The Content-Focused Methods Course: A Model for Integrating Pedagogy and Mathematics Content*

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In the majority of secondary mathematics teacher preparation programs, the work of learning mathematics and the work of learning to teach mathematics are separated, leaving open the question of when and how teachers integrate their knowledge of content and pedagogy. We present a model for a content-focused methods course, which systematically develops a slice of mathematics content in the context of typical methods course activities. Three design principles are posited that undergird the design of such a course, addressing the nature of the mathematics content, the sequencing and design of activities, and the ways in which the course addresses the needs of diverse learners. Data from an instantiation of one such course is presented to illustrate the ways in which the course design framed teachers' opportunities to learn about both content and pedagogy.

Key words: Teacher education; Mathematical knowledge for teaching; Mathematics methods.

As Linda Darling-Hammond (2010) points out, teacher education in the United States finds itself in a Dickensian conundrum. On one side, a great deal of political attention has been paid to improving the quality of teaching and learning, particularly in the fields of science, technology, engineering, and mathematics. On the other, pointed questions are being raised about the specific value of formal teacher education. In light of studies criticizing the mathematical training that US teachers receive (Schmidt et al., 2007; Schmidt, Houang, & Cogan, 2011), there has been a return to favoring mathematical preparation over education coursework for teacher certification in many states. This press has largely focused on the preparation of secondary teachers, with the notion that disciplinary specialists with some basic pedagogical instruction might be well equipped to teach. While the research community generally agrees that teachers certified through formal teacher preparation programs effect stronger learning outcomes in students (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2008; Darling-Hammond, 2006; Darling-Hammond & Bransford, 2005), little consensus exists regarding the features of mathematics teacher preparation that promote teacher and student learning. The call for common school mathematics standards has cascaded into calls for common mathematics teacher education standards (Simon, 2000; Wilson, 2011) consistent with student-centered instruction, and frameworks that support the development of such a knowledge base. Several researchers have taken up this call, describing the knowledge base for teaching mathematics (e.g., Ball, Thames, & Phelps, 2008; Stein, Engle, Smith, & Hughes, 2008), creating instruments for measuring teacher knowledge (e.g., Brown, Bush, & McGatha, 2006; Hill & Ball, 2004; Izsák, 2008), and linking those measures to student outcomes (Hill, Rowan, & Ball, 2005). While the field has made substantial progress in describing mathematical knowledge for teaching and in linking that knowledge to student outcomes, little work has been done to describe features of mathematics teacher education that support the development of this knowledge. In this paper, we describe design principles that undergird a model for a mathematics methods course for secondary teachers that systematically integrates mathematics content in ways that provide opportunities to learn mathematical knowledge for teaching.

Researchers have conceptualized the complex knowledge base for teaching in ways that incorporate content, pedagogy, and several conceptualizations of the intersections between the two across the K–16 spectrum (e.g., Ball, Thames & Phelps, 2008; Shulman, 1986; Speer & Wagner, 2009; Steele, 2005). A common thread across this work is that pedagogical knowledge is neither discrete nor conceptually separable from the knowledge of the mathematics content being taught. Knowledge of how to teach a particular slice of mathematics rests on one's knowledge of the mathematics in question; however, research that has investigated the development of mathematical knowledge for teaching has shown this process to be less additive (e.g., learn the content, then learn to teach it) and more

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iterative. For example, Steele (2008) demonstrated the ways in which engaging in mathematical and pedagogical tasks can enhance different aspects of both knowledge bases; Speer and Wagner (2009) identified pedagogical dilemmas that arise during teaching that spark re-examination of the content and the further development of pedagogical capacity. Yet in both policy and practical circles, the work of learning mathematics content and learning to teach mathematics are bifurcated. Prospective teachers receive content and pedagogical instruction in different courses, often separated both temporally and organizationally within teacher education systems. While many elementary preparation programs feature mathematics for teachers courses that sometimes attempt to integrate these learning experiences, few such opportunities exist for secondary teacher candidates (for exceptions, see Hill, 2006; Senk, Keller, & Ferrini-Mundy, 2000).

One model integrating the study of content and methods for secondary teachers is what Markovits & Smith (2008) term a *content-focused methods course*. Content-focused methods courses (CFMC) situate the systematic development of mathematical knowledge for teaching in the context of the typical activities in a methods course (Markovits & Smith, 2008). Whereas a methods course might treat content opportunistically through isolated tasks or lesson plans that teachers prepare, and a content course might provide plausible connections to pedagogical practice, the content-focused methods course features discernible mathematical and pedagogical storylines that are tightly connected. In this article, we look back at a content-focused methods course intended to enhance teachers' mathematical knowledge for teaching and articulate a set of design principles common to the work. These principles can serve as a framework for the design of teacher education experiences that target mathematical knowledge for teaching across a wide range of mathematical content and in a variety of contexts: both preservice and practicing, both elementary and secondary teachers.

Content-Focused Methods Course Design Principles

1. Focuses on a narrow slice of mathematical content or process central to developing mathematical proficiency in secondary school.
2. Uses a guiding inquiry to frame and motivate the course and provide a unifying thread.
3. Organizes content and pedagogical activities into sequences that engage teachers across the continuum from learner to teacher.

Illustrating the Model With a Specific Example: A Content-Focused Methods Course on Function

A content-focused methods course centered on function (herein referred to as *the functions course* for simplicity) was designed using the three principles. We begin with a description of the course and context, followed by a discussion of the ways in which each of the principles influenced course design. We then describe in general the learning evident from the teachers who participated in the course and relate those data to the design principles.

Description of the Functions Course

The course was intended to enhance teachers' mathematical knowledge for teaching functions and to develop their capacity for enacting meaningful student-centered learning experiences around these ideas for secondary students. It was taught as a graduate-level course at a large urban university in the Midwestern United States. Course development and implementation were part of a larger research project whose goals were to design and study courses around case-based mathematics education materials. The goals of the course are shown in Table 1.

The course targeted preservice and practicing secondary teachers and was promoted as an "advanced methods" course. It was a required course for preservice secondary teachers at the end of a yearlong master of arts in teaching program and was offered as an elective for practicing teachers pursuing master's-level study. In addition, a number of elementary preservice teachers and in-service special educators with particular interests in mathematics took the course as an elective. (We reflect on the impact of the diverse teacher population later in this article.) The background of the 21 teachers enrolled in the course is shown in Table 2.

The principal investigator of the research project served as the lead designer and course instructor with support from a research team made up of teacher education researchers and graduate students. The authors of this article were graduate students on the research team and have subsequently refined and enacted the course as faculty members at other institutions. The research team (RT) began by selecting sets of mathematical tasks and narrative or video cases (featuring the same or a similar task) that represented rich learning opportunities related to functions, drawing primarily from Smith, Silver, and Stein's (2005a) set of algebra tasks and cases. The RT then created or adapted additional activities related to the mathematical tasks and assembled activity sequences

Table 1
Goals for the Functions Course

Mathematical goals	Pedagogical goals
Develop a mathematically accurate definition of function and use it to distinguish examples and nonexamples of function	Support the development of students' understanding of functions by encouraging and facilitating rich mathematical discussions
Distinguish linear and nonlinear and proportional and nonproportional functions	Identify and enact cognitively challenging mathematical tasks
Solve a variety of problems involving functions, using recursive or closed form terminology and notation	Identify factors that impact the maintenance and decline of cognitive demands during implementation
Create and make connections among multiple representations of functions	

Table 2
Demographic Data on Course Participants

	Preservice: postbacc MAT program	In-service: masters of education	Secondary education doctorate	TOTAL
Elementary (K–6, all subjects)	3	1		4
Secondary (7–12, mathematics)	10	5		15
Deaf Education		1	1	2
TOTAL	13	7	1	21

(called constellations) centered on a particular aspect of the mathematics of function. Figure 1 shows this collection of activities, with the colors representing the constellations, the shapes representing different activity types, and grey borders representing activities closely related to the guiding inquiry. (Figure 1 shows an enactment of the course during a 6-week summer term meeting 3 hours twice a week. The course has also been enacted during a typical 16-week semester.) Activities above the horizontal bar were enacted in class, with those below the bar representing homework assignments. We next describe the ways in which the design in Figure 1 reflected the three principles, and discuss how the team anticipated those principles and supported teachers' opportunities to learn.

