Using “Lack of Fidelity” to Improve Teaching

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The author presents a procedure for learning from variations that occur when instructors implement lesson plans designed by others. This kind of variation, occurring in many classrooms every day, can provide a source of information for improving curriculum, both in terms of instructional activities for students and especially in terms of clarifications for instructors to support more effective implementation. The author provides detailed descriptions, in the context of a mathematics course for preservice K–8 teachers, for using implementation variations in a practical, research-based way to study and improve teaching. The goal is to build an accumulating knowledge base for teacher education. Examples are presented to illustrate how increasingly rich lesson plans, based on observing implementation variations, can move toward achieving this goal.

Key words: Curriculum Implementation; Continuous Improvement of Teaching; Teacher Preparation

The purpose of this article is to describe a method for studying variation in teaching resulting from lack of fidelity in implementing a curriculum to create evidence-based improvements in teaching. The method uses observations of implementations to identify details of the curriculum that could be changed to increase the chances that all instructors who use the curriculum will provide the intended learning opportunities for students. A key assumption is that repeated cycles of observing and revising the *details* of curriculum implementation are essential for building knowledge for teaching that leads to cumulating and lasting improvements in classroom instruction.

Lack of fidelity in implementing curricula might be a surprising setting in which to conduct cycles of observing and revising teaching. Lack of fidelity has usually been interpreted in one of two ways. Either it is viewed as an obstacle to measuring the effects of an intended curriculum on student achievement (Fullan, 2008; Huntley, 2009; National Research Council, 2004; O’Donnell, 2008) or it has been interpreted as an unavoidable, and fully appropriate, mediation by the teacher to fit the local conditions and connect the curriculum with the students in a particular classroom (Fullan, 2008; Lloyd, Remillard, & Herbel-Eisenmann, 2009; Remillard, 2005).

Rarely has the lack of fidelity been tapped as a source of natural variation that can be used as a comparatively inexpensive way to improve the curriculum and, ultimately, improve the instruction that arises from implementing the curriculum. But lack of fidelity in implementing a curriculum introduces variation in teaching that either decreases or enhances learning opportunities for students. Such variation can be treated as data that suggest improvements to the curriculum. The purpose of this article is to explore a method for treating lack of fidelity as a learning opportunity for teacher educators and to illustrate the way in which the method can be used to improve a curriculum for preservice teachers.

Background

Reducing Variation and Raising the Mean

The method I describe for studying variation in teaching due to lack of fidelity in implementing a curriculum is based on an assumption not universally endorsed in the United States. Simply put, the assumption is that the goal of improving classroom teaching requires reducing variation in teaching from classroom to classroom and, at the same time, raising the mean level of teaching quality (e.g., Raudenbush, 2009).

The assumption more commonly accepted in the U.S. is that variation in teaching is necessary and sometimes even desirable. Variation in teaching is sometimes seen as a wise response to different local conditions and, in turn, a recognition of professional respect for individual teachers (e.g., Duffy & Hoffman, 1999; Lloyd et al., 2009; Remillard, 2005). In this view, reducing variation implies diminishing teachers’ roles in making professional judgments about their own classrooms. I argue for a different point of view. I agree with Nicolas Kristof who, in his *New York Times* column, described the variation in teaching from classroom to classroom as a major national problem in education, often diminishing the learning opportunities for students (2009).

In the setting I describe, teachers are teacher educators, students are K–8 preservice teachers, and classroom teaching is the teaching of preservice mathematics courses. But the issues of implementing a planned curriculum, the lack of fidelity in doing so, and the resulting variation across classrooms are otherwise the same as those arising in school settings. In other words, I believe the method I describe can be used by teachers (teacher educators or classroom teachers) who are teaching from the same curriculum with the same student learning goals in mind. This is true for a range of curricula, from traditional to reform. The only requirement is that the person(s) conducting the observations and proposing revisions understand well the learning goals *and* the intent of the curriculum being improved.

Working Toward a Theory of Implementation

Studying the lack of fidelity when implementing a curriculum is best guided by a theory of implementation. Over the past several decades, it has come to be recognized that there is a large gap between intended instructional treatments and the outcomes of the treatments. Implementing treatments has become an object of study in its own right (Fullan & Pomfret, 1977; Lipsey, 1993; O'Donnell, 2008; Remillard, Herbel-Eisenmann, & Lloyd, 2009). But theories of implementation are not well established. To set the stage for the method I describe, I present here the beginnings of a theory of implementation for the case I will examine: implementing lesson plans designed for mathematics content courses for preservice K–8 teachers.

It is reasonable to expect theories of curriculum implementation to address two key questions: (1) What does a curriculum (in this case a set of lesson plans) need to contain to be implemented as intended? (2) How will we know whether it has been implemented as intended? That is, how can we measure its implementation?

I hypothesize that a lesson plan for a preservice content course should include the following features. These features seem to be essential for both prescribing a lesson designed to help students achieve the learning goals and helping instructors implement the lesson as intended (Hiebert & Morris, 2009; Morris & Hiebert, 2011).

- A complete and precise statement of the learning goal(s).
- Explanations (rationales) for how each instructional activity is designed to help students achieve the learning goal(s).
- Descriptions of each instructional activity, including descriptions of the activities themselves and descriptions of the pedagogical approach that should be used. Descriptions of pedagogy explain how the recommended instructional moves derive from the theory of learning on which the lessons are based (a theory described below).
- Responses that students are expected to give to the instructional tasks, suggestions for how the instructor might respond, and rationales for the suggested instructor responses.
- Samples of verbal explanations that instructors can present at key moments in the lesson.
- Review of content that instructors might need if they haven't encountered this content or how it is treated in the lesson.

An example of a lesson plan with these features is provided on page 84.

How can the implementation of a lesson be measured? In other words, how can one tell whether a lesson has been implemented as intended? Answering this question in general is beyond the scope of this paper because there are numerous issues that must be considered (Fullan, 2008; Huntley, 2009; Remillard, 2005; Remillard et al., 2009). But it is possible to offer a brief answer to the measurement question that serves as a working definition for the method I describe.

I define lack of fidelity as follows: Implementations of individual lessons lack fidelity if they include teacher moves, not prescribed in the lesson, that represent (1) *significant variations* of the lesson, or (2) *positive adaptations* of the lesson. Significant variations of the lesson include teacher explanations, class discussions, or instructional activities that (a) do not appear to help students achieve the learning goal(s) as effectively as those prescribed in the lesson, or (b) violate the learning theory on which the lessons are based. Positive adaptations of the lesson are teacher explanations, class discussions, or instructional activities that (a) appear to help students achieve the learning goal(s) more effectively than those prescribed in the lesson, or (b) change the lesson to make it more compatible with the learning theory on which the lessons are based.

It should be noted that not all variations are significant variations or positive adaptations. Only variations that fit the definitions just presented are considered to be instances of lack of fidelity. Many variations can occur that are not considered to be implementation infidelities. For example, class discussions can take a variety of forms without containing significant variations or positive adaptations.

