the use of patterning tasks in the classroom. The next class session looped back to talking about the mathematics by considering the mathematical standards in the hexagon task and posing the guiding inquiry (what is a function?) for the first time. The next three activities, analyzing student work, reading a practitioner article on teaching algebra, and interviewing a student around one of the tasks, represented a strong push toward teaching practice.

Activity sequences that keep the mathematics constant and traverse the continuum between learner and teacher provide teachers with a range of different opportunities to learn. First, the work begins in a relatively comfortable space for discussion—doing mathematics—and gradually moves to more sensitive spaces of a teacher's classroom practice. Along the way, teacher participants build under-

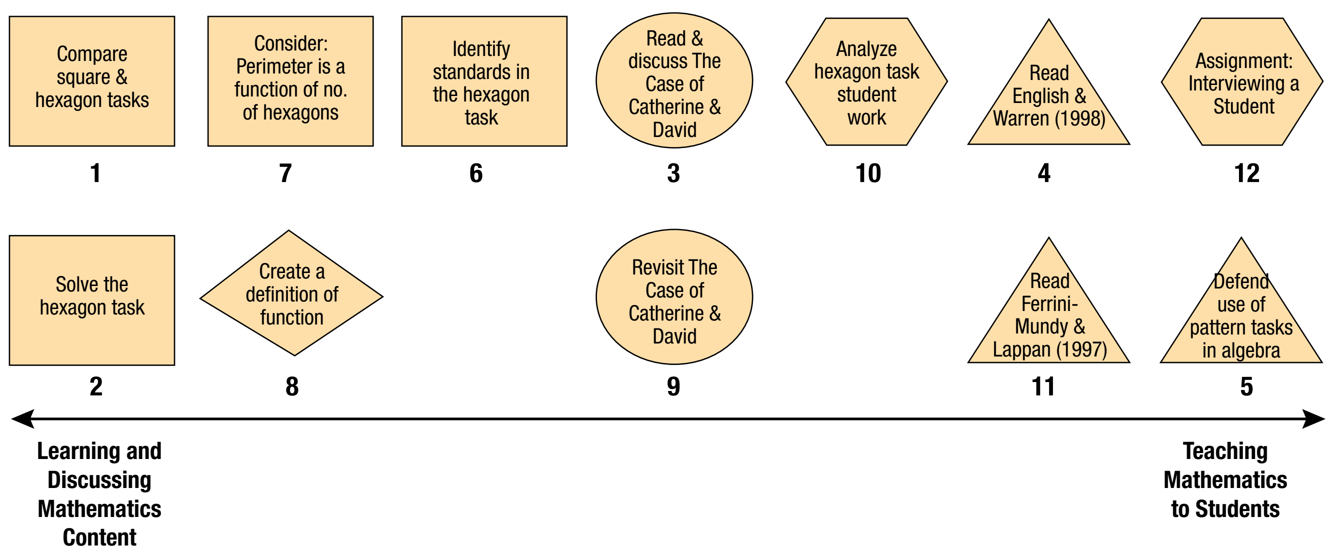


Figure 2. **The tasks in Constellation 1 on the learning-teaching continuum.**

2 The use of at least one exemplary case as a comparative measure can be particularly helpful. For a more extensive discussion of the selection and use of cases, see Smith and Friel, 2008.

standings of how other teachers and their students make sense of the mathematics content, and these understandings can then be applied to participants' own classrooms. Like turning a gemstone in the light to see its different facets, considering multiple perspectives on the mathematics creates a more nuanced and robust sense of the intertwined package of teaching and learning. Moreover, the sequence also provides teachers with a model of what student-centered pedagogy might look like around a particular mathematical topic. While not every aspect of the mathematical work will transfer directly to practice, a strong socially constructed mathematical conversation is likely to include useful features adaptable to the classroom (Hillen & Hughes, 2008).

These three design principles together frame the opportunities teachers have to learn about content and pedagogy in a content-focused methods course. In the section that follows, we briefly describe teacher learning in the course with respect to both content and pedagogy. We then use data from the course to illustrate the ways in which the design principles may have afforded teachers particular sorts of opportunities to learn.

Teacher Learning in the Functions Course

The research team collected data to assess teacher learning about content and pedagogy in the functions course in several ways. Through written assessments and semi-structured interviews at the start and end of the course teachers were asked to solve mathematical tasks, analyze cases of teaching and student work artifacts, and plan lessons. The postcourse interview used a course map similar to Figure 1 and asked teachers to reflect on their learning of (a) mathematics; (b) students as learners of mathematics; and (c) teaching mathematics, and to identify activities that contributed to their learning. Course meetings were videotaped and transcribed, and all instructional artifacts were retained, which provided data related to opportunities to learn. All written assessment items were coded by both authors, with an inter-rater reliability of at least 92%.

In general, our analysis of the data suggests that teachers added to both their knowledge of content and of pedagogy. Prior to the course, many of the teachers struggled to produce a correct definition of function as well as an example and nonexample. Performance in generating the definition, example, and nonexample improved significantly on the postcourse assessment. Teachers were also asked to solve a number of mathematical tasks, both on written assessments and during course meetings, that involved functions. The use of representations and the ways in which teachers made connections between them improved from the start to the end of the course as well. From a pedagogical standpoint, teachers were better able to select high-cognitive demand tasks related to functions and plan for them in ways that supported the maintenance of the cognitive demand. They came to understand the ways in which one might systematically plan for and support work on multiple representations of functions with students, with a particular focus on meaningful questions that supported conceptual understanding. Teachers also considered the utility of having and supporting multiple mathematically correct definitions for function rather than a single canonical definition.

In the section that follows, we explore this data set in greater detail. Our goal is to use the three design principles as lenses through which to consider data on teacher learning and ways in which the course provided teachers with opportunities to learn about both content and pedagogy.

Making connections among multiple representations: Using the lens of Principle 1. One of the mathematical goals of the course was for teachers to make connections among multiple representations of functions. In this section, we use the lens of Principle 1—the focus on the content of function—to consider the ways in which course design using this principle offered teachers opportunities to learn related to connections between representations. Teachers had numerous opportunities to make connections between visual geometric patterns, symbolic equations, tables, graphs, mathematical language, and real-world contexts. The choice of specific tasks related to function and the design of a specific progression through those tasks contributed to these opportunities to learn.

Table 3 lists the mathematical tasks related to function that were used in the course and describes both the family of function (e.g., linear, quadratic, rational) and the starting representation used in each task. By holding the

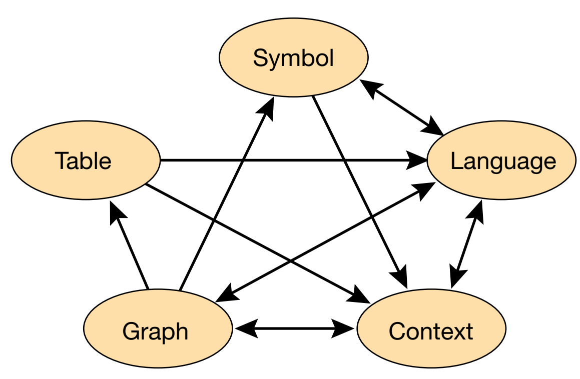


Figure 3. **Five representations of function and the connections made in Class 7.**

Table 3
Range of Examples Used in Course Tasks

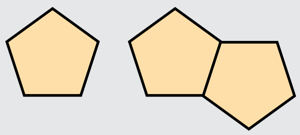
	Example (starting representation in parentheses)						
	Functions						Nonfunctions
Class	Linear proportional	Linear non-proportional	Quadratic	Piecewise	Rational	Non-numeric	
1							
2		Hexagon task (context)					
3			S-pattern task (context)				
4		Square/pool border task (context)					
5		*Paul's hair growth* (*context/table*)		*Sonya's hair growth* (*context*)			
6		Supermarket carts (context)					
7	Car wash (context) Cal's Dinner Cards: Regular Plan (graph)	Cal's Dinner Cards: Plans A and B (graph)				*Mail carrier* (*context*) *Students & test scores* (*graph*)	Weight and height (graph)
8		Cal's Cost Per Meal: Regular Plan (table)			Cal's Cost Per Meal: Plans A and B (table)		
9	*Graphs of functions: Functions 1 & 2* (*symbolic*)	*Graphs of functions: Functions 3 & 4* (*symbolic*)					
10		Calling Plans (context)					
11							
12			S-pattern task (context)				

Note. Italics indicate situations that are or could be considered continuous; nonitalics indicate discrete situations.

content of function constant, teachers were able to experience the ways in which different representations made salient different features of the function relationship. As the course progressed, the teachers were using different representations spontaneously in their mathematical work and talk to make sense of the underlying mathematical constructs. For example, in a discussion of the Cal's Dinner Card Deals task in Class 7 in which teachers were asked to make sense of the slope and y-intercept, teachers made 13 different connections between the 5 core representations of function (symbol, language, context, graph, table). Figure 3 shows the connections between the representations made by teachers in a 20-minute discussion.

Changes in teachers' abilities to make connections between mathematical representations were assessed in part through their performance on the visual pattern

The first train in this pattern consists of one regular pentagon. For each subsequent train, one additional pentagon is added. The first three trains in the pattern are shown below.



Train 1 **Train 2** **Train 3**

a. Determine the perimeter for the 4th train.

b. Determine the perimeter for the 100th train.

c. Write a description that could be used to find the perimeter of any train in the pattern. Explain how you know. How does your description relate to the visual representation of the trains?

Figure 4. **The pentagon pattern task (adapted from Schifter, 1996).**

task shown in Figure 4. Results from the pre/post written assessment show evidence of changes in teacher capacity to make connections between a visual geometric pattern and a symbolic equation. Teachers' responses to the task were coded using the rubric shown in Table 4. Initially, a majority of teachers determined an equation that generalized the pattern (rubric scores of 1 or higher), but the extent to which their explanations related their equation to the visual pattern varied. About 25% fully related their explanation to the pattern (score 4), with another 25% making no connections (score 1), and the remaining teachers making some connections (scores 2 and 3). By contrast, on the posttest, the majority (80%) completely explained how their equation related to the visual pattern, a significant difference (Wilcoxon sign-rank test; $W = -73$, $ns/r = 12$, $p = 0.0045$).

In addition to the changes in performance on the written assessment, our analysis of the postcourse interview data showed that teachers saw an important pedagogical use for connecting representations. When asked what they learned in the course, 13 of 21 teachers discussed connections between representations as a topic that helped them better understand students as learners of mathematics, and 11 of 21 named connections between representations as an important priority in the teaching of mathematics.

In following Design Principle 1, the RT held the content of function as a consistent thread throughout the course. Doing so allowed the instructor to focus on the ways in which representations helped to illuminate different aspects of functions. Teachers developed both a stronger fluency with the mathematical representations and a clearer sense of why and how one might use different representations as pedagogical tools.

Defining function: Using the lens of Principle 2. Another goal of the course was for teachers to develop a mathematically accurate definition of function and use it to distinguish examples and nonexamples of function. Teachers considered the guiding inquiry, *What is a function, and what are examples and nonexamples of functions?*, which provided opportunities to meet this goal, at several points throughout the course, as shown in Figure 1. The prominence of the guiding inquiry, combined with the repeated opportunities to revisit and revise thinking about the definition of function and the nature of examples and nonexamples, provided teachers with opportunities to deepen their content knowledge related to function. The pre/post written assessment measured that learning, asking teachers, *What is a function? Give an example of a function and a nonfunction.*

Teachers' responses to the first part of this task (*What is a function?*) were coded as correct, incorrect, or inconclusive. A correct definition included the idea of univalence (i.e., the mapping of each element of the domain to exactly one element of the range) and did not explicitly rule out arbitrariness (i.e., elements of the domain and range do not need to be numeric). Correct definitions could use different terminology for x and y (e.g., input and output; domain and range; independent variable and dependent variable). Definitions that did not include univalence or made erroneous statements (e.g., functions must be linear relationships) were coded as incorrect. Definitions were coded as inconclusive if there was not enough information present to suggest the univalence criterion. For example, several definitions included correct statements (e.g., functions pass the vertical line test) but provided no further explanation regarding why the statement(s) implied that a relationship was a function. Teachers' responses to the second part of the task (*Give an example of a function and a nonfunction*) were also coded as correct, incorrect, or inconclusive.[3] Examples and nonexamples were also coded by family (e.g., linear; quadratic) and representation(s) used.

In general, teachers' initial definitions were problematic, although their examples were not. Fewer than half of the 21 teachers in the course provided a correct definition

3 The inconclusive code was used for responses that provided a correct example and nonexample of function, but the teacher did not label which was which.

Table 4
Rubric for the Pentagon Pattern Task

Score	Description	Example
4	Full explanation; well connected to visual pattern A generalization is evident (verbally or symbolically) All aspects of the generalization are explained accurately with respect to the visual pattern	For each pentagon on the end of the train you count 4 sides, so that is always $4 \times 2 = 8$. There are two less pentagons in the middle of the train than the train number itself, and each of these has 3 sides counted as part of the perimeter (3 exterior sides)$\rightarrow 8 + 3(n - 2)$, where n is the train number.
3	Some explanation; partially connected to visual pattern A generalization is evident (verbally or symbolically) At least one aspect of the generalization is explained accurately with respect to the visual pattern Remaining aspects of the generalization are either explained incorrectly, inaccurately, vaguely, or not explained at all with respect to the visual pattern	$n(5) - (n - 1)(2)$ n refers to the number of the train, multiply this number by 5 then subtract one less than the total number multiplied by 2. From the visual representation we can see that 2 pentagons will share one side. This shared side will be on the inside of the shape and will not be included in the perimeter. This shared side must be subtracted from each pentagon.
2	Weak explanation; some connection to visual pattern A generalization is evident (verbally or symbolically) At least one aspect of the generalization is explained, but the explanation is incorrect, inaccurate, or vague	$(3x) + 2$ When a new train is added only three units sides two sides of that train are actually added. The $(3x)$ is 3 sides of the trains from before multiplied by the train number.
1	Numeric explanation only; no connection to visual pattern A generalization is evident (verbally or symbolically) The elements of the generalization are explained but not connected to the visual pattern in any way	$3n + 2$ Multiply the number of trains by 3 and then add 2. I know this works because it fits my pattern. My description is independent of the visual representation. I had to make a table—the pictures did not help me in finding the patterns.
0	No explanation present	

on the pretest, with incorrect definitions cutting across all grade levels taught and experience. Most of these incorrect definitions did not include mention of univalence and included features that suggested a narrow conception of function (e.g., functions are linear; all functions can be graphed). Nearly all teachers (20 of 21)—including all of the teachers who provided an incorrect definition—provided a correct example of function, with most being linear or quadratic relationships presented as equations or graphs. Over half the teachers (13 of 21) provided a correct nonexample of function on the pretest; however, 6 of these teachers provided an incorrect definition of function.