Design Principle #1: A narrow focus: Algebra as the study of patterns and functions. The first principle for the content-focused methods course prescribes a narrow focus on an aspect of mathematical content central to the secondary mathematics curriculum. This principle establishes relevance for the mathematical content to be explored with respect to the work of teachers in their classrooms and affords an in-depth exploration of the content rather than a surface-level treatment of a variety of mathematical ideas. The content focus should cut across grade levels in some important way, be identified in standards documents as important to secondary mathematics, and be complex and challenging for both teachers and their students.

Therefore, the RT selected algebra as the study of patterns and functions as the focus for this course because it met the preceding criteria well. Function is an important cornerstone of secondary mathematics, which has become even more prominent with the rise of secondary mathematics curricula that explicitly use function as the grounding concept for the development of algebraic thinking (Alper, Fendel, Fraser, & Resek, 1997; Center for Mathematics Education, 2009; Cooney, 1996; Coxford et al., 1997). The Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010) underscore this importance by positioning functions as a high school content strand alongside algebra, geometry, modeling, and statistics and probability.

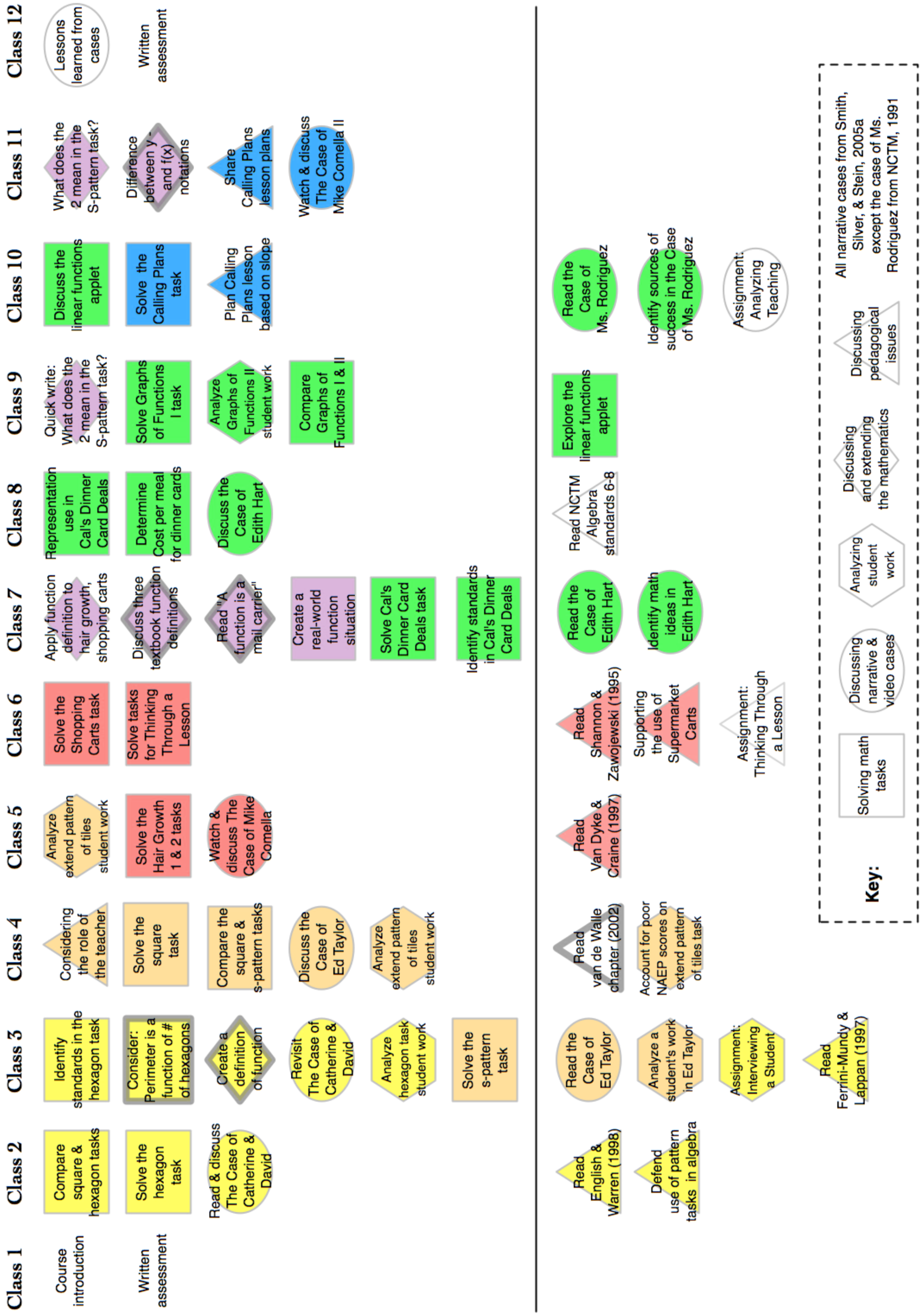


Figure 1. **Activities in the functions CFMC.**

Researchers investigating function have consistently found that students and teachers alike are frequently able to identify and generate examples at the heart of the function concept, such as continuous linear and quadratic functions, but struggle to identify more exotic functions, cannot always provide a mathematically complete definition of function, and are limited in their representational fluency (e.g., Even, 1998; Pitts, 2003; Sánchez & Llinares, 2003; Stein, Baxter, & Leinhardt, 1990). These characteristics together suggested that a content-focused methods course designed around functions would be a rich site for sustained mathematical inquiry.

Specifically, the RT conceived of the course as focusing on families of functional relationships that were familiar—largely linear and quadratic—to provide a diverse group of teachers with entry to the topic. At the same time, the team selected tasks that were mathematically extensible, allowing teachers to explore noncontinuous linear functions and rational functions built by transforming a simple linear function. The RT also recognized the importance of representational fluency—generating multiple representations of functions, moving flexibly between them, and describing the ways in which different representations are useful for noticing and analyzing specific features of a function. This set of ideas is important for developing both content knowledge and pedagogical practice.

Design Principle #2: A guiding inquiry: What is a function, and what are examples and nonexamples? A guiding inquiry is a question (similar to an essential question in K–12) designed to frame the course-long content focus. This inquiry establishes the importance of the in-depth study of a particular slice of content within mathematics and mathematics teaching. This sort of big-picture view is often lacking in curricula and standards documents and is an important aspect of teachers' curricular vision, which guides their decision making about what content is taught in the classroom (Drake & Sherin, 2008). The guiding inquiry also represents an opportunity to generalize from the set of particular mathematical tasks in the course to a larger mathematical structure and concept. The guiding inquiry should be about a topic for which most teachers will have initial ideas but one for which it is reasonable to believe teachers and students might have a limited understanding or misconceptions. In this spirit, the guiding inquiry should also be posed early in the course so as to reveal teachers' initial conceptions and revisited at key moments in the course to provide opportunities to refine and elaborate those initial understandings.

The RT selected the guiding inquiry of *What is a function, and what are examples and nonexamples of functions?* for the reasons outlined above: Teachers would likely have some fluency with functions, yet a rich understanding of the concept can elude teachers and students. A significant body of research has demonstrated that even if teachers and students can work with examples of functions, they may not have a clear definition of a function and the specific criteria that distinguish functions from nonfunctions (Pitts, 2003; Vinner & Dreyfus, 1989). Lack of a clear definition of function can lead to the over- or under-generalization of the function concept and can engender a limited view of function and obscure its mathematical utility. For example, teachers who conceive of a function as something that can always be represented graphically potentially miss important function examples such as the Dirichlet function or nonnumeric functions such as the relationship between letters and mailboxes (Sand, 1996). A reliance on a graphing requirement on the Cartesian plane also obscures geometrically-based functions such as transformations.

Motivating a course-long inquiry into a topic for which teachers may feel as if they already have a great deal of knowledge can be a challenge. To motivate deep consideration of the definition of function, the RT positioned the definition of function as something to be constructed and revisited over time rather than simply stated and taken as shared. The language of function was used in the discussion of the first mathematical task, and teachers were then asked to define function individually and in small groups. They were able to state their initial ideas about functions, providing the instructional team with a baseline gauge of what the teachers knew. The course instructor assembled a list that captured all publicly shared ideas, including incomplete or vague conceptions, and this list was posted for all subsequent class sessions. This list was then used both as a resource when considering future examples and as a living document to be modified over the course.[1] These recurring discussions helped to problematize the work on function.