The learning theory on which the lessons in this study are based is actually a pair of learning principles rather than a full theory. The two principles, as described by Hiebert and Grouws (2007), are especially relevant for learning goals that have a heavy conceptual component: (1) conceptual relationships among mathematical ideas, representations, and procedures must be made clear, and (2) students must be given an opportunity to grapple or struggle with the critical mathematical concepts. Given these two principles, it is possible to define significant variations of type (b) above as teacher actions that remove one or both of these principles from the intended learning opportunities, and to define positive adaptations of type (b) above as teacher actions that improve the learning opportunities for students in ways that are relevant to the learning goals and are consistent with these two learning principles.

Implicit in the theory of implementation just presented is an assumption that studying variation in teaching when implementing a curriculum does not yield claims of *best* teaching practices. The intent is to yield *better* teaching practices by building into the enacted curriculum the positive adaptations observed and to eliminate from the enacted curriculum the significant variations. It is assumed that best teaching practices for the learning goals specified in a curriculum are an ideal that teachers work toward by incrementally improving the curriculum through studying its implementation through methods like those described next.

Method

Setting

I am a mathematics teacher educator at a large university in the mid-Atlantic region of the United States. The K–8 teacher certification program is completed in four years and graduates about 150 students per year. The mathematics portion of the program includes three mathematics content courses and one mathematics methods course. If students wish to obtain an endorsement to teach mathematics in middle school, they can take an additional four mathematics courses and one mathematics methods course.

The work I describe centers on the first of the three mathematics content courses required for all K–8 preservice teachers. This course focuses on whole numbers and decimal numbers. Classes are limited to 35 students, so multiple sections of the course are offered each semester. The instructors consist of faculty, doctoral students, and adjunct instructors.

The curriculum for the course consists of detailed lesson plans for each class session in the semester. The lesson plans were developed over time by mathematics education faculty and doctoral students working together. Each semester, instructors for a particular course meet weekly to develop, test, and refine the lessons (Hiebert & Morris, 2009). Critical for the work reported here is that the lesson plans contain the features identified earlier (e.g., learning goals, rationales for activities, detailed descriptions of activities, predicted student responses and suggested instructor responses, teacher explanations, and reviews of content).

I am an author of the curriculum and an experienced instructor of the course. This means that I understand well the learning goals for the course and the intentions of the curriculum. This allowed me to develop strong hypotheses during my observations about which changes to the intended curriculum were significant variations, which were positive adaptations, and which were neither. As noted earlier, the method depends on at least one person possessing deep knowledge of the learning goals and the curriculum. It is this knowledge that allowed me to generate hypotheses about implementation variations.

The instructors observed in this study were adjunct instructors who had not been involved in the original process of lesson development and did not have frequent interactions with instructors who had been involved in this process. This meant it was likely that these instructors would implement the curriculum in some ways that varied from the intent of the authors. The variations could be better or worse adaptations—they could increase or decrease students' opportunities to achieve the learning goals. Because these data provide the key opportunities to learn how to improve the curriculum, it is important that at least some of the observed instructors are less familiar and less experienced with the curriculum than the observer. These conditions often exist in teacher education programs and schools. Although the instructors in this study met weekly with each other to review past and future lessons to enrich their interpretation of the curriculum, I assumed that their relative inexperience would yield variations in implementing the curriculum that would be worth recording.

Procedure

I observed 24 of the 27 sessions in each of the two sections of the first mathematics content course, one section for each of the two instructors. Observing more than one instructor was useful for sorting out whether significant variations were due to the written lesson plan or idiosyncrasies of the instructor. I recorded written notes on all teacher statements and student statements intended for the whole class, including student responses to the instructional tasks. I flagged places in the lesson plan where, in my judgment, positive adaptations and significant variations occurred and wrote notes in the margin of the lesson plan that would help me reconstruct the nature of the positive adaptations and significant variations. For significant variations, I noted the feature of the lesson plan that was likely responsible. For example, if instructors carried out an instructional activity differently than described in the plan, I marked the activity itself and, depending on the nature of the significant variation, the statement of the learning goal or the description of the pedagogical approach that was apparently misinterpreted or ignored. Within 1 or 2 days of observing a lesson, I reviewed all written notes, typed detailed descriptions of the instances of lack of fidelity, and made changes to the written lesson plan that either captured the positive adaptations or elaborated or corrected the lesson plans to reduce the likelihood of the significant variations in the future.

Findings

I will present findings from this work by reporting three examples in considerable detail. For findings to be useful, they need to inform several aspects of the lesson improvement process driven by documenting lack of fidelity of lesson implementation. In particular, the findings must identify the source of a hypothesized significant variation in terms of the lesson features posited in the theory of implementation, they must explain why a teacher move was hypothesized to be a significant variation or positive adaptation, they must suggest fixes to the lesson plan, and they must provide a basis for hypothesizing that the fix will lead to reduced variation across instructors and a more effective level of teaching for everyone. The examples presented below illustrate these features of the findings.

Example 1: Misinterpreting the Learning Goal

Source of the Significant Variation

I assumed that the general source for all significant variations was the written lesson plan (rather than the instructor) because written plans are always imperfect and incomplete, and significant variations result from a misinterpretation or selective interpretation of the plan. The first example stems from variations to the lessons I interpreted as significant that occurred in both instructors’ sections during Lessons 2–4. Because of the nature of the variations, I attributed the problem to a misinterpretation, or more accurately a selective interpretation, of the learning goals for these lessons. As stated in the lessons, the learning goals for Lessons 2 and 3 were:

1. Preservice teachers will understand the terms *numeration system*, *quantity*, *numeral*, and *number* and the relationships among them.
2. Preservice teachers will recognize the properties of numeration systems: additive, multiplicative, subtractive, positional, place-valued, and the meaning of zero.
3. Preservice teachers will understand that the symbolic representation of a quantity in any numeration system is determined by decomposing it into parts equal in size to the measuring units of the numeration system, and representing the total amount of equal-sized parts with symbols, according to certain rules. The size of these parts as well as the symbols and rules used to represent them vary from system to system.

The learning goals for Lesson 4 were:

1. Preservice teachers will understand the properties of based place-valued numeration systems. Preservice teachers will understand that a based place-valued numeration system consists of a set of measuring units, a finite set of symbols, and a collection of rules that determine the structure of the system.
2. Preservice teachers will construct a set of measuring units associated with the place values for any based place-valued numeration system.
3. Preservice teachers will be able to represent the same quantity with different based place-valued numeration systems.

Although the authors of the lessons intended the preservice teachers to work with actual quantities and pictures of quantities to develop an understanding of the concepts underlying numeration systems, and Lessons 2–4 included instructional activities that engaged preservice teachers in doing just that, both instructors eliminated many of the activities that involved breaking quantities into parts equal in size to the measuring units of a given numeration system and activities that involved creating pictures to represent different-sized units. Instructors taught the lessons using primarily written words, numerals, and arithmetic calculations. For example, the lesson plans ask the teacher to repeatedly engage students in instructional activities that involve making place-value charts that show the measuring units of a numeration system with pictures of quantities. In contrast, the instructors placed measuring units in place-value charts but usually labeled the positions only with words and numerals (e.g., for the Babylonian system, “ones,” “60s, “60 × 60,” and so on).