By contrast, nearly all teachers (20 of 21) provided a correct definition on the posttest, a significant difference (Fisher's exact test, $p < 0.001$). It is also important to note that all 12 teachers who did not provide a correct definition on the pretest improved in some way: 7 moved from incorrect to correct, 4 moved from inconclusive to correct, and 1 moved from incorrect to inconclusive. All teachers provided a correct example of function on the posttest,[4] mostly linear or quadratic in nature. However, there was a significant increase in the number of correct nonexamples of function (Fisher's exact test, $p < 0.01$). Given the background of these teachers, it is no surprise that the majority could easily produce and identify examples of functions. The types of examples provided by teachers were relatively straightforward relationships central to secondary mathematics. However, it is interesting to note that prior to the course, the majority of teachers were able to provide correct examples of functions, but not all teachers were able to correctly produce a definition or a nonexample of a function.

The second design principle that specifies a guiding inquiry, problematized early in the course and revisited throughout, provided teachers with repeated opportunities

4 One teacher did not respond to this item; all 20 teachers who did respond to this item provided a correct example of function.

to learn the definition of function. By considering multiple examples, examining narrative cases of teachers seeking to support their students in understanding the construct of function, and thinking about tasks to use in their classrooms related to function, teachers had the opportunity to rise above simply learning a correct definition of function. By thinking through the definition and examples as learners and teachers, teachers developed knowledge of both content and pedagogy through this sustained inquiry.

Two cases of learning about content and pedagogy: Using the lens of principle 3. In this section, we look at the learning of two teachers through the lens of Principle 3, which describes the ways in which activities that span the continuum of learning mathematics and teaching mathematics can support the development of teachers' knowledge of content and pedagogy. We present cases of two teachers with differing backgrounds and prior knowledge to describe the ways in which the learning-teaching continuum provided opportunities to learn about the content and pedagogy of function. Olivia was an experienced elementary teacher whose knowledge of function was relatively thin at the start of the course, and Carl was a preservice secondary teacher with strong mathematical knowledge. We consider the ways in which the course addressed differing needs based on each teacher's initial conceptions of function and how the diverse set of activities on the continuum from learner to teacher provided them with opportunities to learn that matched their backgrounds.

The case of Olivia. Olivia was a practicing elementary teacher completing her sixth year of teaching who enrolled in the course as an elective. Olivia was known as a thoughtful teacher-learner who had taken part in many high-quality professional development experiences, including a similar content-focused methods course on proportional reasoning in her masters of education program.

At the beginning of the course, Olivia's knowledge of function was limited (see Figure 5). The definition of function she provided on her pretest did not include univalence and implicitly ruled out arbitrariness. Although she provided a correct example of function on the pretest, her work during the first interview (conducted after Class 3) revealed that she struggled to explain *why* her example was a function, even though a correct definition of function had been made public in class by the time of her interview:

Interviewer: I have the example of a function that you gave on the pretest. So you gave $y = 2x$. And I wanted to ask you why this is an example of a function?

Olivia: Well, I think that looking at that, there would be one y for every x, and one x for every y, so I think that that's why it's a function.

Interviewer: OK. And that's based on the discussion in (the third class)?

Olivia: Yes, I mean, doesn't—I don't think it would have to be one value for each of them, I mean, every time you have y, or x, um, y is going to be two times that.

Interviewer: OK. What could you do to make it not a function?

Olivia: Make it x squared, or something. If I made it x squared, then, I think you'd have more than one value for x. And [long pause], I don't really know. I think that because it's x squared, I think you'd have more than one value. But I'm not really sure what I'm doing.

In this excerpt, Olivia attempted to use univalence to explain why her example is a function and to create a nonexample of a function but grappled with its meaning and determining the variable (x or y) to which she should attend.

By the end of the course, however, Olivia had a more robust understanding of function and its definition. Her posttest function definition (Figure 5) satisfied both conditions for a correct definition. During the postcourse interview, Olivia successfully classified a set of relationships as functions and nonfunctions and explained her classifications drawing on the definition. In the excerpt below, she explained why the graphs of $x = 2$ and $y = \pm x^{1/2}$ are not functions, using univalence as the justification:

Olivia: [$x = 2$ is a nonfunction] because x would be 2, but on that line, you could have any value for y. And also because of the multiple values of y, and also because if you think of drawing a vertical line through it, it is a vertical line, it'd hit the whole line. So it wouldn't be just one spot. And for [$y = \pm x^{1/2}$], if you do the vertical line test, it goes through the graph twice.

Interviewer: What is the vertical line test?

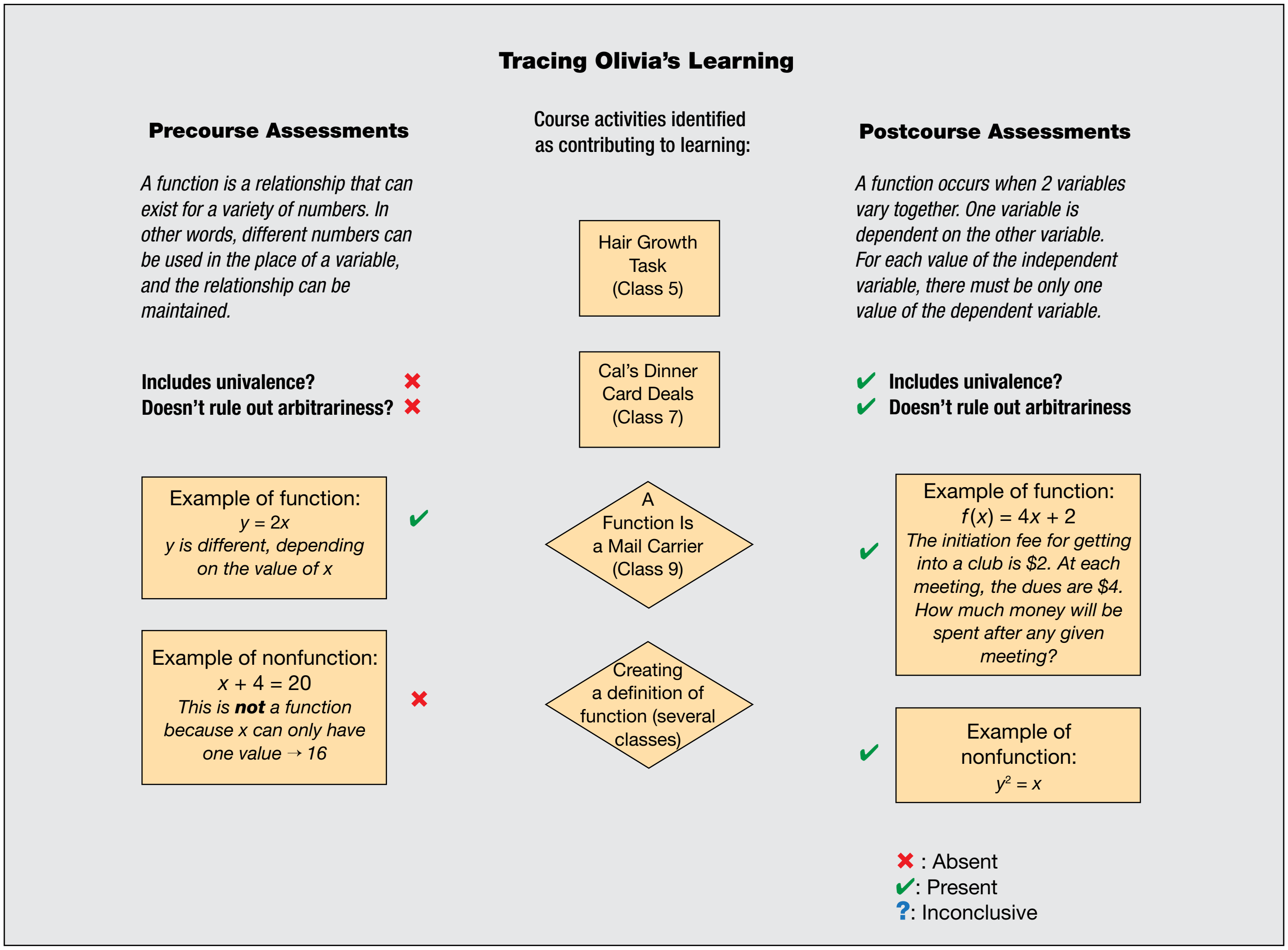


Figure 5. **Tracing Olivia's Learning (teacher responses shown in italics)**

Olivia: Well, if you draw a vertical line through, you should only cross the graph once because if you cross it more than once, it means for that particular value of *x*, there's more than one value for *y*. Like here, say my value for *x* was 2. I could have this value of *y* and this value of *y*...,[5] say (2, 1) or (2, –1). So because you have those two values, it means it's not a function.

Olivia's responses in the second interview suggest that she had not simply memorized the definition discussed during the course; rather, she understood the key characteristics and drew upon them to classify relationships as functions or nonfunctions. In addition, she described the vertical line test and connected it to univalence, suggesting that she did not merely memorize the procedure of using the vertical line test to determine whether a relationship is a function. Olivia acknowledged her narrow view of function at the beginning of the course and described how her understandings changed through engagement in particular course activities:

Olivia: I have a much broader understanding of functions… a broader view of what a function is and what it involves… And also, thinking about what a function was. But I don't know that I could really define that before. And I try to think, "Could I have done that when I was maybe in 8th or 9th grade, when I was taking algebra classes?" And I don't really know that I could have.

5 An ellipsis indicates deleted words.

Interviewer: So were there places that helped you with defining a function and thinking about what a function really is?

Olivia: Well, creating a definition of a function. I remember "Function as a Mail Carrier" because the idea that every *x* can only have one—that idea that when you put in a value for *x*, you should always get one *y*. It shouldn't be you put in 4 one time, and you get 6 for *y*, and you put in 4 another time, the same number, and get 8. You can't have that. So I think that was important, too.

Olivia noted that the course enhanced her understanding of function and identified four particular activities as being instrumental in her learning. She recognized that her broadened view of function was influenced by participating in discussions based on the guiding inquiry and by various types of mathematical and pedagogical activities. When asked about the pedagogy of the course as a factor in her learning, she described specific features related to Principle 3:

Olivia: One thing I also liked about the class is that we really worked on developing our understanding of math AND connecting it to teaching, like through the case studies. And there aren't very many classes that do that... I think the two go hand in hand—really learning about the math and understanding it, then looking at how is that taught in the classroom? We looked at the tasks first, so we understood... what this task was about, the math that was involved, and then how a teacher was presenting the task, and how students in the task interpreted it, and maybe compare in your mind, "Well, you know, that's how I thought of it." It's effective, I think, for teachers because both of them are really important and connecting them [is] important.

In sum, Olivia entered the course with substantial confusion about function from a content standpoint. Her work in interacting with the mathematical tasks and the development of the definition enabled her to successfully define and identify examples and nonexamples by the end of the course. Moreover, she linked the content learning to the work in considering the cases and student work, describing the ways in which moving between learner and teacher was important to her development as a teacher.

We now consider the ways that the same course supported a teacher with a different background by looking at the case of Carl, a preservice secondary mathematics teacher.

The case of Carl. Carl was a preservice secondary teacher completing a yearlong internship in a suburban middle school. He had earned a bachelor's degree in mathematics from a major public university and took the course as the capstone of a fifth-year master of arts in teaching program. Despite Carl's mathematical background, his work early in the course suggested a muddled understanding of function. Carl's pretest definition allowed for arbitrariness but did not reference univalence, as shown in Figure 6. In distinguishing examples and nonexamples in his pre-interview, he used the univalence criterion but incorrectly described it as "one-to-one correspondence."

At the end of the course, Carl held a deeper and better connected understanding of function. His posttest definition fulfilled both criteria for a correct function definition. Interestingly, Carl did not use the input/output language that was often used in class discussions of the definition. This suggests that Carl had not merely memorized the class definition but held a conception of function that made sense to him. In reflecting on his learning, he described the differences in his understandings since the beginning of the course:

Carl: I have a clearer definition of what a function is… I think most of us came into the class having worked with functions before, obviously, and doing vertical line tests to see if something in the function maybe is not a function. But I don't think a lot of us had a really solid definition in our heads of what a function is. And I think that the class kinda helped us revise our own inkling of what a function is. The thing about functions is that correspondence between two different sets of quantities. Before the course, if someone had asked me "What is a function?" I couldn't say… I would've said something about the vertical line test. I would've said something about an equation [or] function notation. But I don't think I could have really given a really direct answer. After the course, I think I can.

Carl noted that while he entered the course with ideas about the definition of function, these ideas were incomplete, and the course provided an opportunity for him

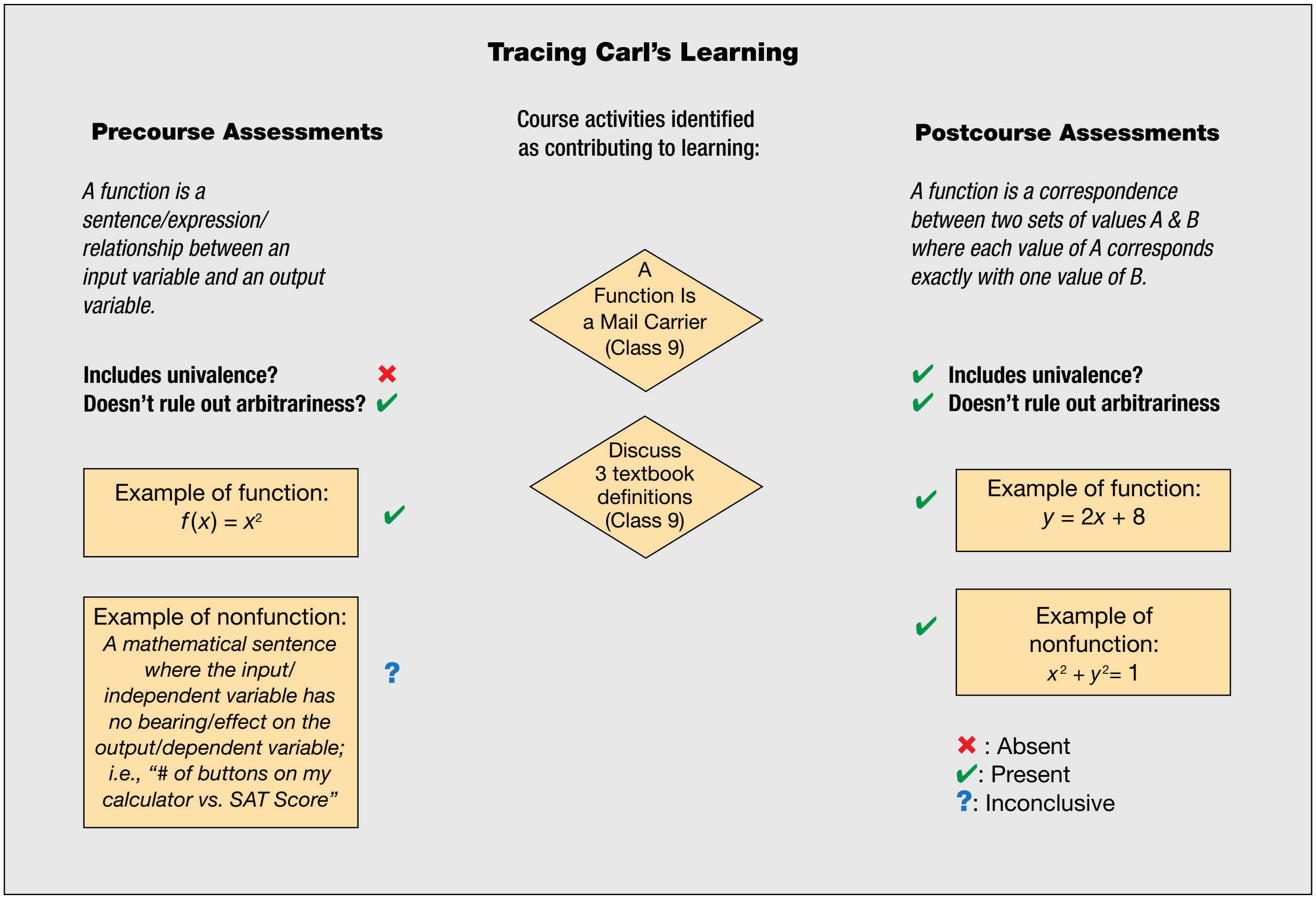


Figure 6. **Tracing Carl's Learning (teacher responses shown in italics).**

to revise his thinking and develop a clearer definition of function. When pressed to identify specific activities in the course that supported his learning, Carl described the discussion in which teachers created a definition of function (Class 3):

Interviewer: So you just (identified) creating a definition of a function. How did that help you come to clarify the definition of function?