Design Principle #3: Engage teachers across the continuum from learner to teacher. The notion of systematically developing content knowledge in a mathematics methods course is an important feature of the content-focused methods course model. In addition, the methods course must also develop pedagogical knowledge and link content and pedagogy in ways that are useful to the work of teaching. The use of authentic artifacts of practice (e.g., mathematical tasks, narrative and video cases of

1 While it did not happen in this particular instantiation, we have had teachers spontaneously request to edit an idea recorded on the list in subsequent iterations of the course.

teaching, student work, and lesson plans) is an important design consideration that supports the integration of the mathematical and pedagogical knowledge bases and connections to practice (Ball & Cohen, 1999). The content-focused methods course takes this connection a step further, using specific mathematical tasks as a grounding experience and starting point in an exploration of the mathematics as learner and teacher. Engaging in a mathematical task provides common ground to discuss the nuances of making sense of the mathematics. From this place, teachers can move back and forth between positions of learner and teacher, first in a protected way that may include analyzing third-party teaching artifacts such as narrative or video cases, sets of student work, or related mathematical tasks. As teachers develop deeper and more nuanced thinking about the content, they can move further on the continuum to consider the implications of taking different perspectives on the mathematics content on their teaching practice.

This principle led the RT to use particular activity structures in the content-focused methods course. For each mathematical task solved, teachers were asked to analyze the teaching of that task in some way (either through narrative or video cases), to consider students' thinking about that mathematics in some way, and to make connections to their classroom practice. Beginning with solving the mathematical task as a learner is a critical element; in grappling with the mathematics themselves, teachers are better positioned to analyze students' mathematical thinking and to consider how to support that thinking (Steele, 2008). The cases of teaching considered do not necessarily have to be exemplary cases but should raise important dilemmas about the teaching and learning of the content in question.[2]

Figure 2 shows the activities from the first constellation in the functions course placed along the learning-teaching continuum, with the numbering representing the order of activities. The constellation began with comparing the square and hexagon tasks and solving the hexagon task from a learner's perspective, followed by reading and discussing the teaching of the tasks in The Case of Catherine Evans & David Young (Smith, Silver, & Stein, 2005a). Activities 4 and 5, both homework, pushed teachers to consider the implications of the use of patterning tasks in the classroom. The next class session looped back to talking about the mathematics by considering the mathematical standards in the hexagon task and posing the guiding inquiry (what is a function?) for the first time. The next three activities, analyzing student work, reading a practitioner article on teaching algebra, and interviewing a student around one of the tasks, represented a strong push toward teaching practice.

Activity sequences that keep the mathematics constant and traverse the continuum between learner and teacher provide teachers with a range of different opportunities to learn. First, the work begins in a relatively comfortable space for discussion—doing mathematics—and gradually moves to more sensitive spaces of a teacher's classroom practice. Along the way, teacher participants build under-

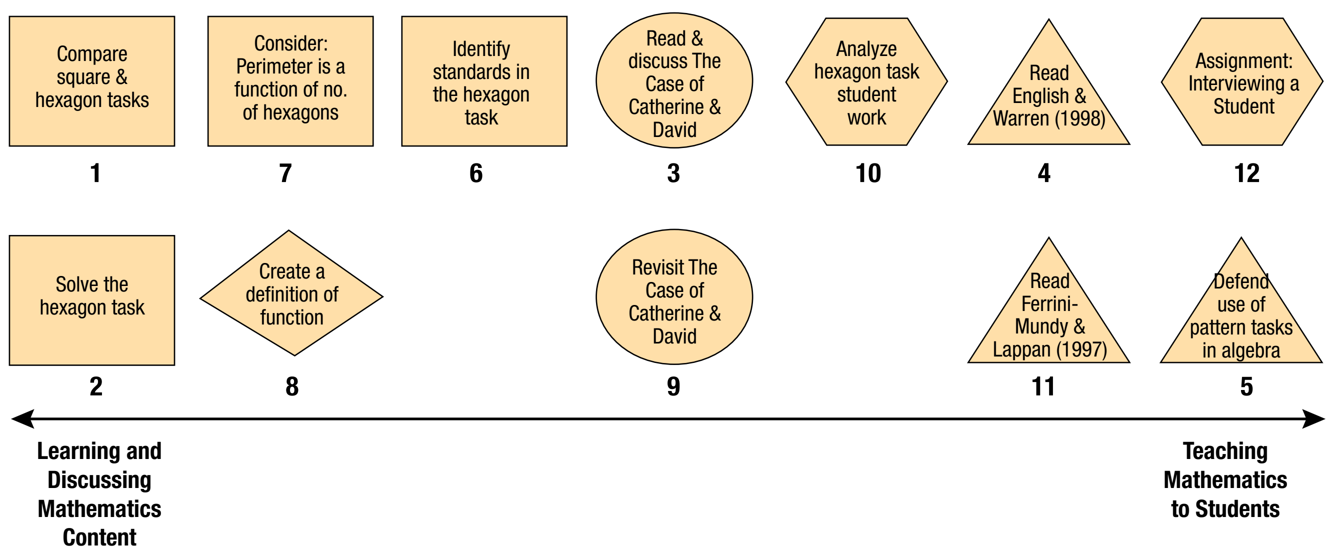


Figure 2. **The tasks in Constellation 1 on the learning-teaching continuum.**

2 The use of at least one exemplary case as a comparative measure can be particularly helpful. For a more extensive discussion of the selection and use of cases, see Smith and Friel, 2008.

standings of how other teachers and their students make sense of the mathematics content, and these understandings can then be applied to participants' own classrooms. Like turning a gemstone in the light to see its different facets, considering multiple perspectives on the mathematics creates a more nuanced and robust sense of the intertwined package of teaching and learning. Moreover, the sequence also provides teachers with a model of what student-centered pedagogy might look like around a particular mathematical topic. While not every aspect of the mathematical work will transfer directly to practice, a strong socially constructed mathematical conversation is likely to include useful features adaptable to the classroom (Hillen & Hughes, 2008).

These three design principles together frame the opportunities teachers have to learn about content and pedagogy in a content-focused methods course. In the section that follows, we briefly describe teacher learning in the course with respect to both content and pedagogy. We then use data from the course to illustrate the ways in which the design principles may have afforded teachers particular sorts of opportunities to learn.

Teacher Learning in the Functions Course

The research team collected data to assess teacher learning about content and pedagogy in the functions course in several ways. Through written assessments and semi-structured interviews at the start and end of the course teachers were asked to solve mathematical tasks, analyze cases of teaching and student work artifacts, and plan lessons. The postcourse interview used a course map similar to Figure 1 and asked teachers to reflect on their learning of (a) mathematics; (b) students as learners of mathematics; and (c) teaching mathematics, and to identify activities that contributed to their learning. Course meetings were videotaped and transcribed, and all instructional artifacts were retained, which provided data related to opportunities to learn. All written assessment items were coded by both authors, with an inter-rater reliability of at least 92%.

In general, our analysis of the data suggests that teachers added to both their knowledge of content and of pedagogy. Prior to the course, many of the teachers struggled to produce a correct definition of function as well as an example and nonexample. Performance in generating the definition, example, and nonexample improved significantly on the postcourse assessment. Teachers were also asked to solve a number of mathematical tasks, both on written assessments and during course meetings, that involved functions. The use of representations and the ways in which teachers made connections between them improved from the start to the end of the course as well. From a pedagogical standpoint, teachers were better able to select high-cognitive demand tasks related to functions and plan for them in ways that supported the maintenance of the cognitive demand. They came to understand the ways in which one might systematically plan for and support work on multiple representations of functions with students, with a particular focus on meaningful questions that supported conceptual understanding. Teachers also considered the utility of having and supporting multiple mathematically correct definitions for function rather than a single canonical definition.

In the section that follows, we explore this data set in greater detail. Our goal is to use the three design principles as lenses through which to consider data on teacher learning and ways in which the course provided teachers with opportunities to learn about both content and pedagogy.