When asked to identify the most important learning goal for these lessons, one of the instructors chose this goal:

> **Preservice teachers will understand that the symbolic representation of a quantity in any numeration system is determined by decomposing**

> **it into parts equal in size to the measuring units of the numeration system, and representing the total amount of equal-sized parts with symbols, according to certain rules. The size of these parts as well as the symbols and rules used to represent them vary from system to system.**

This was indeed the most important learning goal that guided the writing of the lessons. The instructor was not ignoring the goal, but rather interpreting it differently than the lesson writers did. The instructor emphasized the symbolic aspects of the goal, whereas the lesson writers emphasized the quantitative aspects.

The instructors made decisions to omit or modify many of the activities that were intended to engage students in concrete or pictorial work and concentrated instead on symbolic presentations and manipulations. I classified these changes as significant variations because they violated the principle of learning that calls for conceptual relationships among representations to be made clear and therefore did not appear to help students achieve the quantitative aspects of the learning goal. Based on past experience with these lessons, I know that students who have not developed physical, quantitative images for units of different sizes with the relationship of, for example, "10 times as big," will have difficulty when they are asked in future lessons to extend their knowledge of whole-number systems to decimal fractions less than 1. So, these instances of lack of fidelity will limit students' opportunities to achieve the later learning goals.

Fixing the Lesson

Fixing the lesson means revising the lesson plan to communicate more clearly to the instructors the feature of the lesson that seemed to be the source of the problem. Because I attribute the misunderstanding that prompted the significant variations to the lessons just described to a selective interpretation of the learning goals, I focused my attention on restating the learning goals more completely and clearly. I will use the phrase *elaborated learning goal* to signify these revised versions of the learning goal.

Rather than just restate the learning goals in clearer language, a fix that might have little effect, I decided to elaborate the learning goals to include a description of how achievement of the goal will be measured plus a scoring rubric, of sorts, that provides an unambiguous standard against which students' performance can be assessed. This is not a new idea. The assessment literature argues that including "performance objectives" with "content objectives" helps clarify the intent of the content objectives (Cook, 2008; Kapfer, 1971; Mager, 1997). The elaborated learning goals I developed are consistent with this general idea but were designed with a very specific correction in mind. In particular, I wanted to ensure that when future instructors read the learning goals, they would not be able to ignore the quantitative aspects of the goal. To accomplish this, I described actions and explanations that students are expected to display as they work toward achieving the learning goals. My purpose is to develop among instructors (a) a deeper shared understanding of the learning goals, and (b) a clearer sense of what to do in the instructional activities to help students achieve these goals. Because the actions and explanations I describe for students are about quantities, not just symbols, I hypothesize that future instructors will no longer be able to ignore this aspect of the learning goals. The elaborated learning goals clearly establish the intended emphasis of the instructional activities. Ironically, if these elaborated learning goals are taken seriously by instructors, the instructors have more freedom in how they implement the suggested instructional activities. In a real sense, the prescription for the lesson moves from the instructional activities into the statement of the learning goals.

Compare, as an example, the original learning goals for Lesson 2 (presented earlier and restated below) with the elaborated learning goals I created as the lesson fix to reduce variation across instructors. The original learning goals were the following:

1. Preservice teachers will understand the terms *numeration system*, *quantity*, *numeral*, and *number* and the relationships among them.
2. Preservice teachers will recognize the properties of numeration systems: additive, multiplicative, subtractive, positional, place-valued, and the meaning of zero.
3. Preservice teachers will understand that the symbolic representation of a quantity in any numeration system is determined by decomposing it into parts equal in size to the measuring units of the numeration system, and representing the total amount of equal-sized parts with symbols, according to certain rules. The size of these parts as well as the symbols and rules used to represent them vary from system to system.

The elaborated learning goals for this lesson are these:

1. Preservice teachers will distinguish between numerals (symbols) and quantities (physical amounts of stuff). They will distinguish between actions on numerals (arithmetic) and actions on quantities. Why do we want preservice teachers to make this distinction? Ideas about quantities will be emphasized throughout the course. This emphasis will help preservice

teachers recognize that mathematics is not just about symbols and calculation—that mathematical ideas are often about quantities and actions on quantities and that quantities can serve as concrete referents for mathematical ideas.

2. Preservice teachers will understand the terms "measuring unit" and "basic symbol" and will be able to use these terms when they explain how a quantity is assigned a numeral. They will understand that each numeration system has a set of measuring units and a set of basic symbols that it uses to represent all quantities. Preservice teachers will understand that, in any numeration system, a quantity (amount of stuff) is assigned a numeral by decomposing the quantity into parts equal in size to the measuring units and representing with the basic symbols of the numeration system how many measuring units of each type fit in.

3. Preservice teachers will understand that measuring units are quantities (physical amounts of stuff) that are used to measure other quantities, whereas basic symbols are symbols. Understanding that measuring units are quantities will make the study of decimals (later in this course) less abstract for the preservice teachers and will allow them to reconceptualize and be successful with decimals, whereas in their previous mathematical experiences, most of them were not.

4. Preservice teachers will develop and show these understandings by carrying out the following mathematical actions and giving explanations that involve these actions:

   a. Using any numeration system, preservice teachers will be able to physically measure quantities by physically partitioning the quantity to be measured into parts equal in size to the measuring units and determining how many measuring units of each type fit into the quantity. For example, preservice teachers will be able to measure a set of 23 dots in the Hindu-Arabic system by circling 2 separate measuring units of size 10 dots each, and 3 separate measuring units of size 1 dot each. In the Babylonian system, preservice teachers will measure 437 straws by physically bundling 7 separate measuring units of size 60 straws each and 17 separate measuring units of size 1 straw each.

   b. After physically measuring a quantity in this way, preservice teachers will be able to represent the measured quantity numerically by using the basic symbols of the numeration system to show how many measuring units of each type fit into the measured quantity.

   c. For place-valued numeration systems, preservice teachers will be able to make a place value chart that represents this process of measuring and assigning a numerical value to a quantity. The first row of the place value chart should show the measuring units as pictures of physical amounts. For example, in a place value chart in the Babylonian numeration system, a measuring unit of size 1 could be shown as 1 dot, a measuring unit of size 60 as 60 dots, a measuring unit of size 3600 as 3600 dots. (Because it is too hard to draw large quantities, preservice teachers can use the notation [3600] to represent 3600 dots.) In the Hindu-Arabic system, a measuring unit of size 1 might be shown as the area of 10 squares on graph paper. The measuring unit of size 10 would then be shown as 100 squares, a measuring unit of size 100 would then be shown as 1000 squares, and so on. The purpose of drawing the measuring units as amounts of stuff in the place value charts is to emphasize to preservice teachers that measuring units are amounts of stuff, not numerals. (This will allow a smooth transition to decimal numbers and operations in future lessons.)

   The place value chart should also show the multiplicative relationship between the measuring units. For example, when using the Babylonian numeration system, preservice teachers should draw an arrow from each measuring unit in the place value chart to the next largest measuring unit, label the arrow "× 60," and be able to explain (and to demonstrate with quantities) that this means that each measuring unit is 60 times as big as the measuring unit that is associated with the place to the right, that 60 copies of the smaller measuring unit will fit into the larger measuring unit, that we can find the larger measuring unit by making 60 copies of the smaller measuring unit, and that we can find the size of the smaller measuring unit by partitioning the larger measuring unit into 60 equal parts. Finally, in the second row of the place value chart, preservice teachers should show the number of measuring units of each type that fit into the measured quantity, represented with the basic symbols of the system. For example, Figure 1, a place value chart for base six (where the measuring unit of size 1_{six} is a circle), shows that four measuring units of size 1000_{six}, two measuring units of size 100_{six}, zero measuring units of size 10_{six}, and five measuring units of size 1_{six} fit into a measured quantity.