Carl: Well, I mean we were just really brainstorming the definition of a function, and I think what it really did was made me analyze the specific kinds of things that make up a function. It doesn't necessarily have to be an equation. You know, it could just be the two sets. . . . But what it really did is it really made me scrutinize my own definition of a function that I had coming into the class, and we could change it and alter it a little bit, due to the discussion of the definition. Having that up there throughout class made me go back and see, "Well, is this a function? Is this a function?" Go through the criteria that we came up with ourselves.

When asked to reflect on how the structure of the course supported his learning, Carl's answers differed from Olivia's. Carl focused on the enactment of the tasks in the course as a model for his own future classroom and the cases as reinforcing the real-time modeling:

Carl: Class time was a good example of how a pattern task could be implemented in the classroom, and the level of mathematics was high. [The instructor] had us solve in groups, was able to ask some open-ended questions that didn't necessarily guide the group directly to an answer. . . . [When] group discussion was over, she was able to bring the class together and have a whole-class discussion [and] pick out certain solutions that were beneficial

> to the class as a whole to see. It was really an example and another reinforcement of how to use pattern tasks and to keep the level of mathematics high in the classroom. [And] the cases that we read are examples.

Carl's background differed from Olivia's in that he entered the course with a stronger conception of function but with a definition that was in need of clarification. Through the same set of activities, Carl was able to make repairs to his definition rather than adopting the co-constructed class definition. Carl also entered as a preservice teacher looking for models of how to enact student-centered tasks, models that had been lacking in his internship placement. For Carl, moving between doing mathematics and considering cases of teaching helped him develop pedagogical knowledge related to how he might support his students in developing conceptual understanding.

For these two teachers, traversing the continuum of content and pedagogy created a common set of learning opportunities that led to different learning outcomes. Olivia's mathematics background was such that she took advantage of opportunities to learn related to the content of function, integrating the new content knowledge into an existing framework about her own well-developed pedagogical practices. She specifically noted the cases as a place in which the content and pedagogy come together, and one might anticipate that seeing this connection would make her better able to integrate new content understandings into her teaching. Carl, with a stronger mathematical background but at the very beginning of his teaching career, was able to take note of the ways that larger-grained pedagogical structures can support the learning of content. His ideas about the ways in which the group discussions and sharing of solutions modeled in the course led to new mathematical understandings provided useful models for Carl's early practice as a beginning teacher. As such, Principle 3 provided these teachers with opportunities to learn about teaching mathematics that fit their differing needs at the time.

Discussion

The content-focused methods course is a promising model for supporting teachers in developing mathematical and pedagogical knowledge and integrating those knowledge bases in ways that build knowledge needed for teaching mathematics. The examples presented from the functions course demonstrate how the design principles can come together to provide diverse groups of teachers with opportunities to learn. The functions course, however, is only one instantiation of the content-focused methods course model and was enacted in a specific institutional context that may differ from your own.

So how does one begin designing a content-focused methods course? Selecting a mathematical focus (Principle 1) is a good starting point. Identifying narrative and/or video cases (e.g., Barnett, Goldenstein, & Jackson, 1994; Boaler & Humphreys, 2005; Merseth, 2003; Smith, Silver, & Stein, 2005a, b, c) and student work (e.g., Lamon, 2005; Parke, Lane, Silver, & Magone, 2003) that relate to the mathematical focus can suggest a specific guiding inquiry. In the sections that follow, we discuss ways in which the principles could be implemented in different contexts and the affordances and constraints of bringing a content focus to an existing mathematics methods course.

Varying the Mathematical Focus and Guiding Inquiry: Principles 1 and 2

By varying the mathematical focus, and in turn, the guiding inquiry, additional content-focused methods courses for teachers of grades 7-12 could be developed. For example, content-focused methods courses on proportional reasoning (using the guiding inquiries *What is proportional reasoning?* and *Are all fractions ratios? Are all ratios fractions?*; Hillen, 2005) and geometry and measurement (using the guiding inquiry *What is a proof?*; Steele, 2006, 2008) have been developed and studied. A content-focused methods course on *reasoning-and-proving* (Smith & Stylianides, 2010; Hillen, Smith, & Arbaugh, 2011) is currently under development. The guiding inquiries to frame this course will include a mathematical question (*What is reasoning-and-proving?*) as well as ones that could be considered more pedagogical in nature (*How do secondary students benefit from engaging in reasoning-and-proving? How can teachers support the development of students' capacity to reason-and-prove?*). While the current principles reflect a secondary population, the content-focused methods course model could be used in courses for teachers of the elementary grades. Similar principles have also been used to structure professional development opportunities for teachers and their principals (Steele, Johnson, Herbel-Eisenmann, & Carver, 2010).

Varying the Focus on the Learner-Teacher Continuum: Principle 3

By shifting the focus on the learner-teacher continuum, additional courses could be created that would serve a variety of purposes. By placing an emphasis on the *learner* end of the continuum, a mathematics content course could be created, which could consist primarily of activities in which teachers solved mathematical tasks and occasionally examined student work or read a practitioner-oriented article. Such a course could conceivably meet the forthcoming recommendations from the

Conference Board of the Mathematical Sciences (CBMS, 2012), stipulating three courses focused on content knowledge for teaching for future secondary mathematics teachers. These courses would address the need for systematic ways to develop specialized content knowledge, identified as a pressing research priority (King & Thames, 2011). For teachers who have already had opportunities to carefully consider mathematics content, a course focusing on the *teacher* end of the continuum, making additional connections to practice (e.g., lesson study cycles) or providing opportunities to do action research (e.g., collecting and analyzing data from their own classrooms), might be useful (Boston & Smith, 2009). Such a course might be particularly appropriate for the master's-level or district-based professional development, where teachers have more fluency inquiring into and reflecting on their own practice.

Such courses also provide a rich site for studying teacher learning. Given the few formal classroom learning opportunities that currently exist for teachers to learn specialized content knowledge and pedagogical content knowledge, little research exists on the ways in which teachers learn these ideas at the intersection of content and pedagogy. A content-focused methods course that is specifically designed with the goals of developing these aspects of mathematical knowledge for teaching could serve as an important site for studying the way these knowledge bases grow in teachers. Courses run in conjunction with a field component would also provide opportunities to study the ways in which such knowledge is used in context.

The content-focused methods course provides a generalizable, adaptable model for integrating the study of content and pedagogy. The course described here resulted in teacher learning that varied in beneficial ways across teachers. Activities that traverse the content-pedagogy spectrum, grounded in a specific slice of mathematical content, can provide teachers opportunities to enhance areas of their knowledge across that spectrum.

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Authors' Note

This article is based on work supported by National Science Foundation grant ESI-0101799 for the ASTEROID (A Study of Teacher Education: Research on Instructional Design) Project, Margaret S. Smith, Principal Investigator. Any opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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Using "Lack of Fidelity" to Improve Teaching

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The author presents a procedure for learning from variations that occur when instructors implement lesson plans designed by others. This kind of variation, occurring in many classrooms every day, can provide a source of information for improving curriculum, both in terms of instructional activities for students and especially in terms of clarifications for instructors to support more effective implementation. The author provides detailed descriptions, in the context of a mathematics course for preservice K–8 teachers, for using implementation variations in a practical, research-based way to study and improve teaching. The goal is to build an accumulating knowledge base for teacher education. Examples are presented to illustrate how increasingly rich lesson plans, based on observing implementation variations, can move toward achieving this goal.

Key words: Curriculum Implementation; Continuous Improvement of Teaching; Teacher Preparation

The purpose of this article is to describe a method for studying variation in teaching resulting from lack of fidelity in implementing a curriculum to create evidence-based improvements in teaching. The method uses observations of implementations to identify details of the curriculum that could be changed to increase the chances that all instructors who use the curriculum will provide the intended learning opportunities for students. A key assumption is that repeated cycles of observing and revising the *details* of curriculum implementation are essential for building knowledge for teaching that leads to cumulating and lasting improvements in classroom instruction.

Lack of fidelity in implementing curricula might be a surprising setting in which to conduct cycles of observing and revising teaching. Lack of fidelity has usually been interpreted in one of two ways. Either it is viewed as an obstacle to measuring the effects of an intended curriculum on student achievement (Fullan, 2008; Huntley, 2009; National Research Council, 2004; O'Donnell, 2008) or it has been interpreted as an unavoidable, and fully appropriate, mediation by the teacher to fit the local conditions and connect the curriculum with the students in a particular classroom (Fullan, 2008; Lloyd, Remillard, & Herbel-Eisenmann, 2009; Remillard, 2005).

Rarely has the lack of fidelity been tapped as a source of natural variation that can be used as a comparatively inexpensive way to improve the curriculum and, ultimately, improve the instruction that arises from implementing the curriculum. But lack of fidelity in implementing a curriculum introduces variation in teaching that either decreases or enhances learning opportunities for students. Such variation can be treated as data that suggest improvements to the curriculum. The purpose of this article is to explore a method for treating lack of fidelity as a learning opportunity for teacher educators and to illustrate the way in which the method can be used to improve a curriculum for preservice teachers.

Background

Reducing Variation and Raising the Mean

The method I describe for studying variation in teaching due to lack of fidelity in implementing a curriculum is based on an assumption not universally endorsed in the United States. Simply put, the assumption is that the goal of improving classroom teaching requires reducing variation in teaching from classroom to classroom and, at the same time, raising the mean level of teaching quality (e.g., Raudenbush, 2009).

The assumption more commonly accepted in the U.S. is that variation in teaching is necessary and sometimes even desirable. Variation in teaching is sometimes seen as a wise response to different local conditions and, in turn, a recognition of professional respect for individual teachers (e.g., Duffy & Hoffman, 1999; Lloyd et al., 2009; Remillard, 2005). In this view, reducing variation implies diminishing teachers' roles in making professional judgments about their own classrooms. I argue for a different point of view. I agree with Nicolas Kristof who, in his *New York Times* column, described the variation in teaching from classroom to classroom as a major national problem in education, often diminishing the learning opportunities for students (2009).

In the setting I describe, teachers are teacher educators, students are K–8 preservice teachers, and classroom teaching is the teaching of preservice mathematics courses. But the issues of implementing a planned curriculum, the lack of fidelity in doing so, and the resulting variation across classrooms are otherwise the same as those arising in school settings. In other words, I believe the method I describe can be used by teachers (teacher educators or classroom teachers) who are teaching from the same curriculum with the same student learning goals in mind. This is true for a range of curricula, from traditional to reform. The only requirement is that the person(s) conducting the observations and proposing revisions understand well the learning goals *and* the intent of the curriculum being improved.

Working Toward a Theory of Implementation

Studying the lack of fidelity when implementing a curriculum is best guided by a theory of implementation. Over the past several decades, it has come to be recognized that there is a large gap between intended instructional treatments and the outcomes of the treatments. Implementing treatments has become an object of study in its own right (Fullan & Pomfret, 1977; Lipsey, 1993; O'Donnell, 2008; Remillard, Herbel-Eisenmann, & Lloyd, 2009). But theories of implementation are not well established. To set the stage for the method I describe, I present here the beginnings of a theory of implementation for the case I will examine: implementing lesson plans designed for mathematics content courses for preservice K–8 teachers.

It is reasonable to expect theories of curriculum implementation to address two key questions: (1) What does a curriculum (in this case a set of lesson plans) need to contain to be implemented as intended? (2) How will we know whether it has been implemented as intended? That is, how can we measure its implementation?

I hypothesize that a lesson plan for a preservice content course should include the following features. These features seem to be essential for both prescribing a lesson designed to help students achieve the learning goals and helping instructors implement the lesson as intended (Hiebert & Morris, 2009; Morris & Hiebert, 2011).

- A complete and precise statement of the learning goal(s).
- Explanations (rationales) for how each instructional activity is designed to help students achieve the learning goal(s).
- Descriptions of each instructional activity, including descriptions of the activities themselves and descriptions of the pedagogical approach that should be used. Descriptions of pedagogy explain how the recommended instructional moves derive from the theory of learning on which the lessons are based (a theory described below).
- Responses that students are expected to give to the instructional tasks, suggestions for how the instructor might respond, and rationales for the suggested instructor responses.
- Samples of verbal explanations that instructors can present at key moments in the lesson.
- Review of content that instructors might need if they haven't encountered this content or how it is treated in the lesson.

An example of a lesson plan with these features is provided on page 84.

How can the implementation of a lesson be measured? In other words, how can one tell whether a lesson has been implemented as intended? Answering this question in general is beyond the scope of this paper because there are numerous issues that must be considered (Fullan, 2008; Huntley, 2009; Remillard, 2005; Remillard et al., 2009). But it is possible to offer a brief answer to the measurement question that serves as a working definition for the method I describe.

I define lack of fidelity as follows: Implementations of individual lessons lack fidelity if they include teacher moves, not prescribed in the lesson, that represent (1) *significant variations* of the lesson, or (2) *positive adaptations* of the lesson. Significant variations of the lesson include teacher explanations, class discussions, or instructional activities that (a) do not appear to help students achieve the learning goal(s) as effectively as those prescribed in the lesson, or (b) violate the learning theory on which the lessons are based. Positive adaptations of the lesson are teacher explanations, class discussions, or instructional activities that (a) appear to help students achieve the learning goal(s) more effectively than those prescribed in the lesson, or (b) change the lesson to make it more compatible with the learning theory on which the lessons are based.

It should be noted that not all variations are significant variations or positive adaptations. Only variations that fit the definitions just presented are considered to be instances of lack of fidelity. Many variations can occur that are not considered to be implementation infidelities. For example, class discussions can take a variety of forms without containing significant variations or positive adaptations.

The learning theory on which the lessons in this study are based is actually a pair of learning principles rather than a full theory. The two principles, as described by Hiebert and Grouws (2007), are especially relevant for learning goals that have a heavy conceptual component: (1) conceptual relationships among mathematical ideas, representations, and procedures must be made clear, and (2) students must be given an opportunity to grapple or struggle with the critical mathematical concepts. Given these two principles, it is possible to define significant variations of type (b) above as teacher actions that remove one or both of these principles from the intended learning opportunities, and to define positive adaptations of type (b) above as teacher actions that improve the learning opportunities for students in ways that are relevant to the learning goals and are consistent with these two learning principles.