Making connections among multiple representations: Using the lens of Principle 1. One of the mathematical goals of the course was for teachers to make connections among multiple representations of functions. In this section, we use the lens of Principle 1—the focus on the content of function—to consider the ways in which course design using this principle offered teachers opportunities to learn related to connections between representations. Teachers had numerous opportunities to make connections between visual geometric patterns, symbolic equations, tables, graphs, mathematical language, and real-world contexts. The choice of specific tasks related to function and the design of a specific progression through those tasks contributed to these opportunities to learn.

Table 3 lists the mathematical tasks related to function that were used in the course and describes both the family of function (e.g., linear, quadratic, rational) and the starting representation used in each task. By holding the

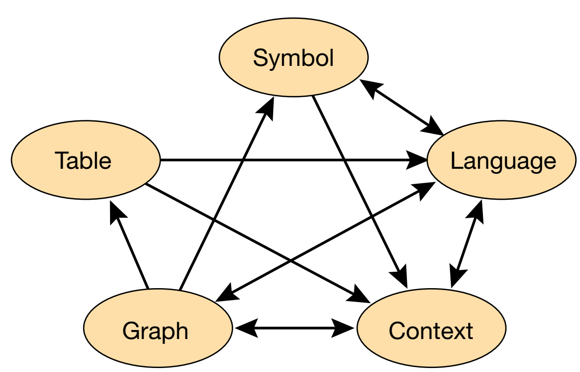


Figure 3. **Five representations of function and the connections made in Class 7.**

Table 3
Range of Examples Used in Course Tasks

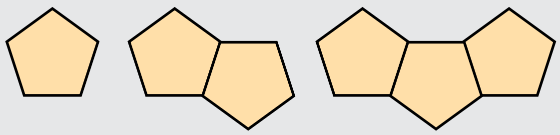
	Example (starting representation in parentheses)						
	Functions						Nonfunctions
Class	**Linear proportional**	**Linear non-proportional**	**Quadratic**	**Piecewise**	**Rational**	**Non-numeric**	
1							
2		Hexagon task (context)					
3			S-pattern task (context)				
4		Square/pool border task (context)					
5		*Paul's hair growth* (*context/table*)		*Sonya's hair growth* (*context*)			
6		Supermarket carts (context)					
7	Car wash (context) Cal's Dinner Cards: Regular Plan (graph)	Cal's Dinner Cards: Plans A and B (graph)				*Mail carrier* (*context*) *Students & test scores* (*graph*)	Weight and height (graph)
8		Cal's Cost Per Meal: Regular Plan (table)			Cal's Cost Per Meal: Plans A and B (table)		
9	*Graphs of functions: Functions 1 & 2* (*symbolic*)	*Graphs of functions: Functions 3 & 4* (*symbolic*)					
10		Calling Plans (context)					
11							
12			S-pattern task (context)				

Note. Italics indicate situations that are or could be considered continuous; nonitalics indicate discrete situations.

content of function constant, teachers were able to experience the ways in which different representations made salient different features of the function relationship. As the course progressed, the teachers were using different representations spontaneously in their mathematical work and talk to make sense of the underlying mathematical constructs. For example, in a discussion of the Cal's Dinner Card Deals task in Class 7 in which teachers were asked to make sense of the slope and y-intercept, teachers made 13 different connections between the 5 core representations of function (symbol, language, context, graph, table). Figure 3 shows the connections between the representations made by teachers in a 20-minute discussion.

Changes in teachers' abilities to make connections between mathematical representations were assessed in part through their performance on the visual pattern

The first train in this pattern consists of one regular pentagon. For each subsequent train, one additional pentagon is added. The first three trains in the pattern are shown below.



a. Determine the perimeter for the 4th train.

b. Determine the perimeter for the 100th train.

c. Write a description that could be used to find the perimeter of any train in the pattern. Explain how you know. How does your description relate to the visual representation of the trains?

Figure 4. **The pentagon pattern task (adapted from Schifter, 1996).**

task shown in Figure 4. Results from the pre/post written assessment show evidence of changes in teacher capacity to make connections between a visual geometric pattern and a symbolic equation. Teachers' responses to the task were coded using the rubric shown in Table 4. Initially, a majority of teachers determined an equation that generalized the pattern (rubric scores of 1 or higher), but the extent to which their explanations related their equation to the visual pattern varied. About 25% fully related their explanation to the pattern (score 4), with another 25% making no connections (score 1), and the remaining teachers making some connections (scores 2 and 3). By contrast, on the posttest, the majority (80%) completely explained how their equation related to the visual pattern, a significant difference (Wilcoxon sign-rank test; $W = -73$, $ns/r = 12$, $p = 0.0045$).

In addition to the changes in performance on the written assessment, our analysis of the postcourse interview data showed that teachers saw an important pedagogical use for connecting representations. When asked what they learned in the course, 13 of 21 teachers discussed connections between representations as a topic that helped them better understand students as learners of mathematics, and 11 of 21 named connections between representations as an important priority in the teaching of mathematics.

In following Design Principle 1, the RT held the content of function as a consistent thread throughout the course. Doing so allowed the instructor to focus on the ways in which representations helped to illuminate different aspects of functions. Teachers developed both a stronger fluency with the mathematical representations and a clearer sense of why and how one might use different representations as pedagogical tools.

Defining function: Using the lens of Principle 2. Another goal of the course was for teachers to develop a mathematically accurate definition of function and use it to distinguish examples and nonexamples of function. Teachers considered the guiding inquiry, *What is a function, and what are examples and nonexamples of functions?*, which provided opportunities to meet this goal, at several points throughout the course, as shown in Figure 1. The prominence of the guiding inquiry, combined with the repeated opportunities to revisit and revise thinking about the definition of function and the nature of examples and nonexamples, provided teachers with opportunities to deepen their content knowledge related to function. The pre/post written assessment measured that learning, asking teachers, *What is a function? Give an example of a function and a nonfunction.*

Teachers' responses to the first part of this task (*What is a function?*) were coded as correct, incorrect, or inconclusive. A correct definition included the idea of univalence (i.e., the mapping of each element of the domain to exactly one element of the range) and did not explicitly rule out arbitrariness (i.e., elements of the domain and range do not need to be numeric). Correct definitions could use different terminology for x and y (e.g., input and output; domain and range; independent variable and dependent variable). Definitions that did not include univalence or made erroneous statements (e.g., functions must be linear relationships) were coded as incorrect. Definitions were coded as inconclusive if there was not enough information present to suggest the univalence criterion. For example, several definitions included correct statements (e.g., functions pass the vertical line test) but provided no further explanation regarding why the statement(s) implied that a relationship was a function. Teachers' responses to the second part of the task (*Give an example of a function and a nonfunction*) were also coded as correct, incorrect, or inconclusive.[3] Examples and nonexamples were also coded by family (e.g., linear; quadratic) and representation(s) used.

In general, teachers' initial definitions were problematic, although their examples were not. Fewer than half of the 21 teachers in the course provided a correct definition

3 The inconclusive code was used for responses that provided a correct example and nonexample of function, but the teacher did not label which was which.

Table 4
Rubric for the Pentagon Pattern Task

Score	Description	Example
4	Full explanation; well connected to visual pattern A generalization is evident (verbally or symbolically) All aspects of the generalization are explained accurately with respect to the visual pattern	For each pentagon on the end of the train you count 4 sides, so that is always $4 \times 2 = 8$. There are two less pentagons in the middle of the train than the train number itself, and each of these has 3 sides counted as part of the perimeter (3 exterior sides)→$8 + 3(n - 2)$, where n is the train number.
3	Some explanation; partially connected to visual pattern A generalization is evident (verbally or symbolically) At least one aspect of the generalization is explained accurately with respect to the visual pattern Remaining aspects of the generalization are either explained incorrectly, inaccurately, vaguely, or not explained at all with respect to the visual pattern	$n(5) - (n - 1)(2)$ n refers to the number of the train, multiply this number by 5 then subtract one less than the total number multiplied by 2. From the visual representation we can see that 2 pentagons will share one side. This shared side will be on the inside of the shape and will not be included in the perimeter. This shared side must be subtracted from each pentagon.
2	Weak explanation; some connection to visual pattern A generalization is evident (verbally or symbolically) At least one aspect of the generalization is explained, but the explanation is incorrect, inaccurate, or vague	$(3x) + 2$ When a new train is added only three units sides two sides of that train are actually added. The $(3x)$ is 3 sides of the trains from before multiplied by the train number.
1	Numeric explanation only; no connection to visual pattern A generalization is evident (verbally or symbolically) The elements of the generalization are explained but not connected to the visual pattern in any way	$3n + 2$ Multiply the number of trains by 3 and then add 2. I know this works because it fits my pattern. My description is independent of the visual representation. I had to make a table—the pictures did not help me in finding the patterns.
0	No explanation present	

on the pretest, with incorrect definitions cutting across all grade levels taught and experience. Most of these incorrect definitions did not include mention of univalence and included features that suggested a narrow conception of function (e.g., functions are linear; all functions can be graphed). Nearly all teachers (20 of 21)—including all of the teachers who provided an incorrect definition—provided a correct example of function, with most being linear or quadratic relationships presented as equations or graphs. Over half the teachers (13 of 21) provided a correct nonexample of function on the pretest; however, 6 of these teachers provided an incorrect definition of function.