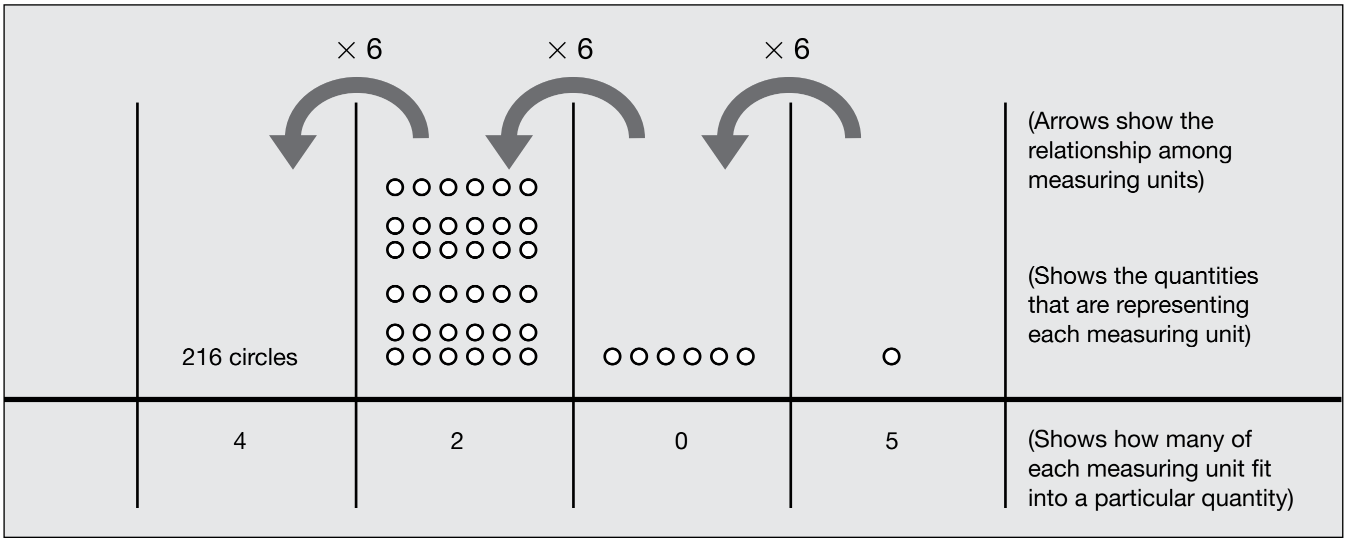


Figure 1. **The place value chart for base six.**

Thus, the place value chart is a concrete picture of the meaning of a numeral; place value charts provide a picture of the measuring units that are associated with the digits in a numeral. The use of a place value chart over multiple lessons increases the probability that preservice teachers will understand the meaning of each digit in a numeral after they move to representing numerals without the aid of a place value chart and will allow a smooth transition to decimal numbers less than one and to measuring units smaller than the measuring unit of size one.

d. Given a numeral in a given numeration system, preservice teachers will be able to represent the numeral with a quantity.

Why the Lesson Fix Should Reduce Variation

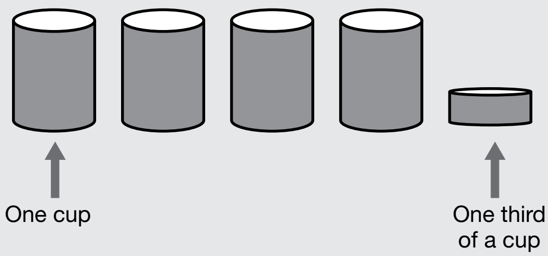
Obviously, the elaborated learning goals are much longer and more detailed than the original learning goals. But it is not just the length or detail that I believe is critical; it is the prescription of students' actions (described in 4a, 4b, 4c, and 4d above) that will be taken as evidence of students' achievement of the learning goals that greatly increases the chances that future instructors will implement the lesson without these significant variations.

To understand why this might be true, consider a lesson designed to help students achieve strictly procedural learning goals, with no conceptual component. It is easy to see that it would be straightforward to write such a goal, with no ambiguity, and that there would be little question about the actions students should take to show competence. This means it is likely instructors would interpret the procedural goal in the same way and follow similar instructional paths. In other words, a shared interpretation of a learning goal is likely to channel instructors onto a similar instructional path (at least, the variations they display are less likely to be significantly different).

With respect to clarity and shared interpretation, the challenge is to write conceptual learning goals like procedural learning goals. Elaborated learning goals are designed to meet this challenge. If instructors have the same understanding of the actions and explanations that students must master to demonstrate competence in a conceptual learning goal, then it is likely instructors will interpret the conceptual goal in the same way and, in turn, follow similar instructional paths.

But readers might be asking whether describing the actions and explanations that students must provide to demonstrate competence will encourage instructors to teach in a rote, procedural way targeted toward the desired outcomes. Will instructors just teach to the test? Two features of the lesson mitigate the danger of this happening. First, the actions and explanations that provide the goals for instruction are sufficiently complex that it is difficult to imagine instructors getting students to memorize these and use them flexibly on a range of problems. For example, by Lesson 5, students are asked to apply the actions and explanations described in the elaborated learning goals above to solve problems like the one from Lesson 5 shown in Figure 2. If students can apply the actions and explanations to flexibly solve a range of problems, they probably have developed some level of conceptual understanding. A second feature of

Suppose you are trying to represent this quantity of liquid (below) in base ten. You let the measuring unit of size 1 be one cup. Make a place value chart that shows all the measuring units that you need as you try to find a base ten numeral for this quantity, and explain how you found a base ten numeral for this quantity. What is the base ten numeral for this quantity?



From your work on this problem, explain why we get "repeating decimals."

***Figure 2.* A problem presented in Lesson 5.**

the lessons that reduces the likelihood of rote instruction is that the instructional activities described in the lessons are consistent with the two learning principles identified earlier that support conceptual learning.

Example 2: Not Recognizing Students' Lack of Understanding

Source of the Significant Variation

The second example is drawn from the same set of lessons but targets a different instance of lack of fidelity. As I observed Lessons 2–4, I noticed that the instructors were receiving no feedback from students regarding students' lack of understanding of the quantitative aspects of the learning goals. Students were following the instructors' lead, creating rules governing numerical manipulations, and completing the problems they were assigned correctly, but without understanding the relationships among quantities. Their lack of understanding was obvious during their small-group discussions, but the answers they produced were correct frequently enough for the instructors to presume that the students were achieving the learning goals.

Practically, the problem with the lesson plans was that the student assignments for Lessons 2–3 could be completed by interpreting them through either the instructors' interpretation of the learning goals (e.g., calculating numerical values for the positions in a place value chart) or the lesson authors' intended learning goals (e.g., forming units for the positions by partitioning and combining quantities). Students used numerical approaches to solve the assigned problems, instructors accepted these responses as indicating achievement of the learning goals, and there was no conflict that would have otherwise warned the instructors that something was wrong.