Implicit in the theory of implementation just presented is an assumption that studying variation in teaching when implementing a curriculum does not yield claims of *best* teaching practices. The intent is to yield *better* teaching practices by building into the enacted curriculum the positive adaptations observed and to eliminate from the enacted curriculum the significant variations. It is assumed that best teaching practices for the learning goals specified in a curriculum are an ideal that teachers work toward by incrementally improving the curriculum through studying its implementation through methods like those described next.

Method

Setting

I am a mathematics teacher educator at a large university in the mid-Atlantic region of the United States. The K–8 teacher certification program is completed in four years and graduates about 150 students per year. The mathematics portion of the program includes three mathematics content courses and one mathematics methods course. If students wish to obtain an endorsement to teach mathematics in middle school, they can take an additional four mathematics courses and one mathematics methods course.

The work I describe centers on the first of the three mathematics content courses required for all K–8 preservice teachers. This course focuses on whole numbers and decimal numbers. Classes are limited to 35 students, so multiple sections of the course are offered each semester. The instructors consist of faculty, doctoral students, and adjunct instructors.

The curriculum for the course consists of detailed lesson plans for each class session in the semester. The lesson plans were developed over time by mathematics education faculty and doctoral students working together. Each semester, instructors for a particular course meet weekly to develop, test, and refine the lessons (Hiebert & Morris, 2009). Critical for the work reported here is that the lesson plans contain the features identified earlier (e.g., learning goals, rationales for activities, detailed descriptions of activities, predicted student responses and suggested instructor responses, teacher explanations, and reviews of content).

I am an author of the curriculum and an experienced instructor of the course. This means that I understand well the learning goals for the course and the intentions of the curriculum. This allowed me to develop strong hypotheses during my observations about which changes to the intended curriculum were significant variations, which were positive adaptations, and which were neither. As noted earlier, the method depends on at least one person possessing deep knowledge of the learning goals and the curriculum. It is this knowledge that allowed me to generate hypotheses about implementation variations.

The instructors observed in this study were adjunct instructors who had not been involved in the original process of lesson development and did not have frequent interactions with instructors who had been involved in this process. This meant it was likely that these instructors would implement the curriculum in some ways that varied from the intent of the authors. The variations could be better or worse adaptations—they could increase or decrease students' opportunities to achieve the learning goals. Because these data provide the key opportunities to learn how to improve the curriculum, it is important that at least some of the observed instructors are less familiar and less experienced with the curriculum than the observer. These conditions often exist in teacher education programs and schools. Although the instructors in this study met weekly with each other to review past and future lessons to enrich their interpretation of the curriculum, I assumed that their relative inexperience would yield variations in implementing the curriculum that would be worth recording.

Procedure

I observed 24 of the 27 sessions in each of the two sections of the first mathematics content course, one section for each of the two instructors. Observing more than one instructor was useful for sorting out whether significant variations were due to the written lesson plan or idiosyncrasies of the instructor. I recorded written notes on all teacher statements and student statements intended for the whole class, including student responses to the instructional tasks. I flagged places in the lesson plan where, in my judgment, positive adaptations and significant variations occurred and wrote notes in the margin of the lesson plan that would help me reconstruct the nature of the positive adaptations and significant variations. For significant variations, I noted the feature of the lesson plan that was likely responsible. For example, if instructors carried out an instructional activity differently than described in the plan, I marked the activity itself and, depending on the nature of the significant variation, the statement of the learning goal or the description of the pedagogical approach that was apparently misinterpreted or ignored. Within 1 or 2 days of observing a lesson, I reviewed all written notes, typed detailed descriptions of the instances of lack of fidelity, and made changes to the written lesson plan that either captured the positive adaptations or elaborated or corrected the lesson plans to reduce the likelihood of the significant variations in the future.

Findings

I will present findings from this work by reporting three examples in considerable detail. For findings to be useful, they need to inform several aspects of the lesson improvement process driven by documenting lack of fidelity of lesson implementation. In particular, the findings must identify the source of a hypothesized significant variation in terms of the lesson features posited in the theory of implementation, they must explain why a teacher move was hypothesized to be a significant variation or positive adaptation, they must suggest fixes to the lesson plan, and they must provide a basis for hypothesizing that the fix will lead to reduced variation across instructors and a more effective level of teaching for everyone. The examples presented below illustrate these features of the findings.

Example 1: Misinterpreting the Learning Goal

Source of the Significant Variation

I assumed that the general source for all significant variations was the written lesson plan (rather than the instructor) because written plans are always imperfect and incomplete, and significant variations result from a misinterpretation or selective interpretation of the plan. The first example stems from variations to the lessons I interpreted as significant that occurred in both instructors' sections during Lessons 2–4. Because of the nature of the variations, I attributed the problem to a misinterpretation, or more accurately a selective interpretation, of the learning goals for these lessons. As stated in the lessons, the learning goals for Lessons 2 and 3 were:

1. Preservice teachers will understand the terms *numeration system*, *quantity*, *numeral*, and *number* and the relationships among them.
2. Preservice teachers will recognize the properties of numeration systems: additive, multiplicative, subtractive, positional, place-valued, and the meaning of zero.
3. Preservice teachers will understand that the symbolic representation of a quantity in any numeration system is determined by decomposing it into parts equal in size to the measuring units of the numeration system, and representing the total amount of equal-sized parts with symbols, according to certain rules. The size of these parts as well as the symbols and rules used to represent them vary from system to system.

The learning goals for Lesson 4 were:

1. Preservice teachers will understand the properties of based place-valued numeration systems. Preservice teachers will understand that a based place-valued numeration system consists of a set of measuring units, a finite set of symbols, and a collection of rules that determine the structure of the system.
2. Preservice teachers will construct a set of measuring units associated with the place values for any based place-valued numeration system.
3. Preservice teachers will be able to represent the same quantity with different based place-valued numeration systems.

Although the authors of the lessons intended the preservice teachers to work with actual quantities and pictures of quantities to develop an understanding of the concepts underlying numeration systems, and Lessons 2–4 included instructional activities that engaged preservice teachers in doing just that, both instructors eliminated many of the activities that involved breaking quantities into parts equal in size to the measuring units of a given numeration system and activities that involved creating pictures to represent different-sized units. Instructors taught the lessons using primarily written words, numerals, and arithmetic calculations. For example, the lesson plans ask the teacher to repeatedly engage students in instructional activities that involve making place-value charts that show the measuring units of a numeration system with pictures of quantities. In contrast, the instructors placed measuring units in place-value charts but usually labeled the positions only with words and numerals (e.g., for the Babylonian system, "ones," "60s, "60 × 60," and so on).

When asked to identify the most important learning goal for these lessons, one of the instructors chose this goal:

> **Preservice teachers will understand that the symbolic representation of a quantity in any numeration system is determined by decomposing**

> **it into parts equal in size to the measuring units of the numeration system, and representing the total amount of equal-sized parts with symbols, according to certain rules. The size of these parts as well as the symbols and rules used to represent them vary from system to system.**

This was indeed the most important learning goal that guided the writing of the lessons. The instructor was not ignoring the goal, but rather interpreting it differently than the lesson writers did. The instructor emphasized the symbolic aspects of the goal, whereas the lesson writers emphasized the quantitative aspects.

The instructors made decisions to omit or modify many of the activities that were intended to engage students in concrete or pictorial work and concentrated instead on symbolic presentations and manipulations. I classified these changes as significant variations because they violated the principle of learning that calls for conceptual relationships among representations to be made clear and therefore did not appear to help students achieve the quantitative aspects of the learning goal. Based on past experience with these lessons, I know that students who have not developed physical, quantitative images for units of different sizes with the relationship of, for example, "10 times as big," will have difficulty when they are asked in future lessons to extend their knowledge of whole-number systems to decimal fractions less than 1. So, these instances of lack of fidelity will limit students' opportunities to achieve the later learning goals.

Fixing the Lesson

Fixing the lesson means revising the lesson plan to communicate more clearly to the instructors the feature of the lesson that seemed to be the source of the problem. Because I attribute the misunderstanding that prompted the significant variations to the lessons just described to a selective interpretation of the learning goals, I focused my attention on restating the learning goals more completely and clearly. I will use the phrase *elaborated learning goal* to signify these revised versions of the learning goal.

Rather than just restate the learning goals in clearer language, a fix that might have little effect, I decided to elaborate the learning goals to include a description of how achievement of the goal will be measured plus a scoring rubric, of sorts, that provides an unambiguous standard against which students' performance can be assessed. This is not a new idea. The assessment literature argues that including "performance objectives" with "content objectives" helps clarify the intent of the content objectives (Cook, 2008; Kapfer, 1971; Mager, 1997). The elaborated learning goals I developed are consistent with this general idea but were designed with a very specific correction in mind. In particular, I wanted to ensure that when future instructors read the learning goals, they would not be able to ignore the quantitative aspects of the goal. To accomplish this, I described actions and explanations that students are expected to display as they work toward achieving the learning goals. My purpose is to develop among instructors (a) a deeper shared understanding of the learning goals, and (b) a clearer sense of what to do in the instructional activities to help students achieve these goals. Because the actions and explanations I describe for students are about quantities, not just symbols, I hypothesize that future instructors will no longer be able to ignore this aspect of the learning goals. The elaborated learning goals clearly establish the intended emphasis of the instructional activities. Ironically, if these elaborated learning goals are taken seriously by instructors, the instructors have more freedom in how they implement the suggested instructional activities. In a real sense, the prescription for the lesson moves from the instructional activities into the statement of the learning goals.

Compare, as an example, the original learning goals for Lesson 2 (presented earlier and restated below) with the elaborated learning goals I created as the lesson fix to reduce variation across instructors. The original learning goals were the following:

1. Preservice teachers will understand the terms *numeration system*, *quantity*, *numeral*, and *number* and the relationships among them.
2. Preservice teachers will recognize the properties of numeration systems: additive, multiplicative, subtractive, positional, place-valued, and the meaning of zero.
3. Preservice teachers will understand that the symbolic representation of a quantity in any numeration system is determined by decomposing it into parts equal in size to the measuring units of the numeration system, and representing the total amount of equal-sized parts with symbols, according to certain rules. The size of these parts as well as the symbols and rules used to represent them vary from system to system.

The elaborated learning goals for this lesson are these:

1. Preservice teachers will distinguish between numerals (symbols) and quantities (physical amounts of stuff). They will distinguish between actions on numerals (arithmetic) and actions on quantities. Why do we want preservice teachers to make this distinction? Ideas about quantities will be emphasized throughout the course. This emphasis will help preservice

teachers recognize that mathematics is not just about symbols and calculation—that mathematical ideas are often about quantities and actions on quantities and that quantities can serve as concrete referents for mathematical ideas.

2. Preservice teachers will understand the terms "measuring unit" and "basic symbol" and will be able to use these terms when they explain how a quantity is assigned a numeral. They will understand that each numeration system has a set of measuring units and a set of basic symbols that it uses to represent all quantities. Preservice teachers will understand that, in any numeration system, a quantity (amount of stuff) is assigned a numeral by decomposing the quantity into parts equal in size to the measuring units and representing with the basic symbols of the numeration system how many measuring units of each type fit in.

3. Preservice teachers will understand that measuring units are quantities (physical amounts of stuff) that are used to measure other quantities, whereas basic symbols are symbols. Understanding that measuring units are quantities will make the study of decimals (later in this course) less abstract for the preservice teachers and will allow them to reconceptualize and be successful with decimals, whereas in their previous mathematical experiences, most of them were not.

4. Preservice teachers will develop and show these understandings by carrying out the following mathematical actions and giving explanations that involve these actions:

    a. Using any numeration system, preservice teachers will be able to physically measure quantities by physically partitioning the quantity to be measured into parts equal in size to the measuring units and determining how many measuring units of each type fit into the quantity. For example, preservice teachers will be able to measure a set of 23 dots in the Hindu-Arabic system by circling 2 separate measuring units of size 10 dots each, and 3 separate measuring units of size 1 dot each. In the Babylonian system, preservice teachers will measure 437 straws by physically bundling 7 separate measuring units of size 60 straws each and 17 separate measuring units of size 1 straw each.

    b. After physically measuring a quantity in this way, preservice teachers will be able to represent the measured quantity numerically by using the basic symbols of the numeration system to show how many measuring units of each type fit into the measured quantity.

    c. For place-valued numeration systems, preservice teachers will be able to make a place value chart that represents this process of measuring and assigning a numerical value to a quantity. The first row of the place value chart should show the measuring units as pictures of physical amounts. For example, in a place value chart in the Babylonian numeration system, a measuring unit of size 1 could be shown as 1 dot, a measuring unit of size 60 as 60 dots, a measuring unit of size 3600 as 3600 dots. (Because it is too hard to draw large quantities, preservice teachers can use the notation [3600] to represent 3600 dots.) In the Hindu-Arabic system, a measuring unit of size 1 might be shown as the area of 10 squares on graph paper. The measuring unit of size 10 would then be shown as 100 squares, a measuring unit of size 100 would then be shown as 1000 squares, and so on. The purpose of drawing the measuring units as amounts of stuff in the place value charts is to emphasize to preservice teachers that measuring units are amounts of stuff, not numerals. (This will allow a smooth transition to decimal numbers and operations in future lessons.)

    The place value chart should also show the multiplicative relationship between the measuring units. For example, when using the Babylonian numeration system, preservice teachers should draw an arrow from each measuring unit in the place value chart to the next largest measuring unit, label the arrow "× 60," and be able to explain (and to demonstrate with quantities) that this means that each measuring unit is 60 times as big as the measuring unit that is associated with the place to the right, that 60 copies of the smaller measuring unit will fit into the larger measuring unit, that we can find the larger measuring unit by making 60 copies of the smaller measuring unit, and that we can find the size of the smaller measuring unit by partitioning the larger measuring unit into 60 equal parts. Finally, in the second row of the place value chart, preservice teachers should show the number of measuring units of each type that fit into the measured quantity, represented with the basic symbols of the system. For example, Figure 1, a place value chart for base six (where the measuring unit of size 1_{six} is a circle), shows that four measuring units of size 1000_{six}, two measuring units of size 100_{six}, zero measuring units of size 10_{six}, and five measuring units of size 1_{six} fit into a measured quantity.