By contrast, nearly all teachers (20 of 21) provided a correct definition on the posttest, a significant difference (Fisher's exact test, $p < 0.001$). It is also important to note that all 12 teachers who did not provide a correct definition on the pretest improved in some way: 7 moved from incorrect to correct, 4 moved from inconclusive to correct, and 1 moved from incorrect to inconclusive. All teachers provided a correct example of function on the posttest,[4] mostly linear or quadratic in nature. However, there was a significant increase in the number of correct nonexamples of function (Fisher's exact test, $p < 0.01$). Given the background of these teachers, it is no surprise that the majority could easily produce and identify examples of functions. The types of examples provided by teachers were relatively straightforward relationships central to secondary mathematics. However, it is interesting to note that prior to the course, the majority of teachers were able to provide correct examples of functions, but not all teachers were able to correctly produce a definition or a nonexample of a function.

The second design principle that specifies a guiding inquiry, problematized early in the course and revisited throughout, provided teachers with repeated opportunities

4 One teacher did not respond to this item; all 20 teachers who did respond to this item provided a correct example of function.

to learn the definition of function. By considering multiple examples, examining narrative cases of teachers seeking to support their students in understanding the construct of function, and thinking about tasks to use in their classrooms related to function, teachers had the opportunity to rise above simply learning a correct definition of function. By thinking through the definition and examples as learners and teachers, teachers developed knowledge of both content and pedagogy through this sustained inquiry.

Two cases of learning about content and pedagogy: Using the lens of principle 3. In this section, we look at the learning of two teachers through the lens of Principle 3, which describes the ways in which activities that span the continuum of learning mathematics and teaching mathematics can support the development of teachers' knowledge of content and pedagogy. We present cases of two teachers with differing backgrounds and prior knowledge to describe the ways in which the learning-teaching continuum provided opportunities to learn about the content and pedagogy of function. Olivia was an experienced elementary teacher whose knowledge of function was relatively thin at the start of the course, and Carl was a preservice secondary teacher with strong mathematical knowledge. We consider the ways in which the course addressed differing needs based on each teacher's initial conceptions of function and how the diverse set of activities on the continuum from learner to teacher provided them with opportunities to learn that matched their backgrounds.

The case of Olivia. Olivia was a practicing elementary teacher completing her sixth year of teaching who enrolled in the course as an elective. Olivia was known as a thoughtful teacher-learner who had taken part in many high-quality professional development experiences, including a similar content-focused methods course on proportional reasoning in her masters of education program.

At the beginning of the course, Olivia's knowledge of function was limited (see Figure 5). The definition of function she provided on her pretest did not include univalence and implicitly ruled out arbitrariness. Although she provided a correct example of function on the pretest, her work during the first interview (conducted after Class 3) revealed that she struggled to explain *why* her example was a function, even though a correct definition of function had been made public in class by the time of her interview:

Interviewer: I have the example of a function that you gave on the pretest. So you gave $y = 2x$. And I wanted to ask you why this is an example of a function?

Olivia: Well, I think that looking at that, there would be one y for every x, and one x for every y, so I think that that's why it's a function.

Interviewer: OK. And that's based on the discussion in (the third class)?

Olivia: Yes, I mean, doesn't—I don't think it would have to be one value for each of them, I mean, every time you have y, or x, um, y is going to be two times that.

Interviewer: OK. What could you do to make it not a function?

Olivia: Make it x squared, or something. If I made it x squared, then, I think you'd have more than one value for x. And [long pause], I don't really know. I think that because it's x squared, I think you'd have more than one value. But I'm not really sure what I'm doing.

In this excerpt, Olivia attempted to use univalence to explain why her example is a function and to create a nonexample of a function but grappled with its meaning and determining the variable (x or y) to which she should attend.

By the end of the course, however, Olivia had a more robust understanding of function and its definition. Her posttest function definition (Figure 5) satisfied both conditions for a correct definition. During the postcourse interview, Olivia successfully classified a set of relationships as functions and nonfunctions and explained her classifications drawing on the definition. In the excerpt below, she explained why the graphs of $x = 2$ and $y = \pm x^{1/2}$ are not functions, using univalence as the justification:

Olivia: [$x = 2$ is a nonfunction] because x would be 2, but on that line, you could have any value for y. And also because of the multiple values of y, and also because if you think of drawing a vertical line through it, it is a vertical line, it'd hit the whole line. So it wouldn't be just one spot. And for [$y = \pm x^{1/2}$], if you do the vertical line test, it goes through the graph twice.

Interviewer: What is the vertical line test?