It seems reasonable to assert that curricula (e.g., the lesson plans of interest here) should contain student assignments that signal the instructor when students have not yet developed the conceptual understanding that is the intended focus of the learning goals. Curricula should include student tasks that provide feedback to instructors about whether they, the instructors, are on the right track.

That the student assignments in Lessons 2–3 could be completed without providing useful feedback to the instructors prompted me to reconsider the theory of implementation presented earlier. There is no feature of lesson plans in the bulleted list that requires such assignments to be included. I take this example as one that argues for adding this feature to the list, thereby refining the original theory. This example illustrates how the study of lack of fidelity or variation in teaching not only can improve teaching but also can refine the theory of implementation that connects the curriculum with classroom practice.

Fixing the Lesson

The fix for the problem of receiving no useful feedback regarding students' understanding is suggested by the problem itself: include student assignments that explicitly focus the instructors' and students' attention on the quantitative relationships in the learning goals. If students cannot complete the tasks without attending to quantitative relationships, then errors on these tasks would signal to the instructors that quantitative understandings had not been developed sufficiently during instruction. Here is an example of an assigned student task before I fixed the lesson and after I fixed the lesson.

Before revision:

Represent the following quantities using the Babylonian numeration system.

(a) O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O

(b) 780,021 circles

The students tended to solve part (a) by counting the number of dots (91), calculating 91 ÷ 60 = 1 remainder 31, and then writing the Babylonian symbol for 1 in the 60s place and 31 in the ones place. They did not break the quantity of dots into parts equal in size to the measuring units (as the lesson writers intended) nor did they draw a place value chart that showed pictures of the relevant measuring units and the multiplicative relationship between them. The students were unable to solve (b) because they got lost in all the numerical calculations.

After revision:

a. Make a place value chart in the space below. In the first row of the place value chart, draw the first three measuring units for the Babylonian numeration system. Use a circle to represent a measuring unit of size 1. (Remember that when a quantity is too large to draw, you can show it with square brackets. For example, the measuring unit of size 3600 circles can be represented as [3600].) Show the multiplicative relationship between the measuring units using the arrow notation.

b. Now bundle the following quantity using the Babylonian measuring units. To make these bundles, start with the largest measuring unit that fits in and determine how many fit in. Now move to the next smallest measuring unit. See how many fit in. Continue the process until the quantity is completely partitioned into parts equal in size to the measuring units.

O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O O O O O O
O O O O O O O O O O O

c. Now in the second row of your place value chart in part (a), show the number of measuring units of each type that fit in, using the basic symbols of the Babylonian numeration system.

To provide additional feedback to the instructor about whether the appropriate quantitative relationships had been developed during instruction, problems that cannot be solved using only a symbolic numerical solution were added to the assignment. An example follows:

Do you think the Babylonian numeration system could be used to assign a numeral to a quantity that is smaller than a measuring unit of size 1? How could you do that? (Hint: Choose your measuring unit of size 1 very carefully.) Use a place value chart and a quantity to show how this could be done. How do we do it in the Hindu-Arabic numeration system?

Why the Lesson Fix Should Reduce Variation

My observations of later lessons provide the best evidence that including better designed student tasks will provide feedback to the instructors that will help reduce significant variations from the intended lessons. The student tasks assigned after Lesson 4 more clearly ask students to use their knowledge of relationships among quantities rather than just carry out numerical calculations. One of the instructors noticed that the students were having difficulty with these tasks and spontaneously began reinserting some of the activities on building relationships between physical quantities that the instructor had previously dropped.

An open question is whether reducing variation in the lessons will require *both* an elaboration of the learning goals (as described under Example 1) *and* a redesign of the student assignments (as I describe here). I conjecture that both these fixes are needed. Because multiple interpretations of a written lesson plan are possible, redundancy in lesson features that clarify the intentions of the curriculum authors can only serve to reinforce a reduction in significant variations.

Example 3: A Positive Adaptation for Modeling Partitioning Division

Why the Teacher Move Was Hypothesized to Be a Positive Adaptation

The third example illustrates how positive adaptations can be used to improve the quality of a curriculum. In Lessons 11–13, both instructors modeled partitioning division in a way that was not prescribed by the lessons but, in my judgment, better prepared the students for the conceptual development of the long division algorithm in Lesson 18. For the problem 0.8 ÷ 4 = ?, for example, the plans for Lessons 11–13 encouraged students to model the problem by either (a) drawing or making a quantity of size 0.8 and then partitioning it into four equal parts to determine the size of each part or (b) using a doling-out process. In the latter approach, students would model 0.8 with eight longs (base ten blocks), for example, and then give one long to group 1, one long to group 2, one long to group 3, one long to group 4, one long to group 1, and so on until all the longs were distributed equally among the groups. When students extended this type of solution to problems like 0.72 ÷ 4 =?, they thought of 0.72 as 72 measuring units of size .01 and doled the 72 units out, one by one, to the four groups. Both instructors, however, presented an additional approach that was not prescribed in the lesson plans. They represented 0.72 as seven longs and two unit blocks, for example, and explained that only one long could be distributed to each of the four groups. To distribute the remaining three longs, they would have to exchange each long for 10 unit blocks, combine the resulting 30 unit blocks with the other 2 unit blocks, and then distribute the 32 unit blocks equally to the four groups. In general, they started with the largest measuring unit of a quantity, distributed as many as they could to the *n* groups, exchanged the remaining measuring units of that size for the next smallest measuring unit, combined them with the other measuring units of that size, distributed as many as they could to the *n* groups, and so on.

I classified these teacher moves as a positive adaptation because they were more consistent with the principle that conceptual relationships among mathematical ideas, representations, and procedures must be made clear. The instructors' approach to concrete modeling (with blocks, graph paper, straws, etc.) developed the concepts underlying the long division algorithm, whereas in the original lesson plans, there was a conceptual discontinuity between the modeling in Lessons 11–13 and the development and modeling of the long division algorithm in Lesson 18.

Fixing the Lesson

Fixing the lesson means revising the lesson plans to incorporate the additional approach to modeling partitioning division. The approach was also included in the elaborated learning goals for Lesson 12, as part of the described actions and explanations that students are expected to display as they work toward achieving the learning goals. A rationale for the approach and its connection to the long division algorithm was added to make instructors aware of the connection.

Why the Lesson Fix Should Reduce Variation

Traditionally, teaching that deviates from a planned curriculum and creates more effective learning opportunities remains a variant not replicated by other teachers. In fact, U.S. educators often celebrate teachers who invent more effective practices than those suggested by the curriculum. But these practices usually remain the province of the inventor. By writing positive adaptations into the planned curriculum, these practices can be replicated by all teachers. The variation is reduced because it becomes standard practice.

Discussion

Nature of the Evidence Gathered to Conduct This Work

I began the article by claiming that improving teaching requires both reducing the variation in teaching across classes with similar learning goals and raising the mean level of teaching. I would like to conclude by pointing out that different kinds of evidence can be used to address these two linked research and policy goals. Specifically, reducing variation among teachers requires evidence of teaching moves and student responses during instruction, whereas improving the mean level of teaching requires evidence of students' achievement. The importance of this distinction is seen both in the design of research and in the expense of conducting it.