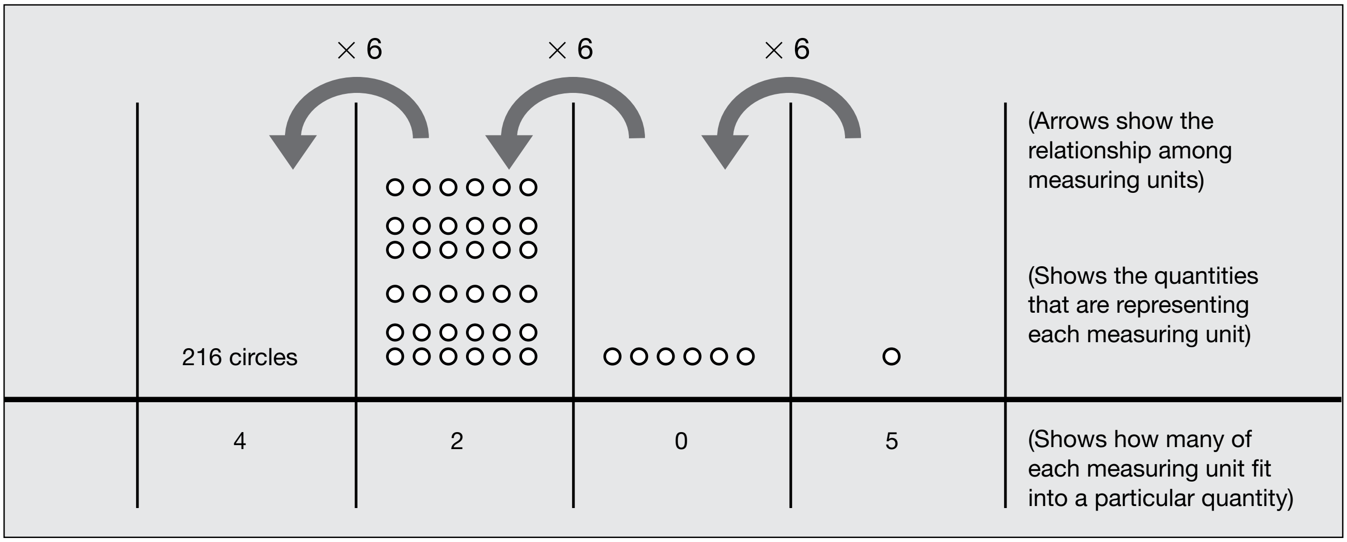


Figure 1. **The place value chart for base six.**

Thus, the place value chart is a concrete picture of the meaning of a numeral; place value charts provide a picture of the measuring units that are associated with the digits in a numeral. The use of a place value chart over multiple lessons increases the probability that preservice teachers will understand the meaning of each digit in a numeral after they move to representing numerals without the aid of a place value chart and will allow a smooth transition to decimal numbers less than one and to measuring units smaller than the measuring unit of size one.

d. Given a numeral in a given numeration system, preservice teachers will be able to represent the numeral with a quantity.

Why the Lesson Fix Should Reduce Variation

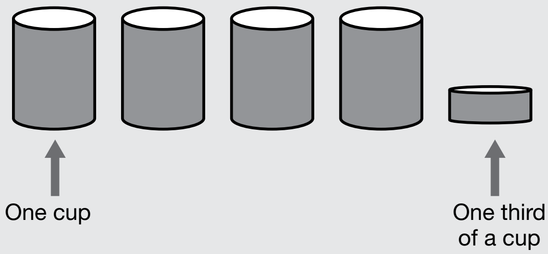
Obviously, the elaborated learning goals are much longer and more detailed than the original learning goals. But it is not just the length or detail that I believe is critical; it is the prescription of students' actions (described in 4a, 4b, 4c, and 4d above) that will be taken as evidence of students' achievement of the learning goals that greatly increases the chances that future instructors will implement the lesson without these significant variations.

To understand why this might be true, consider a lesson designed to help students achieve strictly procedural learning goals, with no conceptual component. It is easy to see that it would be straightforward to write such a goal, with no ambiguity, and that there would be little question about the actions students should take to show competence. This means it is likely instructors would interpret the procedural goal in the same way and follow similar instructional paths. In other words, a shared interpretation of a learning goal is likely to channel instructors onto a similar instructional path (at least, the variations they display are less likely to be significantly different).

With respect to clarity and shared interpretation, the challenge is to write conceptual learning goals like procedural learning goals. Elaborated learning goals are designed to meet this challenge. If instructors have the same understanding of the actions and explanations that students must master to demonstrate competence in a conceptual learning goal, then it is likely instructors will interpret the conceptual goal in the same way and, in turn, follow similar instructional paths.

But readers might be asking whether describing the actions and explanations that students must provide to demonstrate competence will encourage instructors to teach in a rote, procedural way targeted toward the desired outcomes. Will instructors just teach to the test? Two features of the lesson mitigate the danger of this happening. First, the actions and explanations that provide the goals for instruction are sufficiently complex that it is difficult to imagine instructors getting students to memorize these and use them flexibly on a range of problems. For example, by Lesson 5, students are asked to apply the actions and explanations described in the elaborated learning goals above to solve problems like the one from Lesson 5 shown in Figure 2. If students can apply the actions and explanations to flexibly solve a range of problems, they probably have developed some level of conceptual understanding. A second feature of

Suppose you are trying to represent this quantity of liquid (below) in base ten. You let the measuring unit of size 1 be one cup. Make a place value chart that shows all the measuring units that you need as you try to find a base ten numeral for this quantity, and explain how you found a base ten numeral for this quantity. What is the base ten numeral for this quantity?



From your work on this problem, explain why we get “repeating decimals.”

***Figure 2.* A problem presented in Lesson 5.**

the lessons that reduces the likelihood of rote instruction is that the instructional activities described in the lessons are consistent with the two learning principles identified earlier that support conceptual learning.

Example 2: Not Recognizing Students’ Lack of Understanding

Source of the Significant Variation

The second example is drawn from the same set of lessons but targets a different instance of lack of fidelity. As I observed Lessons 2–4, I noticed that the instructors were receiving no feedback from students regarding students’ lack of understanding of the quantitative aspects of the learning goals. Students were following the instructors’ lead, creating rules governing numerical manipulations, and completing the problems they were assigned correctly, but without understanding the relationships among quantities. Their lack of understanding was obvious during their small-group discussions, but the answers they produced were correct frequently enough for the instructors to presume that the students were achieving the learning goals.

Practically, the problem with the lesson plans was that the student assignments for Lessons 2–3 could be completed by interpreting them through either the instructors’ interpretation of the learning goals (e.g., calculating numerical values for the positions in a place value chart) or the lesson authors’ intended learning goals (e.g., forming units for the positions by partitioning and combining quantities). Students used numerical approaches to solve the assigned problems, instructors accepted these responses as indicating achievement of the learning goals, and there was no conflict that would have otherwise warned the instructors that something was wrong.

It seems reasonable to assert that curricula (e.g., the lesson plans of interest here) should contain student assignments that signal the instructor when students have not yet developed the conceptual understanding that is the intended focus of the learning goals. Curricula should include student tasks that provide feedback to instructors about whether they, the instructors, are on the right track.

That the student assignments in Lessons 2–3 could be completed without providing useful feedback to the instructors prompted me to reconsider the theory of implementation presented earlier. There is no feature of lesson plans in the bulleted list that requires such assignments to be included. I take this example as one that argues for adding this feature to the list, thereby refining the original theory. This example illustrates how the study of lack of fidelity or variation in teaching not only can improve teaching but also can refine the theory of implementation that connects the curriculum with classroom practice.

Fixing the Lesson

The fix for the problem of receiving no useful feedback regarding students’ understanding is suggested by the problem itself: include student assignments that explicitly focus the instructors’ and students’ attention on the quantitative relationships in the learning goals. If students cannot complete the tasks without attending to quantitative relationships, then errors on these tasks would signal to the instructors that quantitative understandings had not been developed sufficiently during instruction. Here is an example of an assigned student task before I fixed the lesson and after I fixed the lesson.

Before revision:

Represent the following quantities using the Babylonian numeration system.

(a) O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O

(b) 780,021 circles

The students tended to solve part (a) by counting the number of dots (91), calculating 91 ÷ 60 = 1 remainder 31, and then writing the Babylonian symbol for 1 in the 60s place and 31 in the ones place. They did not break the quantity of dots into parts equal in size to the measuring units (as the lesson writers intended) nor did they draw a place value chart that showed pictures of the relevant measuring units and the multiplicative relationship between them. The students were unable to solve (b) because they got lost in all the numerical calculations.

After revision:

a. Make a place value chart in the space below. In the first row of the place value chart, draw the first three measuring units for the Babylonian numeration system. Use a circle to represent a measuring unit of size 1. (Remember that when a quantity is too large to draw, you can show it with square brackets. For example, the measuring unit of size 3600 circles can be represented as [3600].) Show the multiplicative relationship between the measuring units using the arrow notation.

b. Now bundle the following quantity using the Babylonian measuring units. To make these bundles, start with the largest measuring unit that fits in and determine how many fit in. Now move to the next smallest measuring unit. See how many fit in. Continue the process until the quantity is completely partitioned into parts equal in size to the measuring units.

O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O

c. Now in the second row of your place value chart in part (a), show the number of measuring units of each type that fit in, using the basic symbols of the Babylonian numeration system.

To provide additional feedback to the instructor about whether the appropriate quantitative relationships had been developed during instruction, problems that cannot be solved using only a symbolic numerical solution were added to the assignment. An example follows:

Do you think the Babylonian numeration system could be used to assign a numeral to a quantity that is smaller than a measuring unit of size 1? How could you do that? (Hint: Choose your measuring unit of size 1 very carefully.) Use a place value chart and a quantity to show how this could be done. How do we do it in the Hindu-Arabic numeration system?

Why the Lesson Fix Should Reduce Variation

My observations of later lessons provide the best evidence that including better designed student tasks will provide feedback to the instructors that will help reduce significant variations from the intended lessons. The student tasks assigned after Lesson 4 more clearly ask students to use their knowledge of relationships among quantities rather than just carry out numerical calculations. One of the instructors noticed that the students were having difficulty with these tasks and spontaneously began reinserting some of the activities on building relationships between physical quantities that the instructor had previously dropped.

An open question is whether reducing variation in the lessons will require *both* an elaboration of the learning goals (as described under Example 1) *and* a redesign of the student assignments (as I describe here). I conjecture that both these fixes are needed. Because multiple interpretations of a written lesson plan are possible, redundancy in lesson features that clarify the intentions of the curriculum authors can only serve to reinforce a reduction in significant variations.

Example 3: A Positive Adaptation for Modeling Partitioning Division

Why the Teacher Move Was Hypothesized to Be a Positive Adaptation

The third example illustrates how positive adaptations can be used to improve the quality of a curriculum. In Lessons 11–13, both instructors modeled partitioning division in a way that was not prescribed by the lessons but, in my judgment, better prepared the students for the conceptual development of the long division algorithm in Lesson 18. For the problem 0.8 ÷ 4 = ?, for example, the plans for Lessons 11–13 encouraged students to model the problem by either (a) drawing or making a quantity of size 0.8 and then partitioning it into four equal parts to determine the size of each part or (b) using a doling-out process. In the latter approach, students would model 0.8 with eight longs (base ten blocks), for example, and then give one long to group 1, one long to group 2, one long to group 3, one long to group 4, one long to group 1, and so on until all the longs were distributed equally among the groups. When students extended this type of solution to problems like 0.72 ÷ 4 =?, they thought of 0.72 as 72 measuring units of size .01 and doled the 72 units out, one by one, to the four groups. Both instructors, however, presented an additional approach that was not prescribed in the lesson plans. They represented 0.72 as seven longs and two unit blocks, for example, and explained that only one long could be distributed to each of the four groups. To distribute the remaining three longs, they would have to exchange each long for 10 unit blocks, combine the resulting 30 unit blocks with the other 2 unit blocks, and then distribute the 32 unit blocks equally to the four groups. In general, they started with the largest measuring unit of a quantity, distributed as many as they could to the *n* groups, exchanged the remaining measuring units of that size for the next smallest measuring unit, combined them with the other measuring units of that size, distributed as many as they could to the *n* groups, and so on.

I classified these teacher moves as a positive adaptation because they were more consistent with the principle that conceptual relationships among mathematical ideas, representations, and procedures must be made clear. The instructors' approach to concrete modeling (with blocks, graph paper, straws, etc.) developed the concepts underlying the long division algorithm, whereas in the original lesson plans, there was a conceptual discontinuity between the modeling in Lessons 11–13 and the development and modeling of the long division algorithm in Lesson 18.

Fixing the Lesson

Fixing the lesson means revising the lesson plans to incorporate the additional approach to modeling partitioning division. The approach was also included in the elaborated learning goals for Lesson 12, as part of the described actions and explanations that students are expected to display as they work toward achieving the learning goals. A rationale for the approach and its connection to the long division algorithm was added to make instructors aware of the connection.

Why the Lesson Fix Should Reduce Variation

Traditionally, teaching that deviates from a planned curriculum and creates more effective learning opportunities remains a variant not replicated by other teachers. In fact, U.S. educators often celebrate teachers who invent more effective practices than those suggested by the curriculum. But these practices usually remain the province of the inventor. By writing positive adaptations into the planned curriculum, these practices can be replicated by all teachers. The variation is reduced because it becomes standard practice.

Discussion

Nature of the Evidence Gathered to Conduct This Work

I began the article by claiming that improving teaching requires both reducing the variation in teaching across classes with similar learning goals and raising the mean level of teaching. I would like to conclude by pointing out that different kinds of evidence can be used to address these two linked research and policy goals. Specifically, reducing variation among teachers requires evidence of teaching moves and student responses during instruction, whereas improving the mean level of teaching requires evidence of students' achievement. The importance of this distinction is seen both in the design of research and in the expense of conducting it.

The work I described in this article focused on generating hypotheses about changes to the curriculum that would reduce variation in future enactments and increase students' achievement of the learning goals. I used teacher moves and student responses during the lesson as data to catch places in the lesson where instances of lack of fidelity occurred and to hypothesize whether they were significant variations or positive adaptations. Testing whether changes I proposed based on these hypotheses will reduce variation in the future requires further observations of teaching. This means that repeated observations of teaching, focused on instances of lack of fidelity, can be sufficient to generate *and* test hypotheses about implementation variations and how to reduce them.

A value of this claim is that it allows teacher educators and teachers to study curricula by taking advantage of the natural variation that will occur as different instructors implement a shared curriculum with shared learning goals. Empirically based improvements in curricula require studying the effects of varying the curricula. Ordinarily, researchers plan variations and study the effects of these variations. The method I am describing complements this more expensive approach by simply observing and analyzing the variations that naturally occur in most teaching settings.

Whether the fixes to the lessons that reduce variation raise the mean level of teaching quality across instructors requires, of course, assessments of students' achievement. Do students across all sections of a course (or across any set of classrooms that share the same learning goals and use the same curriculum) achieve the learning goals more effectively after variation has been reduced than before? Collecting these data requires a phase of research not reported in this article but a phase that must follow the work reported here.

It should be noted, however, that reducing variation in the way I have described carries with it strong hypotheses that the mean quality of instruction will, in fact, improve. In simplistic terms, eliminating significant variations eliminates those aspects of the lesson plan that appear to unnecessarily dampen the learning opportunities for students, and inserting positive adaptations increases the learning opportunities for all students, not just those of the instructor who introduced the adaptation. Eliminating the weakest aspects of instruction from all classes and introducing stronger aspects of instruction into all classes should increase the average quality of instruction. But, as noted earlier, these are hypotheses that must be tested empirically.