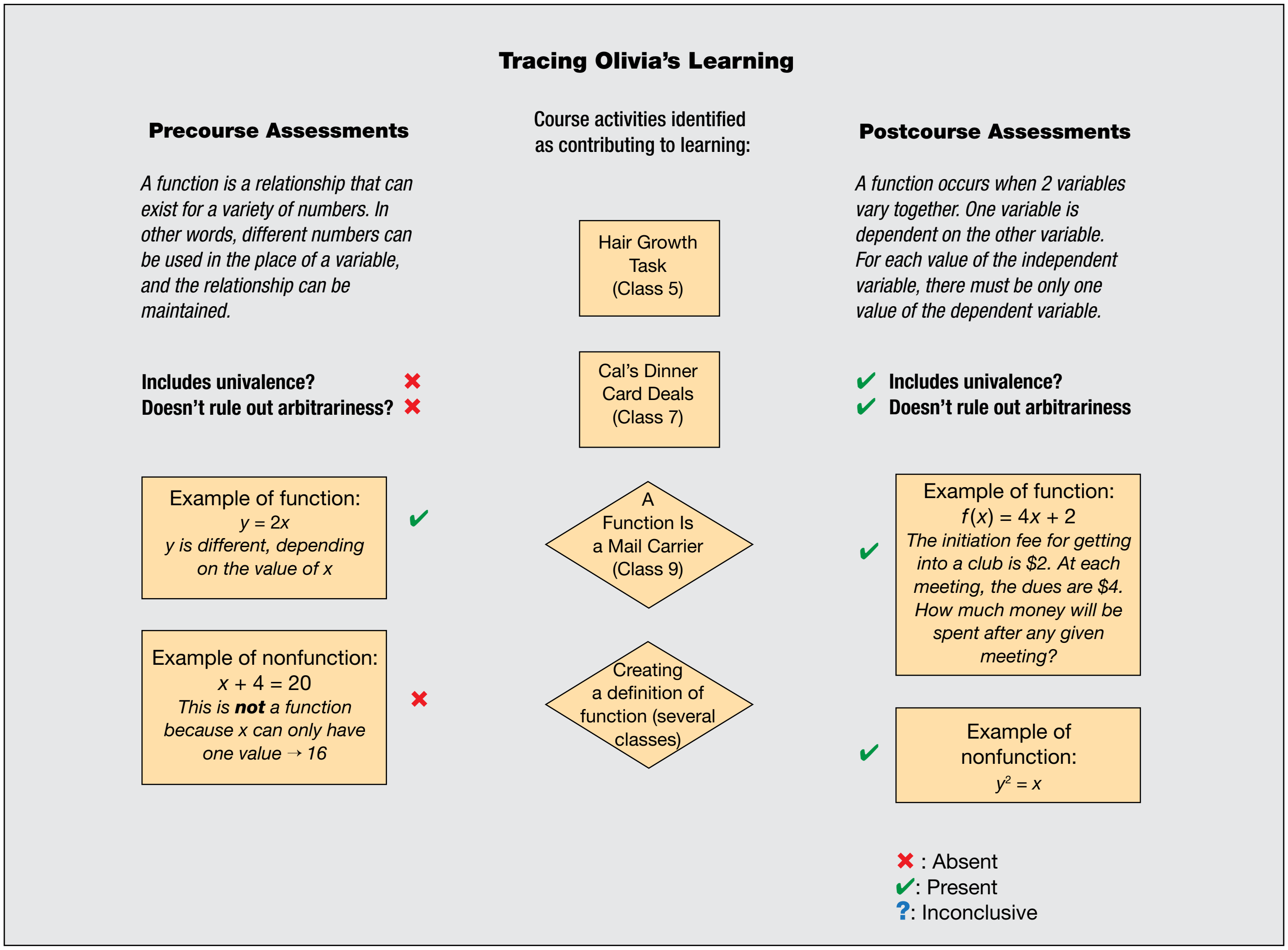


***Figure 5.* Tracing Olivia's Learning (teacher responses shown in italics)**

Olivia: Well, if you draw a vertical line through, you should only cross the graph once because if you cross it more than once, it means for that particular value of *x*, there's more than one value for *y*. Like here, say my value for *x* was 2. I could have this value of *y* and this value of *y*...,[5] say (2, 1) or (2, –1). So because you have those two values, it means it's not a function.

Olivia's responses in the second interview suggest that she had not simply memorized the definition discussed during the course; rather, she understood the key characteristics and drew upon them to classify relationships as functions or nonfunctions. In addition, she described the vertical line test and connected it to univalence, suggesting that she did not merely memorize the procedure of using the vertical line test to determine whether a relationship is a function. Olivia acknowledged her narrow view of function at the beginning of the course and described how her understandings changed through engagement in particular course activities:

Olivia: I have a much broader understanding of functions… a broader view of what a function is and what it involves… And also, thinking about what a function was. But I don't know that I could really define that before. And I try to think, "Could I have done that when I was maybe in 8th or 9th grade, when I was taking algebra classes?" And I don't really know that I could have.

5 An ellipsis indicates deleted words.

Interviewer: So were there places that helped you with defining a function and thinking about what a function really is?

Olivia: Well, creating a definition of a function. I remember "Function as a Mail Carrier" because the idea that every *x* can only have one—that idea that when you put in a value for *x*, you should always get one *y*. It shouldn't be you put in 4 one time, and you get 6 for *y*, and you put in 4 another time, the same number, and get 8. You can't have that. So I think that was important, too.

Olivia noted that the course enhanced her understanding of function and identified four particular activities as being instrumental in her learning. She recognized that her broadened view of function was influenced by participating in discussions based on the guiding inquiry and by various types of mathematical and pedagogical activities. When asked about the pedagogy of the course as a factor in her learning, she described specific features related to Principle 3:

Olivia: One thing I also liked about the class is that we really worked on developing our understanding of math AND connecting it to teaching, like through the case studies. And there aren't very many classes that do that... I think the two go hand in hand—really learning about the math and understanding it, then looking at how is that taught in the classroom? We looked at the tasks first, so we understood... what this task was about, the math that was involved, and then how a teacher was presenting the task, and how students in the task interpreted it, and maybe compare in your mind, "Well, you know, that's how I thought of it." It's effective, I think, for teachers because both of them are really important and connecting them [is] important.

In sum, Olivia entered the course with substantial confusion about function from a content standpoint. Her work in interacting with the mathematical tasks and the development of the definition enabled her to successfully define and identify examples and nonexamples by the end of the course. Moreover, she linked the content learning to the work in considering the cases and student work, describing the ways in which moving between learner and teacher was important to her development as a teacher.

We now consider the ways that the same course supported a teacher with a different background by looking at the case of Carl, a preservice secondary mathematics teacher.

The case of Carl. Carl was a preservice secondary teacher completing a yearlong internship in a suburban middle school. He had earned a bachelor's degree in mathematics from a major public university and took the course as the capstone of a fifth-year master of arts in teaching program. Despite Carl's mathematical background, his work early in the course suggested a muddled understanding of function. Carl's pretest definition allowed for arbitrariness but did not reference univalence, as shown in Figure 6. In distinguishing examples and nonexamples in his pre-interview, he used the univalence criterion but incorrectly described it as "one-to-one correspondence."

At the end of the course, Carl held a deeper and better connected understanding of function. His posttest definition fulfilled both criteria for a correct function definition. Interestingly, Carl did not use the input/output language that was often used in class discussions of the definition. This suggests that Carl had not merely memorized the class definition but held a conception of function that made sense to him. In reflecting on his learning, he described the differences in his understandings since the beginning of the course:

Carl: I have a clearer definition of what a function is… I think most of us came into the class having worked with functions before, obviously, and doing vertical line tests to see if something in the function maybe is not a function. But I don't think a lot of us had a really solid definition in our heads of what a function is. And I think that the class kinda helped us revise our own inkling of what a function is. The thing about functions is that correspondence between two different sets of quantities. Before the course, if someone had asked me "What is a function?" I couldn't say… I would've said something about the vertical line test. I would've said something about an equation [or] function notation. But I don't think I could have really given a really direct answer. After the course, I think I can.

Carl noted that while he entered the course with ideas about the definition of function, these ideas were incomplete, and the course provided an opportunity for him

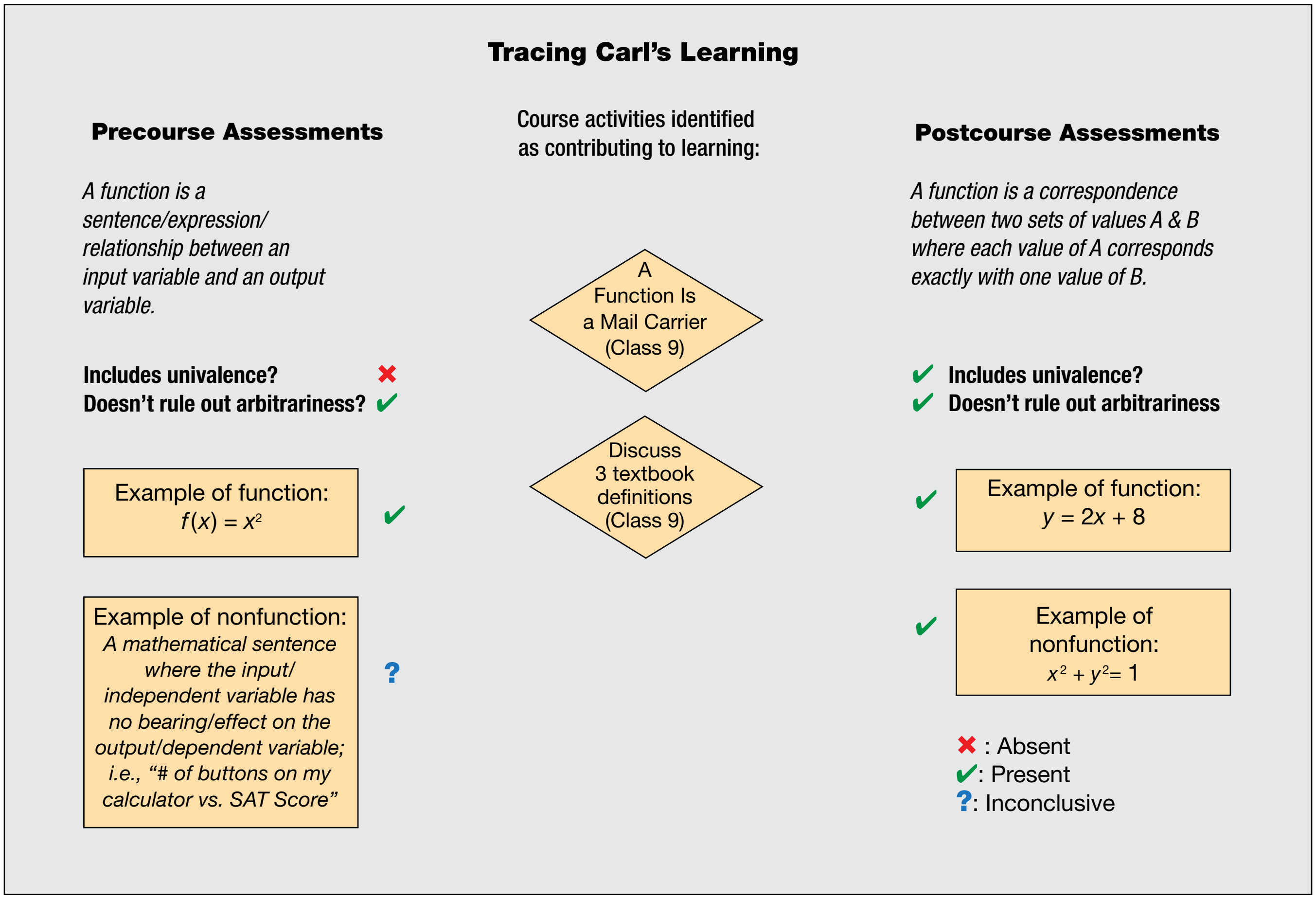


Figure 6. **Tracing Carl's Learning (teacher responses shown in italics).**

to revise his thinking and develop a clearer definition of function. When pressed to identify specific activities in the course that supported his learning, Carl described the discussion in which teachers created a definition of function (Class 3):