The work I described in this article focused on generating hypotheses about changes to the curriculum that would reduce variation in future enactments and increase students' achievement of the learning goals. I used teacher moves and student responses during the lesson as data to catch places in the lesson where instances of lack of fidelity occurred and to hypothesize whether they were significant variations or positive adaptations. Testing whether changes I proposed based on these hypotheses will reduce variation in the future requires further observations of teaching. This means that repeated observations of teaching, focused on instances of lack of fidelity, can be sufficient to generate *and* test hypotheses about implementation variations and how to reduce them.

A value of this claim is that it allows teacher educators and teachers to study curricula by taking advantage of the natural variation that will occur as different instructors implement a shared curriculum with shared learning goals. Empirically based improvements in curricula require studying the effects of varying the curricula. Ordinarily, researchers plan variations and study the effects of these variations. The method I am describing complements this more expensive approach by simply observing and analyzing the variations that naturally occur in most teaching settings.

Whether the fixes to the lessons that reduce variation raise the mean level of teaching quality across instructors requires, of course, assessments of students' achievement. Do students across all sections of a course (or across any set of classrooms that share the same learning goals and use the same curriculum) achieve the learning goals more effectively after variation has been reduced than before? Collecting these data requires a phase of research not reported in this article but a phase that must follow the work reported here.

It should be noted, however, that reducing variation in the way I have described carries with it strong hypotheses that the mean quality of instruction will, in fact, improve. In simplistic terms, eliminating significant variations eliminates those aspects of the lesson plan that appear to unnecessarily dampen the learning opportunities for students, and inserting positive adaptations increases the learning opportunities for all students, not just those of the instructor who introduced the adaptation. Eliminating the weakest aspects of instruction from all classes and introducing stronger aspects of instruction into all classes should increase the average quality of instruction. But, as noted earlier, these are hypotheses that must be tested empirically.

I would like to make a final point about the critical role of empirical data in the process I have described. How to write learning goals so instructors do not misinterpret them, and how to write student assignments so teachers and students do not miss or bypass the intent, are empirical questions. It is impossible to know whether one has succeeded without empirical observations because people are capable of interpreting written text in multiple ways. Writing shared learning goals and creating tasks that provide critical feedback to instructors are not usually thought of as empirical issues. I believe this is especially true of writing learning goals, so I would like to elaborate on this particular claim. Learning to write goals that are interpreted similarly by all instructors requires observing how instructors operationalize the goals during instruction. It is not just a matter of writing out the goals in more detail, or even a matter of including performance objectives with the learning goals. Rather, writing learning goals for which a shared understanding develops among instructors requires an empirical cycle of writing goals at a grain size that reduces misinterpretation *and then* observing multiple instructors and classes to learn whether the goals are enacted as intended *and then* modifying them according to the information gathered *and then* asking instructors to implement the lessons again, and so on. It is impossible to predict beforehand which goals will be interpreted in a common way and which will be interpreted in different ways. The empirical cycle of observations and revisions is essential for writing learning goals that enable shared understandings among instructors.

Professionalizing Teaching

How does variation in teaching influence its professionalization? As noted earlier, some have argued that accepting variation among teachers' practices signals professional respect (e.g., Duffy & Hoffman, 1999; Lloyd et al., 2009; Remillard, 2005). In this view, reducing variation implies diminishing teachers' roles in making professional judgments about their own classrooms. But I, along with others (Shanker, 1997; Stigler & Hiebert, 1999), believe the process of improving teaching by reducing variation supports, rather than undermines, the professionalization of teaching. The goal of reducing variation is to improve the quality of teaching for all students. To paraphrase Al Shanker (1997), the goal is to make the best we know standard practice.

Both cases I described in this report place teachers (teacher educators) in a position of making professional judgments about the relative quality of learning opportunities. In one case, the lesson plan (or curriculum) failed to clarify for instructors the intent or the details of a lesson. The task for teachers studying curriculum enactment is to recognize these deficiencies as they play out in the classroom and to identify the features of the intended curriculum that can be corrected or elaborated more clearly. In the second case, the task is to recognize a richer variant of the lesson as introduced by an instructor and build this into the shared curriculum. In both cases, the aim is to use teachers' professional judgments to create steady and lasting improvements in the practices of teaching, a sure sign that teaching is being treated as a true profession.

The Benefits of Studying Implementation and Making Small Changes

The findings reported in this paper might seem overly focused and narrow, and the changes to the curriculum small and obvious. The findings themselves are probably of little interest to teacher educators who do not share the same learning goals. But the real message of this paper is that this is exactly the nature and grain size of the work that needs to be done to improve the implementation of curricula—collecting details about implementation through empirical observations, using the observations to revise the curriculum and to revise theories of implementation, and repeating the process. This unglamorous nitty-gritty work usually produces only small changes, but these changes can build to produce a curriculum that is implemented in less varying ways across instructors and that eventually improves student achievement.

I believe this work can be conducted by teachers and teacher educators across a range of settings. What is required is that two or more teachers share the same learning goals for their students, teach using the same curriculum, and differ in their understanding of the learning goals and the intention of the curriculum. These conditions exist in many teacher education programs and in many K–12 school settings. Often the most experienced teachers, or the most reflective teachers, will have developed a deep understanding of the learning goals and the curriculum. These teachers are able to develop informed hypotheses about variations from the intended curriculum that either raise or lower the learning opportunities for students. They can then propose curriculum changes to the group to reduce these variations. Finally, the effects of these changes can be tested through repeated observations and student assessments.

This work is not for those wishing for quick fixes. It is ongoing and yields small, incremental improvements. But the improvements are steady, can be preserved across changes in teachers, and can cumulate over time to yield substantial improvements in the quality of teaching and students’ learning.

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SUPPLEMENT

Using "Lack of Fidelity" to Improve Teaching

Anne K. Morris

Lesson 5
Topic: Place Value II

Learning Goals

1. For based place-valued numeration systems, preservice teachers will extend the following understandings to non-whole number numerals and to measuring units smaller than the basic measuring unit (the measuring unit of size 1).
   a. Preservice teachers will understand that measuring units are quantities (physical amounts of stuff).
   b. Preservice teachers will understand that in any base *b* place-valued system, there is a *b* times relationship between the measuring units. They will understand that, for a given measuring unit, we can construct the next largest measuring unit by making *b* copies of the given measuring unit. They will understand that, for a given measuring unit, we can construct the next smallest measuring unit by partitioning the given measuring unit into *b* equal parts. One of these parts will be equal to the next smallest measuring unit. They will understand that *b* copies of a measuring unit fit into the next largest measuring unit.
   c. The preservice teachers will understand that a quantity (amount of stuff) is assigned a numeral by decomposing the quantity into parts equal in size to the measuring units and representing with the basic symbols how many measuring units of each type fit in.
2. The preservice teachers will develop a deeper understanding of the idea of a basic measuring unit.

Preservice teachers will develop and show these understandings by (a) carrying out the following mathematical actions and (b) giving explanations that involve these actions:

(The following actions and explanations are all within the context of based place-valued numeration systems.)