I would like to make a final point about the critical role of empirical data in the process I have described. How to write learning goals so instructors do not misinterpret them, and how to write student assignments so teachers and students do not miss or bypass the intent, are empirical questions. It is impossible to know whether one has succeeded without empirical observations because people are capable of interpreting written text in multiple ways. Writing shared learning goals and creating tasks that provide critical feedback to instructors are not usually thought of as empirical issues. I believe this is especially true of writing learning goals, so I would like to elaborate on this particular claim. Learning to write goals that are interpreted similarly by all instructors requires observing how instructors operationalize the goals during instruction. It is not just a matter of writing out the goals in more detail, or even a matter of including performance objectives with the learning goals. Rather, writing learning goals for which a shared understanding develops among instructors requires an empirical cycle of writing goals at a grain size that reduces misinterpretation *and then* observing multiple instructors and classes to learn whether the goals are enacted as intended *and then* modifying them according to the information gathered *and then* asking instructors to implement the lessons again, and so on. It is impossible to predict beforehand which goals will be interpreted in a common way and which will be interpreted in different ways. The empirical cycle of observations and revisions is essential for writing learning goals that enable shared understandings among instructors.

Professionalizing Teaching

How does variation in teaching influence its professionalization? As noted earlier, some have argued that accepting variation among teachers' practices signals professional respect (e.g., Duffy & Hoffman, 1999; Lloyd et al., 2009; Remillard, 2005). In this view, reducing variation implies diminishing teachers' roles in making professional judgments about their own classrooms. But I, along with others (Shanker, 1997; Stigler & Hiebert, 1999), believe the process of improving teaching by reducing variation supports, rather than undermines, the professionalization of teaching. The goal of reducing variation is to improve the quality of teaching for all students. To paraphrase Al Shanker (1997), the goal is to make the best we know standard practice.

Both cases I described in this report place teachers (teacher educators) in a position of making professional judgments about the relative quality of learning opportunities. In one case, the lesson plan (or curriculum) failed to clarify for instructors the intent or the details of a lesson. The task for teachers studying curriculum enactment is to recognize these deficiencies as they play out in the classroom and to identify the features of the intended curriculum that can be corrected or elaborated more clearly. In the second case, the task is to recognize a richer variant of the lesson as introduced by an instructor and build this into the shared curriculum. In both cases, the aim is to use teachers' professional judgments to create steady and lasting improvements in the practices of teaching, a sure sign that teaching is being treated as a true profession.

The Benefits of Studying Implementation and Making Small Changes

The findings reported in this paper might seem overly focused and narrow, and the changes to the curriculum small and obvious. The findings themselves are probably of little interest to teacher educators who do not share the same learning goals. But the real message of this paper is that this is exactly the nature and grain size of the work that needs to be done to improve the implementation of curricula—collecting details about implementation through empirical observations, using the observations to revise the curriculum and to revise theories of implementation, and repeating the process. This unglamorous nitty-gritty work usually produces only small changes, but these changes can build to produce a curriculum that is implemented in less varying ways across instructors and that eventually improves student achievement.

I believe this work can be conducted by teachers and teacher educators across a range of settings. What is required is that two or more teachers share the same learning goals for their students, teach using the same curriculum, and differ in their understanding of the learning goals and the intention of the curriculum. These conditions exist in many teacher education programs and in many K–12 school settings. Often the most experienced teachers, or the most reflective teachers, will have developed a deep understanding of the learning goals and the curriculum. These teachers are able to develop informed hypotheses about variations from the intended curriculum that either raise or lower the learning opportunities for students. They can then propose curriculum changes to the group to reduce these variations. Finally, the effects of these changes can be tested through repeated observations and student assessments.

This work is not for those wishing for quick fixes. It is ongoing and yields small, incremental improvements. But the improvements are steady, can be preserved across changes in teachers, and can cumulate over time to yield substantial improvements in the quality of teaching and students’ learning.

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Author Note

Preparation of this article was supported, in part, by the National Science Foundation, Grant 0083429 to the Mid-Atlantic Center for Teaching and Learning Mathematics. The opinions expressed in the article are those of the author and not necessarily those of the Foundation. Thanks to James Hiebert for his comments on earlier drafts of the paper.

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SUPPLEMENT

Using "Lack of Fidelity" to Improve Teaching

Anne K. Morris

Lesson 5
Topic: Place Value II

Learning Goals

1. For based place-valued numeration systems, preservice teachers will extend the following understandings to non-whole number numerals and to measuring units smaller than the basic measuring unit (the measuring unit of size 1).
   a. Preservice teachers will understand that measuring units are quantities (physical amounts of stuff).
   b. Preservice teachers will understand that in any base *b* place-valued system, there is a *b* times relationship between the measuring units. They will understand that, for a given measuring unit, we can construct the next largest measuring unit by making *b* copies of the given measuring unit. They will understand that, for a given measuring unit, we can construct the next smallest measuring unit by partitioning the given measuring unit into *b* equal parts. One of these parts will be equal to the next smallest measuring unit. They will understand that *b* copies of a measuring unit fit into the next largest measuring unit.
   c. The preservice teachers will understand that a quantity (amount of stuff) is assigned a numeral by decomposing the quantity into parts equal in size to the measuring units and representing with the basic symbols how many measuring units of each type fit in.
2. The preservice teachers will develop a deeper understanding of the idea of a basic measuring unit.

Preservice teachers will develop and show these understandings by (a) carrying out the following mathematical actions and (b) giving explanations that involve these actions:

(The following actions and explanations are all within the context of based place-valued numeration systems.)

- Preservice teachers will be able to construct measuring units smaller than the basic measuring unit (the measuring unit of size 1).
- Preservice teachers will be able to make a place-value chart that includes measuring units larger and smaller than the basic measuring unit. The place-value charts should show the measuring units as pictures of physical amounts. The preservice teachers should clearly show how they constructed the set of measuring units; they should explain that they made *b* copies of a smaller measuring unit to find the size of the next largest measuring unit, or they should explain how they partitioned a larger measuring unit into *b* equal parts to find the size of the next smallest measuring unit. Measuring units smaller than the basic measuring unit should be shown in the place-value charts as separate quantities, not as shaded parts of a whole where the whole is the next largest measuring unit. The place-value chart should also show the *b*-times relationship between all of the measuring units with the arrow notation.
- Given a non-whole number numeral, preservice teachers will be able to represent the numeral with their constructed set of measuring units. They will represent the non-whole number portion of the numeral with measuring units that are shown as separate quantities, not as shaded parts of a whole, where the whole is the next largest measuring unit.
- For all quantities (i.e., including quantities that are represented by non-whole number numerals), the preservice teachers will be able to physically measure the quantities—that is, physical amounts of stuff—by physically partitioning the quantity to be measured into parts equal in size to measuring units and determining how many of each type of measuring unit fit into the quantity.

- After physically measuring a quantity in this way, preservice teachers will be able to represent the measured quantity numerically by using the basic symbols of the system to show how many measuring units of each type fit into the measured quantity.
- Preservice teachers will be able to flexibly use a variety of different-looking quantities to represent "1."
- The preservice teachers will be able to solve a series of challenging problems, which are posed in the homework for this lesson, that require these ideas.

Equipment

- 100 straws for the instructor
- Rubber bands
- Scissors

Associated Files

- Lesson 4 Homework
- Handout 1 (one copy for each student)
- Handout 2 (one copy for each student)
- Lesson 5 Homework (one copy for each student)

Associated Text

- Handout 2
- *Mathematics for Elementary School Teachers*, Bassarear, Section 2.3, pages 100–115

Time: 0–30 min.
Activity Flow: Part 1; Based Place-Valued Numeration Systems; Sets of Measuring Units

Activity

Assign each group a problem on the Lesson 4 Homework. Ask them to put their solution on the board or on a transparency. Have the groups present their solutions. Go over every problem. Do not skip problems, because the preservice teachers should have numerous opportunities to practice explaining the relevant concepts.

Student responses	Suggested teacher responses
Students may use base ten language when they verbally refer to their numeral. For example, for 203_{five}, they say "two hundred three."	If you say "two hundred three," you are referring to a different quantity than "two zero three base five." *Two hundred three* implies base ten, so the measuring units are different sizes than the measuring units in base five. Even if you use the same basic measuring unit for both bases, the other measuring units are different, so the quantities that are represented by the two numerals are different.
***Discussion of Problems 7 and 9c* [*equivalent quantities represented by different symbolic representations*]**	
Student responses	**Suggested teacher responses**
Students approach the conversions in a very procedural way. For example, they explain that the place values are generated by raising the base to the power of 0, 1, . . . *n*. Their answers may be correct, but they are unable to produce a conceptual explanation.	Procedures that are not developmentally appropriate will cause difficulties for your future students. Younger students will not know about exponents. Your explanation of the meaning of the numerals requires an understanding of the meaning of the numerals. For example, you are explaining that "'10' means one group of '10.'" But if I do not know what the symbol "10" means, the explanation is not helpful. When you are trying to explain the meaning of numerals to young children, it is more appropriate to explain the meaning of numerals in terms of quantities—for example, to show them what measuring units are associated with each place value and what the digits in a numeral tell you about the number of measuring units.
On Problem 9c, some students are unable to convert 102_{four} to base seven without converting it to base ten first and take a numerical approach rather than a quantity-based approach. They write that 102_{four} is equal to $(1 \times 16) + (2 \times 1) = 18$. Then they set up a place-value chart with the numerals 1 and 7 written in for the measuring units. Then they determine that the base seven numeral for 18 is 24_{seven}.	This is an excellent solution, but it is based on understandings about numerical representations and arithmetic. Let's also develop a solution that is based on understandings about quantities and how numerals represent quantities. You begin by choosing an appropriate/convenient basic measuring unit. You might choose a single straw to represent your basic measuring unit. Now let's make a place-value chart and draw the measuring units for base four. (Ask the students to help you create the measuring units for base four and draw them in the first row of the place-value chart.) Let's put the digits of our numeral in the second row of our place-value chart under our measuring units. This helps us to remember what 102_{four} means. The numeral 102_{four} means there are two basic measuring units, zero bundles of the measuring unit of size 10_{four} (the measuring unit that is equal in size to four basic measuring units), and one bundle of the measuring unit of size 100_{four} (the measuring unit that is equal in size to 16 basic measuring units or four measuring units of size 10_{four}.) (Draw this quantity on the board, bundled into the measuring units of base four.) We need to unbundle and rebundle this quantity, using the measuring units for base seven. Let's make another place-value chart, showing the measuring units for base seven. Using the same basic measuring unit of one straw, what is the next largest measuring unit for base seven? (Draw it in the place-value chart.) What is the next largest measuring unit? (Draw it in the chart.) What is the biggest measuring unit that will fit into the [now unbundled] quantity of straws? Yes, the measuring unit of size 10_{seven}. (Circle measuring units of size 10_{seven} in the unbundled quantity to show the measuring of this quantity.) Now we go to the next smallest measuring unit. This is the basic measuring unit. Will any basic measuring units fit into the remaining amount? We see that four basic measuring units fit in. Consequently, our base seven numeral would look like this: 24_{seven}. (Write 24 in the place-value chart under the appropriate measuring units to help the students understand the meaning of the numeral.) So the very same quantity is represented by different numerals in the two bases. The only difference is that we measure the quantity with different-sized measuring units. We bundle it differently, but in both cases, the numeral conveys to others how many bundles we have of each type.

Time: 30–50 min.
Activity Flow: Part 2; Based Place-Valued Numeration Systems; Place Values Less Than One

Rationale

This activity addresses the learning goals. It extends the preservice teachers' understandings about whole-number place values and numerals (understandings about measuring units, the relationship between measuring units, how to construct measuring units from other measuring units, the meaning of the basic symbols in a numeral, and the idea that a quantity is assigned a numeral by decomposing the quantity into parts equal in size to the measuring units and representing with the basic symbols how many measuring units of each type fit in) to non-whole number place values and numerals. The activity is intended to reinforce the idea of bundling or copying *b* measuring units to create the next largest measuring unit in base *b* and the idea of partitioning a measuring unit into *b* equal parts in order to create the next smallest measuring unit in base *b*.

The preservice teachers should see that numerals smaller than 1 are not an anomaly or frightening—that numerals less than 1 reflect the same relationships discussed previously and the basic symbols mean the same thing. By first working with an unfamiliar numeration system (base three), they develop more explicit understandings about based place-valued numeration systems and measuring units smaller than the basic measuring unit. (In addition, the use of an unfamiliar numeration system is designed to develop further preservice teachers' ability to recognize (a) that several component understandings are involved in representing quantities with numerals, in counting, and in computing with the standard algorithms, (b) that these understandings are not trivial or easily acquired, and (c) why children might experience difficulties as they try to develop these understandings.)

Activity

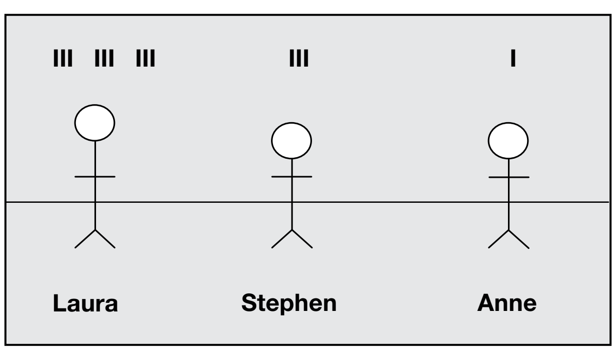
In this activity, students will model the structure of a base three system with straws. After a review of how measuring units are created in a base *b* system, they will figure out how measuring units smaller than the basic measuring unit are generated. The instructor will need straws, rubber bands, and scissors, as students at the board will be physically bundling and cutting straws to form base three measuring units.

> *We have been thinking about how to represent quantities with numerals. However, up to now we have only represented quantities that were equal to or bigger than the basic measuring unit. Today we will discuss how to represent with numerals quantities that are smaller than the basic measuring unit. Let's think about how we count in base three, and how we bundle quantities using the base three measuring units.*

Call three students to the board. Line them up (as shown in the diagram below).

> *Let this spot be for the basic measuring unit* [Anne's place]. (Draw a straw over Anne's head.) *This will be the basic measuring unit. What will the next measuring unit look like?* (Draw the measuring unit of three straws over Stephen's head.) *And the next?*

Draw the measuring unit of nine straws over Laura's head. Your final display on the board should look like the diagram below. Emphasize that they obtained the measuring units by making the next measuring unit three times as big as the last.



I am going to hand straws to Anne. As I hand each additional straw to Anne, tell me how we would represent the new amount using the measuring units of base three. In addition, as I hand the straws to Anne, help me count the amounts in base three.

Hand 14 straws to Anne, one by one. Focus students' attention on the bundling and counting of the straws. As each straw is handed to Anne, the students should say the appropriate word in the counting sequence in base three. In addition, the students should explain what should be done when Anne is given the third, sixth, ninth, and twelfth straws. Attention should be drawn to the need to rebundle the straws to form bigger measuring units. The instructor may want to say something like:

I am giving Anne the first straw. So that's 1_{three}. Now I'm handing her the second straw. That's 2_{three}. Now I'm handing her another straw. How many straws does Anne have? Can she hold this many? Why not? In base three, any given place value can hold no more than two of the corresponding measuring units, so when she is given the third straw, she must bundle them together and pass them to Stephen, who now has one measuring unit. What numeral represents this quantity? [10_{three}]

Anne should put a rubber band around the three straws and pass the bundle to Stephen. Similar statements can be made for subsequent amounts that require rebundling straws. The students will describe how to rebundle the quantities on their own; however, it helps to summarize their responses with these kinds of statements.