> *Interviewer:* So you just (identified) creating a definition of a function. How did that help you come to clarify the definition of function?
>
> *Carl:* Well, I mean we were just really brainstorming the definition of a function, and I think what it really did was made me analyze the specific kinds of things that make up a function. It doesn't necessarily have to be an equation. You know, it could just be the two sets. . . . But what it really did is it really made me scrutinize my own definition of a function that I had coming into the class, and we could change it and alter it a little bit, due to the discussion of the definition. Having that up there throughout class made me go back and see, "Well, is this a function? Is this a function?" Go through the criteria that we came up with ourselves.

When asked to reflect on how the structure of the course supported his learning, Carl's answers differed from Olivia's. Carl focused on the enactment of the tasks in the course as a model for his own future classroom and the cases as reinforcing the real-time modeling:

> *Carl:* Class time was a good example of how a pattern task could be implemented in the classroom, and the level of mathematics was high. [The instructor] had us solve in groups, was able to ask some open-ended questions that didn't necessarily guide the group directly to an answer. . . . [When] group discussion was over, she was able to bring the class together and have a whole-class discussion [and] pick out certain solutions that were beneficial

> to the class as a whole to see. It was really an example and another reinforcement of how to use pattern tasks and to keep the level of mathematics high in the classroom. [And] the cases that we read are examples.

Carl's background differed from Olivia's in that he entered the course with a stronger conception of function but with a definition that was in need of clarification. Through the same set of activities, Carl was able to make repairs to his definition rather than adopting the co-constructed class definition. Carl also entered as a preservice teacher looking for models of how to enact student-centered tasks, models that had been lacking in his internship placement. For Carl, moving between doing mathematics and considering cases of teaching helped him develop pedagogical knowledge related to how he might support his students in developing conceptual understanding.

For these two teachers, traversing the continuum of content and pedagogy created a common set of learning opportunities that led to different learning outcomes. Olivia's mathematics background was such that she took advantage of opportunities to learn related to the content of function, integrating the new content knowledge into an existing framework about her own well-developed pedagogical practices. She specifically noted the cases as a place in which the content and pedagogy come together, and one might anticipate that seeing this connection would make her better able to integrate new content understandings into her teaching. Carl, with a stronger mathematical background but at the very beginning of his teaching career, was able to take note of the ways that larger-grained pedagogical structures can support the learning of content. His ideas about the ways in which the group discussions and sharing of solutions modeled in the course led to new mathematical understandings provided useful models for Carl's early practice as a beginning teacher. As such, Principle 3 provided these teachers with opportunities to learn about teaching mathematics that fit their differing needs at the time.

Discussion

The content-focused methods course is a promising model for supporting teachers in developing mathematical and pedagogical knowledge and integrating those knowledge bases in ways that build knowledge needed for teaching mathematics. The examples presented from the functions course demonstrate how the design principles can come together to provide diverse groups of teachers with opportunities to learn. The functions course, however, is only one instantiation of the content-focused methods course model and was enacted in a specific institutional context that may differ from your own.

So how does one begin designing a content-focused methods course? Selecting a mathematical focus (Principle 1) is a good starting point. Identifying narrative and/or video cases (e.g., Barnett, Goldenstein, & Jackson, 1994; Boaler & Humphreys, 2005; Merseth, 2003; Smith, Silver, & Stein, 2005a, b, c) and student work (e.g., Lamon, 2005; Parke, Lane, Silver, & Magone, 2003) that relate to the mathematical focus can suggest a specific guiding inquiry. In the sections that follow, we discuss ways in which the principles could be implemented in different contexts and the affordances and constraints of bringing a content focus to an existing mathematics methods course.

Varying the Mathematical Focus and Guiding Inquiry: Principles 1 and 2

By varying the mathematical focus, and in turn, the guiding inquiry, additional content-focused methods courses for teachers of grades 7-12 could be developed. For example, content-focused methods courses on proportional reasoning (using the guiding inquiries *What is proportional reasoning?* and *Are all fractions ratios? Are all ratios fractions?*; Hillen, 2005) and geometry and measurement (using the guiding inquiry *What is a proof?*; Steele, 2006, 2008) have been developed and studied. A content-focused methods course on *reasoning-and-proving* (Smith & Stylianides, 2010; Hillen, Smith, & Arbaugh, 2011) is currently under development. The guiding inquiries to frame this course will include a mathematical question (*What is reasoning-and-proving?*) as well as ones that could be considered more pedagogical in nature (*How do secondary students benefit from engaging in reasoning-and-proving? How can teachers support the development of students' capacity to reason-and-prove?*). While the current principles reflect a secondary population, the content-focused methods course model could be used in courses for teachers of the elementary grades. Similar principles have also been used to structure professional development opportunities for teachers and their principals (Steele, Johnson, Herbel-Eisenmann, & Carver, 2010).

Varying the Focus on the Learner-Teacher Continuum: Principle 3

By shifting the focus on the learner-teacher continuum, additional courses could be created that would serve a variety of purposes. By placing an emphasis on the *learner* end of the continuum, a mathematics content course could be created, which could consist primarily of activities in which teachers solved mathematical tasks and occasionally examined student work or read a practitioner-oriented article. Such a course could conceivably meet the forthcoming recommendations from the

Conference Board of the Mathematical Sciences (CBMS, 2012), stipulating three courses focused on content knowledge for teaching for future secondary mathematics teachers. These courses would address the need for systematic ways to develop specialized content knowledge, identified as a pressing research priority (King & Thames, 2011). For teachers who have already had opportunities to carefully consider mathematics content, a course focusing on the *teacher* end of the continuum, making additional connections to practice (e.g., lesson study cycles) or providing opportunities to do action research (e.g., collecting and analyzing data from their own classrooms), might be useful (Boston & Smith, 2009). Such a course might be particularly appropriate for the master's-level or district-based professional development, where teachers have more fluency inquiring into and reflecting on their own practice.

Such courses also provide a rich site for studying teacher learning. Given the few formal classroom learning opportunities that currently exist for teachers to learn specialized content knowledge and pedagogical content knowledge, little research exists on the ways in which teachers learn these ideas at the intersection of content and pedagogy. A content-focused methods course that is specifically designed with the goals of developing these aspects of mathematical knowledge for teaching could serve as an important site for studying the way these knowledge bases grow in teachers. Courses run in conjunction with a field component would also provide opportunities to study the ways in which such knowledge is used in context.

The content-focused methods course provides a generalizable, adaptable model for integrating the study of content and pedagogy. The course described here resulted in teacher learning that varied in beneficial ways across teachers. Activities that traverse the content-pedagogy spectrum, grounded in a specific slice of mathematical content, can provide teachers opportunities to enhance areas of their knowledge across that spectrum.

References

Alper, L., Fendel, D., Fraser, S., & Resek, D. (1997). Designing a high school mathematics curriculum for all students. *American Journal of Education, 106*, 148–178.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Towards a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). San Francisco, CA: Jossey-Bass.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*, 389–407.

Barnett, C., Goldenstein, D., & Jackson, B. (Eds.) (1994). *Fractions, decimals, ratios, & percents: Hard to teach and hard to learn?* Portsmouth, NH: Heinemann.

Boaler, J., & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning*. Portsmouth, NH: Heinemann.