- Preservice teachers will be able to construct measuring units smaller than the basic measuring unit (the measuring unit of size 1).
- Preservice teachers will be able to make a place-value chart that includes measuring units larger and smaller than the basic measuring unit. The place-value charts should show the measuring units as pictures of physical amounts. The preservice teachers should clearly show how they constructed the set of measuring units; they should explain that they made *b* copies of a smaller measuring unit to find the size of the next largest measuring unit, or they should explain how they partitioned a larger measuring unit into *b* equal parts to find the size of the next smallest measuring unit. Measuring units smaller than the basic measuring unit should be shown in the place-value charts as separate quantities, not as shaded parts of a whole where the whole is the next largest measuring unit. The place-value chart should also show the *b*-times relationship between all of the measuring units with the arrow notation.
- Given a non-whole number numeral, preservice teachers will be able to represent the numeral with their constructed set of measuring units. They will represent the non-whole number portion of the numeral with measuring units that are shown as separate quantities, not as shaded parts of a whole, where the whole is the next largest measuring unit.
- For all quantities (i.e., including quantities that are represented by non-whole number numerals), the preservice teachers will be able to physically measure the quantities—that is, physical amounts of stuff—by physically partitioning the quantity to be measured into parts equal in size to measuring units and determining how many of each type of measuring unit fit into the quantity.

- After physically measuring a quantity in this way, preservice teachers will be able to represent the measured quantity numerically by using the basic symbols of the system to show how many measuring units of each type fit into the measured quantity.
- Preservice teachers will be able to flexibly use a variety of different-looking quantities to represent "1."
- The preservice teachers will be able to solve a series of challenging problems, which are posed in the homework for this lesson, that require these ideas.

Equipment

- 100 straws for the instructor
- Rubber bands
- Scissors

Associated Files

- Lesson 4 Homework
- Handout 1 (one copy for each student)
- Handout 2 (one copy for each student)
- Lesson 5 Homework (one copy for each student)

Associated Text

- Handout 2
- *Mathematics for Elementary School Teachers*, Bassarear, Section 2.3, pages 100–115

Time: 0–30 min.
Activity Flow: Part 1; Based Place-Valued Numeration Systems; Sets of Measuring Units

Activity

Assign each group a problem on the Lesson 4 Homework. Ask them to put their solution on the board or on a transparency. Have the groups present their solutions. Go over every problem. Do not skip problems, because the preservice teachers should have numerous opportunities to practice explaining the relevant concepts.

Student responses	Suggested teacher responses
Students may use base ten language when they verbally refer to their numeral. For example, for 203_{five}, they say "two hundred three."	If you say "two hundred three," you are referring to a different quantity than "two zero three base five." *Two hundred three* implies base ten, so the measuring units are different sizes than the measuring units in base five. Even if you use the same basic measuring unit for both bases, the other measuring units are different, so the quantities that are represented by the two numerals are different.
***Discussion of Problems 7 and 9c* [*equivalent quantities represented by different symbolic representations*]**	
Student responses	**Suggested teacher responses**
Students approach the conversions in a very procedural way. For example, they explain that the place values are generated by raising the base to the power of 0, 1, . . . *n*. Their answers may be correct, but they are unable to produce a conceptual explanation.	Procedures that are not developmentally appropriate will cause difficulties for your future students. Younger students will not know about exponents. Your explanation of the meaning of the numerals requires an understanding of the meaning of the numerals. For example, you are explaining that "'10' means one group of '10.'" But if I do not know what the symbol "10" means, the explanation is not helpful. When you are trying to explain the meaning of numerals to young children, it is more appropriate to explain the meaning of numerals in terms of quantities—for example, to show them what measuring units are associated with each place value and what the digits in a numeral tell you about the number of measuring units.
On Problem 9c, some students are unable to convert 102_{four} to base seven without converting it to base ten first and take a numerical approach rather than a quantity-based approach. They write that 102_{four} is equal to $(1 \times 16) + (2 \times 1) = 18$. Then they set up a place-value chart with the numerals 1 and 7 written in for the measuring units. Then they determine that the base seven numeral for 18 is 24_{seven}.	This is an excellent solution, but it is based on understandings about numerical representations and arithmetic. Let's also develop a solution that is based on understandings about quantities and how numerals represent quantities. You begin by choosing an appropriate/convenient basic measuring unit. You might choose a single straw to represent your basic measuring unit. Now let's make a place-value chart and draw the measuring units for base four. (Ask the students to help you create the measuring units for base four and draw them in the first row of the place-value chart.) Let's put the digits of our numeral in the second row of our place-value chart under our measuring units. This helps us to remember what 102_{four} means. The numeral 102_{four} means there are two basic measuring units, zero bundles of the measuring unit of size 10_{four} (the measuring unit that is equal in size to four basic measuring units), and one bundle of the measuring unit of size 100_{four} (the measuring unit that is equal in size to 16 basic measuring units or four measuring units of size 10_{four}.) (Draw this quantity on the board, bundled into the measuring units of base four.) We need to unbundle and rebundle this quantity, using the measuring units for base seven. Let's make another place-value chart, showing the measuring units for base seven. Using the same basic measuring unit of one straw, what is the next largest measuring unit for base seven? (Draw it in the place-value chart.) What is the next largest measuring unit? (Draw it in the chart.) What is the biggest measuring unit that will fit into the [now unbundled] quantity of straws? Yes, the measuring unit of size 10_{seven}. (Circle measuring units of size 10_{seven} in the unbundled quantity to show the measuring of this quantity.) Now we go to the next smallest measuring unit. This is the basic measuring unit. Will any basic measuring units fit into the remaining amount? We see that four basic measuring units fit in. Consequently, our base seven numeral would look like this: 24_{seven}. (Write 24 in the place-value chart under the appropriate measuring units to help the students understand the meaning of the numeral.) So the very same quantity is represented by different numerals in the two bases. The only difference is that we measure the quantity with different-sized measuring units. We bundle it differently, but in both cases, the numeral conveys to others how many bundles we have of each type.

Time: 30–50 min.
Activity Flow: Part 2; Based Place-Valued Numeration Systems; Place Values Less Than One

Rationale

This activity addresses the learning goals. It extends the preservice teachers' understandings about whole-number place values and numerals (understandings about measuring units, the relationship between measuring units, how to construct measuring units from other measuring units, the meaning of the basic symbols in a numeral, and the idea that a quantity is assigned a numeral by decomposing the quantity into parts equal in size to the measuring units and representing with the basic symbols how many measuring units of each type fit in) to non-whole number place values and numerals. The activity is intended to reinforce the idea of bundling or copying *b* measuring units to create the next largest measuring unit in base *b* and the idea of partitioning a measuring unit into *b* equal parts in order to create the next smallest measuring unit in base *b*.

The preservice teachers should see that numerals smaller than 1 are not an anomaly or frightening—that numerals less than 1 reflect the same relationships discussed previously and the basic symbols mean the same thing. By first working with an unfamiliar numeration system (base three), they develop more explicit understandings about based place-valued numeration systems and measuring units smaller than the basic measuring unit. (In addition, the use of an unfamiliar numeration system is designed to develop further preservice teachers' ability to recognize (a) that several component understandings are involved in representing quantities with numerals, in counting, and in computing with the standard algorithms, (b) that these understandings are not trivial or easily acquired, and (c) why children might experience difficulties as they try to develop these understandings.)