After the 14 straws have been handed out and counted, ask for two more volunteers, who will represent 2 measuring units smaller than the basic measuring unit. Have them stand on the other side of Anne. Point to the measuring units for the three students who are holding straws. Ask the class what the fourth student's measuring unit (Sam's measuring unit) should be [the measuring unit for the place to the right of the ones place].

So what does Sam's measuring unit look like?

Sam should make the measuring unit for his place. [Sam should decide to use the scissors, cut a straw into three equal pieces, and explain that one of the equal pieces is the measuring unit for his place because it is three times as small as the next largest measuring unit.] If the student (or class) fails to suggest this, remind the class of the relationship between the sizes of the measuring units in base three. You may use the expression “one third of the basic measuring unit,” but be sure to also use the “3 times as small” language to be consistent with the construction of measuring units that are larger than the basic measuring unit. Now ask the class what the measuring unit for the fifth person (Tom) should look like.

So what does Tom's measuring unit look like?

Tom should make the measuring unit for his place. After Tom has made the measuring unit and explained his solution, draw the measuring units on the chalkboard above Sam and Tom. Next write the numerals for the measuring units for Anne, Stephen, and Laura on the chalkboard next to the quantities that were drawn earlier (i.e., 1_{three}, 10_{three}, and 100_{three}, respectively). Extend the students' understanding of whole-number place values (measuring units, the relationship between the measuring units, and the numerals that are associated with each place value) to non-whole number place values:

Why could we possibly need measuring units smaller than the basic measuring unit? What need could have motivated people to invent the idea of a measuring unit smaller than one? [Elicit the students' answers.]

When based place-valued numeration systems were extended to represent quantities smaller than one, people needed a convention to convey that they were using measuring units smaller than the basic measuring unit. If they wanted to measure and represent quantities smaller than the basic measuring unit, they would need smaller measuring units, and they would have to have a way to represent these smaller quantities with numerals. They solved this problem by placing a dot after the “ones” place value. In our system, we call this a decimal point because we are in base ten. Since we are in base three now, we will call this a “tricimal point.” What is the numeral that is associated with this measuring unit? [Point to the *measuring unit* for the 0.1_{three} place. The students should say, “0.1_{three}.”] *What is the numeral that is associated with this measuring unit?* [Point to the *measuring unit* for the 0.01_{three} place. The students should say, “0.01_{three}.”] Write these numerals on the board over the students' heads.

Recall the labels for the whole-number place values, those to the left of the "tricimal point." We read these as "one base three," "one zero base three," "one zero zero base three." The symbolic representations—that is, the numerals— are 1_{three}, 10_{three}, 100_{three}.

So if we try to keep the same pattern, how can we represent the place values to the right of the "tricimal point" in base three?

The word labels are "zero point one base three," "zero point zero one base three." The numerals are 0.1_{three}, 0.01_{three}.

Do you see how this pattern is identical to the base ten numeration system? Why is that the case? What do the digits mean, and why would the measuring units in any base be assigned the same numerals? [Because one measuring unit of size *x* would fit into a measuring unit of size *x*.]

Next ask the five students to represent the following numeral with straws: 112.22_{three}. Be sure that the students explain how the straws are related to the numeral; their explanation should refer to the measuring units for each place and the meaning of the digits.

Summarize the students' explanation, and emphasize that the numeral *means* 1 of this measuring unit, 1 of this measuring unit, 2 basic measuring units, 2 of this measuring unit, and 2 of this measuring unit. In other words, each digit in the numeral tells you how many measuring units there are of each size.

Now ask students:

What does $112.22_{three} + 1_{three}$ equal?

Write this number sentence on the board. Give the class some time to think about the problem. Then ask the students at the board to show the addition and the rebundling that must occur. [The students are already holding the straws for the numeral 112.22_{three}, so Anne should pick up a straw from the table. Anne cannot hold three straws, so she will have to put a rubber band around them and pass them to Stephen. The students should determine the numeral for the new quantity [120.22_{three}].]

Pose one more task. The students at the board are currently holding straws that are represented by the numeral 120.22_{three}. Ask the students to add 0.01_{three} to this quantity. *What is the numerical representation of this new quantity?*

[The new quantity is 121_{three}.] After the class has had time to find a solution, have the five students show the addition and the regroupings that must occur in the various place values. Tom needs to begin the process by adding 0.01 to the amount he already holds.

Ask them to relate what they just did to the solution process used when solving the problem with the standard algorithm for addition:

$$\begin{array}{r} 120.22_{\text{three}} \\ \underline{+\ 0.01_{\text{three}}} \end{array}$$

First, have the five students start over; they should each hold the correct number of straws to represent 120.22_{three}. Now hand Tom 0.01_{three}. This represents the first step of the algorithm—adding 2 measuring units of size 0.01_{three} to 1 measuring unit of size 0.01_{three}. The instructor should carry out the steps of the algorithm on the board and ask the students to explain and illustrate the steps with the straws.

Instructor: I added 2 measuring units of size 0.01_{three} to one measuring unit of size 0.01_{three}. What happens next?

Tom: I need to bundle the 3 measuring units and give them to Sam, who will exchange them for one measuring unit of size 0.1_{three}. Now I have no measuring units of size 0.01_{three}.

Instructor: How do I indicate that in the algorithm?

Tom: Write a 0 in the 0.01_{three} place of the answer and write a little 1 above the 0.1_{three} place to indicate the exchange of 3 measuring units of size 0.01_{three} for one measuring unit of size 0.1_{three}.

Instructor: OK, I did that. Now what?

Sam: I was already holding 2 measuring units of size 0.1_{three}, so after Tom hands me his, I now have 3 measuring units of size 0.1_{three}. But I can't hold three. So Anne takes them and exchanges them for 1 measuring unit of size 1_{three}.

Instructor: How do I show that in the algorithm?

Sam: Write a 0 in the 0.1_{three} place, because I no longer have any measuring units after the exchange. Write a little 1 above the ones place to indicate the exchange of 3 measuring units of size 0.1_{three} for 1 measuring unit of size 1_{three}.

Instructor: OK, I did that. Now what?

Anne: I had 0 measuring units of size 1_{three}, but I was handed 1 by Sam, so I now have 1 measuring unit of size 1_{three}. So in the algorithm, add 0 and 1 to get 1 in the ones place of the answer.

Stephen: Two measuring units of size 10_{three} and 0 measuring units of size 10_{three} is 2 measuring units of size 10_{three}, so write a 2 in the 10_{three} place.

Laura: One measuring unit of size 100_{three} plus 0 measuring units of size 100_{three} is 1 measuring unit of size 100_{three}, so write a 1 in the 100_{three} place of the answer.

Ask the students at the board to return to their seats.

Time: 50–75 min.
Activity Flow: Part 3; Based Place-Valued Numeration Systems and Measuring Units For Places to the Right of the Point

Rationale

In this activity, students are asked to solve problems in base three and then in base ten. By first working in an unfamiliar system (base three), they develop more explicit understandings about based place-valued numeration systems and measuring units smaller than the basic measuring unit. When the students subsequently work in base ten, they can make connections between the base ten system and what they learned in base three, and transfer the more explicit understandings to base ten.

Activity

In this activity, the students create sets of measuring units, including measuring units smaller than the basic measuring unit, and represent numerals with their measuring units.

In this activity we will apply what we just learned.

Distribute Handout 1.

Have the groups work on these two problems. All groups should put their solutions on the board while they are working so the instructor can monitor their understanding. After the activity, the instructor should choose a solution to go over, modeling appropriate language for the students.

Problem 1	
Student responses	**Suggested teacher responses**
The student is confused because the basic measuring unit is not a straw, and "has parts." That is, the basic measuring unit consists of nine boxes. The student's idea of "one" is one discrete unpartitioned object, and the basic measuring unit in this case does not match that description. The "nineness" is a perceptual distracter that interferes with the student's ability to view the quantity as the "one" and to apply the "3 times as big" relationship idea to this quantity. The student cannot generate the other measuring units.	Ask the student to construct a place-value chart with the measuring units. This helps the student focus on making the basic measuring unit 3 times as big. You should help the student understand that making a quantity 3 times as big always involves copying or repeating the quantity 3 times, and the way that it looks is irrelevant; it is the amount that matters, and this amount must be made 3 times as big.
The student extends vague understandings about base ten and base ten blocks to the problem. The student uses one box to represent the 0.1. *"Wouldn't 0.1 be one of these blocks* [in the rectangle with 9 blocks]*, since the basic measuring unit, or 1, is 9 blocks?"*	Remind the student of the relationship between the measuring units in a base *b* system. What base are you in? So what is the relationship between the measuring units? Remind the student of the construction of the base three measuring units completed earlier: What quantity is serving the same role as the straw?
Problem 2	
Student responses	**Suggested teacher responses**
The most common problem is that students are unable to identify the basic measuring unit.	Direct them to the "times as big/as small" relationship for the set of measuring units. Ask them to construct a place-value chart and to draw the given measuring unit in the chart. How can you find the other measuring units?
The students build the set of measuring units correctly, but their solution for 4.32 is inconsistent with their set of measuring units.	Ask them to construct a place-value chart that shows their measuring units. Then ask them what the digits in 4.32 mean.

Time: 75 min.
Activity Flow: Part 4; Conclusion and Homework

Rationale

Students need to have experiences with many representations of basic measuring units, different bases, and many types of problems (e.g., finding the basic measuring unit when they are given the quantity that is represented by 0.1, and vice versa) to become flexible in their ability to interpret the meanings of decimals. This homework assignment provides these experiences. The students need time to independently grapple with these situations. They also need to work on a number of challenging, nonstandard problems that will help them recognize what they do and do not understand about decimals. (This assignment may be hard for the students.)

Activity

Explain that we will shift our focus to the four basic operations in the next lesson. Therefore, the homework assignment offers more practice with constructing sets of measuring units for whole-number and decimal place values, but it also prepares them for the next topic by asking them to model some addition problems.

Hand out Lesson 5 Homework and Handout 2.

Lesson 4 Homework

1. Make a place-value chart for a base five numeration system. To do this, first choose a basic measuring unit. Now, in the first row of the place-value chart, use your basic measuring unit to draw the measuring units for the 1_{five} place, the 10_{five} place, and the 100_{five} place. These measuring units should be shown as pictures of amounts of stuff, not as numerals. Now show the multiplicative relationship between the measuring units by drawing arrows between the measuring units and labeling the arrows "× 5."

2. Using your measuring units from #1, build the quantity represented by the base five numeral 203_{five}. To do this, first write the basic symbols 2, 0, and 3 in the second row of your place-value chart under the appropriate measuring units. The 2 should be written in the second row, under the drawing of the measuring unit of size 100_{five}.

   The 0 should be written in the second row, under the drawing of the measuring unit of size 10_{five}. The 3 should be written in the second row, under the drawing of the measuring unit of size 1_{five}. Now you can easily see that the numeral 203_{five} means 2 of the drawn measuring units of size 100_{five}, 0 measuring units of size 10_{five}, and 3 measuring units of size 1_{five}. So now draw the quantity represented by the numeral 203_{five}.

Place-value chart with pictures of the measuring units in the first row, arrows in the first row, and basic symbols in the second row:

Now draw the quantity represented by the numeral 203_{five}:

3. Using your measuring units from #1, build the quantity represented by the base five numeral 34_{five}. Use the same steps that were described in #2 above.

Place-value chart with pictures of the measuring units in the first row, arrows in the first row, and basic symbols in the second row:

Now draw the quantity represented by the numeral 34_{five}:

4. Count from 34_{five} to 203_{five}. That is, write the numerals from 34_{five} to 203_{five}. If this is hard for you, how can you use your place-value chart to help you figure out the next numeral?

5. Using the same basic measuring unit that you used in #1 for the base five numeration system, make a place-value chart for a base three numeration system. To do this, in the first row of the place-value chart, use your basic measuring unit to draw the measuring units for the 1_{three} place, the 10_{three} place, the 100_{three} place, and the 1000_{three} place. These measuring units should be shown as pictures of amounts of stuff, not as numerals. Now show the multiplicative relationship between the measuring units by drawing arrows between the measuring units and labeling the arrows "× 3."

6. Using your measuring units from #5, build the quantity represented by the base three numeral 201_{three}. To do this, first write the basic symbols 2, 0, and 1 in the second row of your place-value chart under the appropriate measuring units. The 2 should be written in the second row, under the drawing of the measuring unit of size 100_{three}. The 0 should be written in the second row, under the drawing of the measuring unit of size 10_{three}. The 1 should be written in the second row, under the drawing of the measuring unit of size 1_{three}. Now you can easily see that the numeral 201_{three} means 2 of the drawn measuring units of size 100_{three}, 0 measuring units of size 10_{three}, and 1 measuring unit of size 1_{three}. So now draw the quantity represented by the numeral 201_{three}.

Place-value chart with pictures of the measuring units in the first row, arrows in the first row, and basic symbols in the second row:

Now draw the quantity represented by the numeral 201_{three}:

7. Explain why 34_{five} and 201_{three} represent the same amount (the same quantity or amount of stuff; compare your answers to #3 and #6).

8. Count from 201_{three} to 2201_{three}.

9. Suppose the length below represents the basic measuring unit: ___________

   a. Using this basic measuring unit, make a place-value chart for a base four numeration system. In the first row of the place-value chart, use your basic measuring unit to draw the measuring units for the 1_{four} place, the 10_{four} place, and the 100_{four} place. These measuring units should be shown as pictures of amounts of stuff, not as numerals. Now show the multiplicative relationship between the measuring units by drawing arrows between the measuring units and labeling the arrows "× 4."

   b. Now using your place-value chart, draw the quantity that is represented by the base four numeral 102_{four}.

   c. Now convert 102_{four} to a base seven numeral by drawing pictures. (Hint: Rebundle your quantity in part (b).)

(*Return to p. 85*)

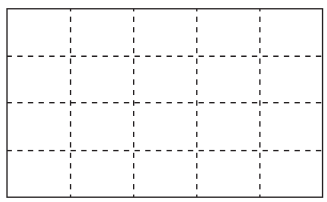
Handout 1

Suppose the basic measuring unit in a base three system is this area:

a. Build a set of measuring units for the two places to the left of, and two places to the right of, the "tricimal point." In other words, the quantity that is represented by 1_{three} is given, and you need to build the quantities that are represented by the numerals 10_{three}, 0.1_{three}, and 0.01_{three}. Show the measuring units in a place-value chart and use an arrow between the measuring units to show the multiplicative relationship between the measuring units.

b. Use the set of measuring units that you constructed in part (a) to build the quantity that is represented by the numeral 12.12_{three}. First, write these basic symbols in the second row of the place-value chart in part (a) under the appropriate measuring units. Then, in the space below, draw the quantity that is represented by 12.12_{three}.