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education*, *40*, 119–156.

Boyd, D., Grossman, P., Lankford, H., Loeb, S., & Wyckoff, J. (2008). *Teacher preparation and student achievement* (NBER Working Paper No. W14314). Cambridge, MA: National Bureau of Economic Research.

Brown, E. T., Bush, W. S., & McGatha, M. (2006, January). *What mathematics do middle school teachers know?* Presentation at the Annual Conference of the Association of Mathematics Teacher Educators, Tampa, FL.

Center for Mathematics Education (2009). *CME Project algebra I*. Upper Saddle River, NJ: Pearson.

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Conference Board of the Mathematical Sciences (CBMS). (2012). *The mathematical education of teachers II*. Retrieved on March 28, 2012, from http://cbmsweb.org/MET2/MET2Draft.pdf

Cooney, T. J. (1996). Developing a topic across the curriculum: Functions. In T. J. Cooney, S. I. Brown, J. A. Dossey, G. Schrage, & E. C. Wittmann (Eds.), *Mathematics, pedagogy, and secondary teacher education* (pp. 27–96). Portsmouth, NH: Heinemann.

Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Burrill, G., Hart, E. W., . . . Watkins, A. (with Messenger, M. J., & Ritsema, B.). (1997) *Contemporary mathematics in context: A unified approach (Course 1)*. Columbus, OH: Glencoe/McGraw-Hill.

Darling-Hammond, L. (2010). Teacher education and the American future. *Journal of Teacher Education*, *61*, 35–47.

Darling-Hammond, L. (with Fickel, L., Macdonald, M., Merseth, K., Miller, L., Ruscoe, G., Silvernail, D., Snyder, J., Whitford, B.L., & Zeichner, K.). (2006). *Powerful teacher education: Lessons from exemplary programs*. San Francisco, CA: Jossey-Bass.

Darling-Hammond, L., & Bransford, J. (Eds.) (2005). *Preparing teachers for a changing world: What teachers should learn and be able to do*. San Francisco, CA: Jossey-Bass.

Drake, C., & Sherin, M. G. (2008). Developing curriculum vision and trust: Changes in teachers' curriculum strategies. In J.T. Remillard, B.A. Herbel-Eisenmann, & G.M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 321–337). New York, NY: Routledge.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*, 105–121.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematical professional development institutes. *Journal for Research in Mathematics Education*, *35*, 330–351.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*, 371–406.

Hill, R. O. (2006). Concerns about the training of future high school math teachers. *MSOR Connections*, *6*, 1–3.

Hillen, A. F. (2005). *Examining preservice secondary mathematics teachers' ability to reason proportionally prior to and upon completion of a practice-based mathematics methods course focused on proportional reasoning*. Unpublished doctoral dissertation, University of Pittsburgh.

Hillen, A. F., & Hughes, E. K. (2008). Developing teachers' abilities to facilitate meaningful classroom discourse through cases: The case of accountable talk. In M. S. Smith & S. N. Friel (Eds.), *Cases in mathematics teacher education: Tools for developing knowledge needed for teaching* (Association of Mathematics Teacher Educators [AMTE] Monograph, Volume 4, pp. 73–88), San Diego, CA: AMTE.

Hillen, A. F., Smith, M. S., & Arbaugh, F. (2011, January). *Tasks, tools, and talk: A framework for supporting students' capacity to reason and prove*. Presentation at the Annual Meeting of the Association of Mathematics Teacher Educators, Irvine, CA.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction*, *26*(1), 95–143.

King, K., & Thames, M. (2011, October). *Research on mathematics for teaching: Taking stock and working collaboratively*. Presentation at Teaching Teachers in the Era of the Common Core, Reston, VA.

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum Associates.

Markovits, Z., & Smith, M.S. (2008). Cases as tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Volume 2, pp. 39–64). Rotterdam, the Netherlands: Sense Publishers.

Merseth, K. K. (Ed.). (2003). *Windows on teaching math: Cases of middle and secondary classrooms*. New York, NY: Teachers College Press.

Parke, C., Lane, S., Silver, E. A., & Magone, M. (2003). *Using assessment to improve middle-grades mathematics teaching and learning: Suggested activities using QUASAR tasks, scoring criteria, and students' work*. Reston, VA: NCTM.

Pitts, V. R. (2003). *Representations of functions: An examination of pre-service mathematics teachers' knowledge of translations between algebraic and graphical representations*. Unpublished doctoral dissertation, University of Pittsburgh.

Sánchez, V., & Llinares, S. (2003). Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teacher Education*, *6*, 5–25.

Sand, M. (1996). A function is a mail carrier. *Mathematics Teacher*, *89*(6), 468–469.

Schifter, D. (1996). What's happening in math class?: Envisioning new practices through teacher narratives. *Series on School Reform, Vol. 1*. New York, NY: Teachers College Press.

Schmidt, W. H., Tatto, M. T., Bankov, K., Blömeke, S., Cedillo, T., Cogan, L., . . . Schwille, J. (2007). *The preparation gap: Teacher education for middle school mathematics in six countries*. East Lansing, MI: Michigan State University Center for Research in Mathematics and Science Education.

Schmidt, W. H., Houang, R., & Cogan, L. S. (2011). Preparing future math teachers. *Science*, *332*(6035), 1266–1267.

Senk, S. L., Keller, B., & Ferrini-Mundy, J. (2000). Teacher preparation: K–12 mathematics. In S. L. Ganter & W. Barker (Eds.), *CRAFTY Curriculum Foundations Project* (pp. 145–155). Washington, DC: Mathematical Association of America.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Simon, M. A. (2000). Research on mathematics teacher development: The teacher development experiment. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 335–359). Hillsdale, NJ: Lawrence Erlbaum Associates.

Smith, M. S., & Friel, S. N. (Eds.) (2008). *Cases in mathematics teacher education: Tools for developing knowledge needed for teaching.* Association of Mathematics Teacher Educators Monograph series, Volume 4. San Diego, CA: AMTE.

Smith, M. S., Silver, E. A., & Stein, M. K. (with Henningsen, M. A., Boston, M., & Hughes, E. K.). (2005a). *Improving instruction in algebra: Using cases to transform mathematics teaching and learning*. New York, NY: Teachers College Press.

Smith, M. S., Silver, E. A., & Stein, M. K. (with Boston, M., & Henningsen, M. A.). (2005b). *Improving instruction in geometry and measurement: Using cases to transform mathematics teaching and learning*. New York, NY: Teachers College Press.

Smith, M. S., Silver, E. A., & Stein, M. K. (with Boston, M., Henningsen, M. A., & Hillen, A. F.). (2005c). *Improving instruction in rational numbers and proportionality: Using cases to transform mathematics teaching and learning.* New York, NY: Teachers College Press.

Smith, M. S., & Stylianides, G. (2010, January). *Redefining the focus of high school mathematics instruction: Developing teachers' capacity to reason and prove*. Presentation at the Annual Meeting of the Association of Mathematics Teacher Educators, Irvine, CA.

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education, 40*(5), 530–562.

Steele, M. D. (2005). Comparing knowledge bases and reasoning structures in discussions of mathematics and pedagogy. *Journal of Mathematics Teacher Education*, *8*(4), 291–328.

Steele, M. D. (2006). *Middle grades geometry and measurement: Examining change in knowledge needed for teaching through a practice-based teacher education experience*. Unpublished doctoral dissertation, University of Pittsburgh.

Steele, M.D. (2008). Building bridges: Cases as catalysts for the integration of mathematical and pedagogical knowledge. In M. S. Smith & S. N. Friel (Eds.), *Cases in mathematics teacher education: Tools for developing knowledge needed for teaching* (Association of Mathematics Teacher Educators Monograph series, Volume 4, pp. 57–72), San Diego, CA: AMTE.

Steele, M.D., Johnson, K.R., Herbel-Eisenmann, B., & Carver, C. (2010). Leadership content knowledge: A case of principal learning in algebra. *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Retrieved from http://www.pmena.org/2010/downloads/PME-NA%202010%20Proceedings%20Book.pdf

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal*, *27*, 639–663.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*, 313–340.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, 356–366.

Wilson, S. M. (2011, October). *The Common Core State Standards: Opportunities and challenges for the mathematical education of teachers*. Presentation at Teaching Teachers in the Era of the Common Core, Reston, VA.

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