2. Suppose that in a base ten system, the measuring unit that is represented by the numeral 10 is this area:



a. Build a set of measuring units for the two places to the left of, and two places to the right of, the "decimal" point. In other words, the quantity that is represented by 10 is given, and you need to build the quantities that are represented by the numerals 1, 0.1, and 0.01. Show the measuring units in a place-value chart and use an arrow between the measuring units to show the multiplicative relationship between the measuring units.

b. Use the set of measuring units that you constructed in part (a) to build the quantity that is represented by the numeral 4.32. First, write these basic symbols in the second row of the place-value chart in part (a) under the appropriate measuring units. Then, in the space below, draw the quantity that is represented by 4.32.

(*Return to p. 85*)

Handout 2

Modern Based Place-Valued Numeration Systems, Basic Measuring Units, and Sets of Measuring Units

Developing a based, place-valued numeration system:

First, some quantity (a straw, a dot, a sheep) is chosen (often because it is convenient) and called the *basic measuring unit* [*BMU*].

A fixed number is chosen that corresponds to the number of basic symbols that will be used in the system. This number is called the *base* [*b*].

Each *place* in a numeral is associated with a *measuring unit [MU]*, which is its *place value*. The *place* directly to the left of the *point* is associated with the *BMU* and is called the *ones place*. In any base, the *BMU* is a particular quantity that is considered "one."

Every other *place* is associated with a quantity that is constructed from the *BMU*. This quantity is called a *measuring unit* [*MU*].

In base *b*, orienting oneself at the *ones place* and the *BMU*, the next largest *measuring unit* [*MU*] is constructed by grouping together *b copies* of the *BMU*. Then, this new *MU* has a magnitude that is *b* times as big as the *BMU*. Each successive *place* to the left of the *ones place* corresponds to the next largest *MU*. Its *place value* is *b* times as big as the *place* to its immediate right.

Again, orienting oneself at the *ones place* and the *BMU*, the next smallest *measuring unit* is constructed by partitioning the *BMU* into the fixed amount (*b*) equal parts and choosing one of these parts. Then this new *MU* has a magnitude that is *b* times as small as the *BMU*. Each successive *place* to the right of the *ones place* corresponds to the next smallest *MU*. Its *place value* is *b* times as small as the *place* to its immediate left.

A set of *MUs* is formed that consists of the *BMU* and the *MUs* constructed as above, and is such that each new *MU* has a magnitude that is *b* times as big or as small as the adjacent *MUs* (on its right and left, respectively).

Suppose one chooses a *BMU* and constructs sets of *MU*s for based place-valued numeration systems with different bases. Once the *BMU* has been identified, then it is the quantity that represents "1" or "one." Its size is completely independent of the chosen base. However, the different based place-valued numeration systems are distinguished by how the corresponding set of *MUs* is constructed. The quantity of the groupings required to move to the next bigger or next smaller *place* is different. That is, the size of the *MU*s associated with each of the *place values* changes among different based systems.

In base *b*, the numeral 10 and the words "one-zero" mean 1 group of the quantity that is *b* times as big as the *BMU* and zero groups of the *BMU*. The numeral 0.1 and the words "zero point one" mean 1 group of the quantity that is *b* times as small as the *BMU* and zero groups of the *BMU*.

Examples: Let a straw be the *BMU*. Then the straw is "one" and is represented by 1_b.

Example 1: Consider a set of MUs for base ten.

- The *BMU* is a straw, so a straw is represented by "1_{ten}" or "one."
- The next larger *place* is associated with a *MU* that is the quantity of 10 straws, and is represented by "10_{ten}" or the "one-zeros" place. Continuing with this construction, then the next largest *place* is associated with the quantity of 10 groups of 10 straws, or 1 group of 100 straws. It is represented by "100_{ten}" or the "one-zero-zeros" place, etc.

- The next smallest *place* is associated with the quantity that is one tenth the size of one straw or 10 times as small as one straw. It is represented by “0.1_{ten}” or the “zero-point-one” place. Continuing with this construction, the next smallest place is associated with an *MU* equal to the quantity that is one tenth of one tenth of one straw, or one hundredth of one straw, or one hundred times as small as one straw. It is represented by “0.01_{ten}” or the “zero-point-zero-ones place.”

Example 2: Consider a set of MUs for base three.

- The *BMU* is a straw, so a straw is represented by “1_{three}” or “one.”
- The next larger *place* is associated with a *MU* that is the quantity of three straws, and is represented by “10_{three}” or the “one-zero base-three” place. Continuing with this construction, then the next largest *place* is associated with the quantity of three groups of three straws or one group of nine straws. It is represented by “100_{three}” or the “one-zero-zero base-three” place, etc.
- The next smallest *place* is associated with the quantity that is one third the size of one straw or three times as small as one straw. It is represented by “0.1_{three}” or the “zero-point-one base-three” place. Continuing with this construction, the next smallest place is associated with an *MU* equal to the quantity that is one third of one third of one straw, or one ninth of one straw, or nine times as small as one straw. It is represented by “0.01_{three}” or the “zero-point-zero-one base-three” place.

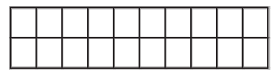
Example 3: Consider a set of MUs for base seven.

- The *BMU* is a straw, so a straw is represented by “1_{seven}” or “one.”
- The next larger *place* is associated with a *MU* that is the quantity of seven straws, and is represented by “10_{seven}” or the “one-zero base-seven” place. Continuing with this construction, then the next largest *place* is associated with the quantity of seven groups of seven straws or one group of 49 straws. It is represented by “100_{seven}” or the “one-zero-zero base-seven” place, etc.
- The next smallest *place* is associated with the quantity that is one seventh the size of one straw or seven times as small as one straw. It is represented by “0.1_{seven}” or the “zero-point-one base-seven” place. Continuing with this construction, the next smallest place is associated with an *MU* equal to the quantity that is one seventh of one seventh of one straw, or one forty-ninth of one straw, or forty-nine times as small as one straw. It is represented by “0.01_{seven}” or the “zero-point-zero-one base-seven” place.

(*Return to p. 85*)

Lesson 5 Homework

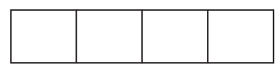
1. Suppose the basic measuring unit is given as this area:



   a. Build the quantity that is represented by 4.2 in a *base ten* system. Make a place-value chart first.

   b. Build the quantity that is represented by 4.2 in a *base five* system. Make a place-value chart first.

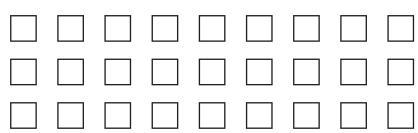
2. Suppose the measuring unit that represents the quantity 0.1 in a *base ten* system is this area:



   a. Construct the measuring units for the ones place, the tens place, and the hundredths place.

   b. Draw the quantity that is assigned the numeral 1.

3. Suppose the measuring unit that represents the quantity 10_{six} in a *base six* system is this area:

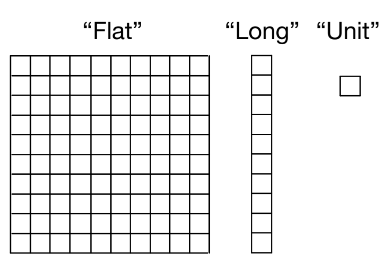


   a. Find and show the basic measuring unit.

   b. Build the quantity that is represented by 13.3_{six}. Make a place-value chart first.

4. Why do the numerals 0.6 and 0.60 represent the same amount?

5. Teachers use *base ten* blocks, shown below, to help children understand the relationship between the measuring units in our system.



a. Suppose the basic measuring unit is a "unit" (which is how the blocks are typically used in the lower grades). How many flats and units would you use to represent the quantity [1210] if you could only use those measuring units?

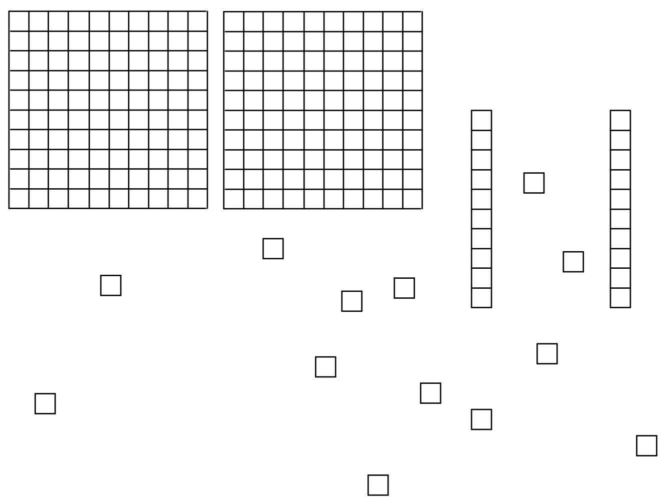
Flats ________________ Units ________________

b. Suppose the basic measuring unit is a "flat." How many "units" would you use to represent the quantity [0.18] if you could only use "units"?

Units ________________

Why is 0.18 called "eighteen hundredths"?

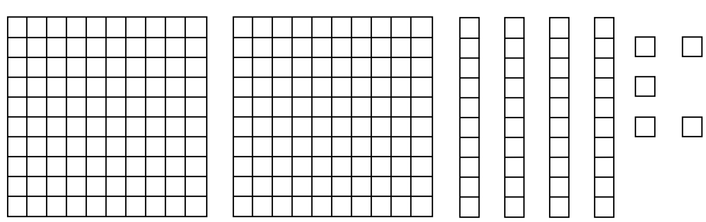
c. Suppose the basic measuring unit is a long. What is the numerical representation of the quantity below? ____________________________________



d. Suppose the basic measuring unit is a flat. What is the numerical representation of the quantity in part (c)? ______________________________

e. Suppose the basic measuring unit is the next-sized block, the "large cube," which is equal to 1000 "units." What is the numerical representation of the quantity in part (c)? ______________________________

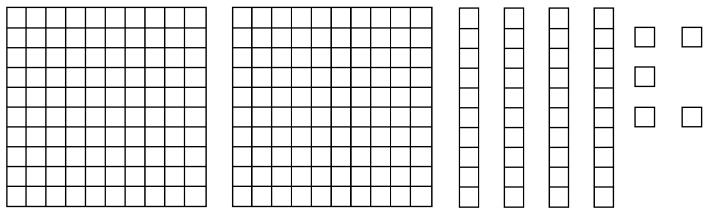
6. Consider the following quantity.



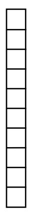
a. Let the area of the long be the basic measuring unit in base ten.

Make a place-value chart that shows your measuring units:

Now in the picture below, show how the quantity would be bundled with your measuring units:

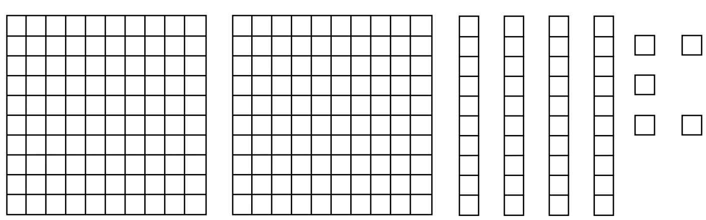


Now write the base ten numeral for this quantity: ______________________________

b. Now represent the same amount of stuff with a base four numeral. **Use the same basic measuring unit from part a.** That is,  is still your basic measuring unit.

Make a place-value chart that shows your measuring units:

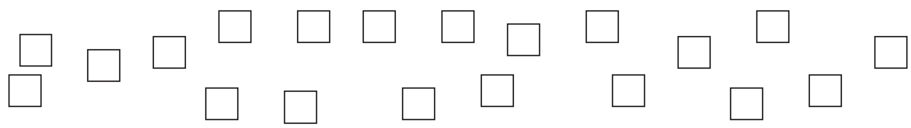
Now in the picture below, show how the quantity would be bundled in the base four system with your measuring units:



Now write the base four numeral for this quantity: ______________________________

7. As we saw in #5, the quantity that you choose for the basic measuring unit (or "one") affects the numerical representation of all other quantities.

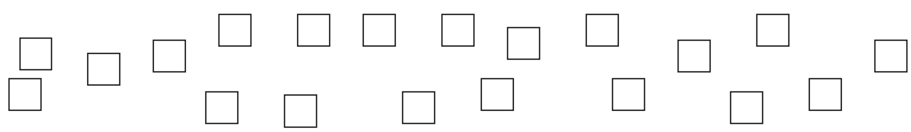
   a. Let □ be the basic measuring unit. Represent the quantity below with a numeral in **base three**. First make a place-value chart that shows your measuring units for base three. Then bundle the quantity using your measuring units. Then write the base three numeral for the quantity.

   Place-value chart that shows your measuring units for base three:

   What is the base three numeral for this quantity?: ___________

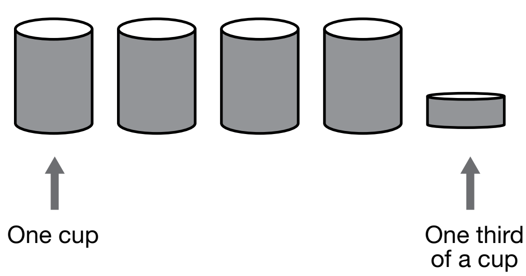
   b. Now let □ □ □ be the basic measuring unit. Again, represent the quantity below with a numeral in **base three** (same quantity as in part a). First make a place-value chart that shows your measuring units for base three. Then bundle the quantity using your measuring units. Then write the base three numeral for the quantity.

   Place-value chart that shows your measuring units for base three:

   What is the base three numeral for this quantity?: ___________

8. Suppose you are trying to represent this quantity of liquid (below) in base ten. You let the basic measuring unit be one cup. Make a place-value chart that shows all of the measuring units that you need as you try to find a base ten numeral for this quantity, and explain how you found a base ten numeral for this quantity. What is the base ten numeral for this quantity?



From your work on this problem, explain why we get repeating decimals.

9. In the next lesson, we will start looking at the meaning of addition and subtraction. Imagine you are a six-year-old. How would you solve the following story problems with objects (e.g., building blocks, pennies, lengths of ribbons, base ten blocks, lengths on a number line, volumes of water)?

For each problem, answer the following questions:

1. Describe the actions that you performed on the objects.

2. What did you use for your basic measuring unit in each problem?

   a. Josh had six cookies. His mom gave him five more. How many cookies does Josh have altogether?

   b. Five cows are in a field. Three are standing and the rest are lying down. How many cows are lying down?

   c. Dave had thirteen gumdrops. He gave four to Cheryl. How many gumdrops does he have left?

   d. Megan had some markers. She gave six to Janet. Now she has nine left. How many markers did she have to start with?

   e. There are four SUVs, two pickup trucks, and six cars in a parking lot. How many vehicles are there in the lot in all?

   f. There were four SUVs in a parking lot. Two pickup trucks and six cars pull into the parking lot. How many vehicles are now in the lot?

   g. Darnell has some red and green grapes. Two pounds are green and three pounds are red. How many pounds of grapes does he have in all?

   h. Joe has six balloons. His sister Connie has nine balloons. How many more balloons does Connie have than Joe?

   i. Sean has four more pennies than Meg. Meg has eight pennies. How many pennies does Sean have?

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