Activity

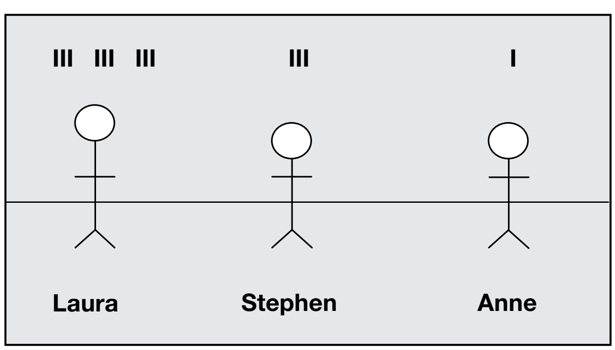
In this activity, students will model the structure of a base three system with straws. After a review of how measuring units are created in a base *b* system, they will figure out how measuring units smaller than the basic measuring unit are generated. The instructor will need straws, rubber bands, and scissors, as students at the board will be physically bundling and cutting straws to form base three measuring units.

> *We have been thinking about how to represent quantities with numerals. However, up to now we have only represented quantities that were equal to or bigger than the basic measuring unit. Today we will discuss how to represent with numerals quantities that are smaller than the basic measuring unit. Let's think about how we count in base three, and how we bundle quantities using the base three measuring units.*

Call three students to the board. Line them up (as shown in the diagram below).

> *Let this spot be for the basic measuring unit* [Anne's place]. (Draw a straw over Anne's head.) *This will be the basic measuring unit. What will the next measuring unit look like?* (Draw the measuring unit of three straws over Stephen's head.) *And the next?*

Draw the measuring unit of nine straws over Laura's head. Your final display on the board should look like the diagram below. Emphasize that they obtained the measuring units by making the next measuring unit three times as big as the last.



I am going to hand straws to Anne. As I hand each additional straw to Anne, tell me how we would represent the new amount using the measuring units of base three. In addition, as I hand the straws to Anne, help me count the amounts in base three.

Hand 14 straws to Anne, one by one. Focus students' attention on the bundling and counting of the straws. As each straw is handed to Anne, the students should say the appropriate word in the counting sequence in base three. In addition, the students should explain what should be done when Anne is given the third, sixth, ninth, and twelfth straws. Attention should be drawn to the need to rebundle the straws to form bigger measuring units. The instructor may want to say something like:

I am giving Anne the first straw. So that's 1_{three}. Now I'm handing her the second straw. That's 2_{three}. Now I'm handing her another straw. How many straws does Anne have? Can she hold this many? Why not? In base three, any given place value can hold no more than two of the corresponding measuring units, so when she is given the third straw, she must bundle them together and pass them to Stephen, who now has one measuring unit. What numeral represents this quantity? [10_{three}]

Anne should put a rubber band around the three straws and pass the bundle to Stephen. Similar statements can be made for subsequent amounts that require rebundling straws. The students will describe how to rebundle the quantities on their own; however, it helps to summarize their responses with these kinds of statements.

After the 14 straws have been handed out and counted, ask for two more volunteers, who will represent 2 measuring units smaller than the basic measuring unit. Have them stand on the other side of Anne. Point to the measuring units for the three students who are holding straws. Ask the class what the fourth student's measuring unit (Sam's measuring unit) should be [the measuring unit for the place to the right of the ones place].

So what does Sam's measuring unit look like?

Sam should make the measuring unit for his place. [Sam should decide to use the scissors, cut a straw into three equal pieces, and explain that one of the equal pieces is the measuring unit for his place because it is three times as small as the next largest measuring unit.] If the student (or class) fails to suggest this, remind the class of the relationship between the sizes of the measuring units in base three. You may use the expression "one third of the basic measuring unit," but be sure to also use the "3 times as small" language to be consistent with the construction of measuring units that are larger than the basic measuring unit. Now ask the class what the measuring unit for the fifth person (Tom) should look like.

So what does Tom's measuring unit look like?

Tom should make the measuring unit for his place. After Tom has made the measuring unit and explained his solution, draw the measuring units on the chalkboard above Sam and Tom. Next write the numerals for the measuring units for Anne, Stephen, and Laura on the chalkboard next to the quantities that were drawn earlier (i.e., 1_{three}, 10_{three}, and 100_{three}, respectively). Extend the students' understanding of whole-number place values (measuring units, the relationship between the measuring units, and the numerals that are associated with each place value) to non-whole number place values:

Why could we possibly need measuring units smaller than the basic measuring unit? What need could have motivated people to invent the idea of a measuring unit smaller than one? [Elicit the students' answers.]

When based place-valued numeration systems were extended to represent quantities smaller than one, people needed a convention to convey that they were using measuring units smaller than the basic measuring unit. If they wanted to measure and represent quantities smaller than the basic measuring unit, they would need smaller measuring units, and they would have to have a way to represent these smaller quantities with numerals. They solved this problem by placing a dot after the "ones" place value. In our system, we call this a decimal point because we are in base ten. Since we are in base three now, we will call this a "tricimal point." What is the numeral that is associated with this measuring unit? [Point to the *measuring unit* for the 0.1_{three} place. The students should say, "0.1_{three}."] *What is the numeral that is associated with this measuring unit?* [Point to the *measuring unit* for the 0.01_{three} place. The students should say, "0.01_{three}."] Write these numerals on the board over the students' heads.

Recall the labels for the whole-number place values, those to the left of the "tricimal point." We read these as "one base three," "one zero base three," "one zero zero base three." The symbolic representations—that is, the numerals— are 1_{three}, 10_{three}, 100_{three}.

So if we try to keep the same pattern, how can we represent the place values to the right of the "tricimal point" in base three?

The word labels are "zero point one base three," "zero point zero one base three." The numerals are 0.1_{three}, 0.01_{three}.

Do you see how this pattern is identical to the base ten numeration system? Why is that the case? What do the digits mean, and why would the measuring units in any base be assigned the same numerals? [Because one measuring unit of size *x* would fit into a measuring unit of size *x*.]

Next ask the five students to represent the following numeral with straws: 112.22_{three}. Be sure that the students explain how the straws are related to the numeral; their explanation should refer to the measuring units for each place and the meaning of the digits.

Summarize the students' explanation, and emphasize that the numeral *means* 1 of this measuring unit, 1 of this measuring unit, 2 basic measuring units, 2 of this measuring unit, and 2 of this measuring unit. In other words, each digit in the numeral tells you how many measuring units there are of each size.

Now ask students:

What does $112.22_{three} + 1_{three}$ *equal?*

Write this number sentence on the board. Give the class some time to think about the problem. Then ask the students at the board to show the addition and the rebundling that must occur. [The students are already holding the straws for the numeral 112.22_{three}, so Anne should pick up a straw from the table. Anne cannot hold three straws, so she will have to put a rubber band around them and pass them to Stephen. The students should determine the numeral for the new quantity [120.22_{three}].]

Pose one more task. The students at the board are currently holding straws that are represented by the numeral 120.22_{three}. Ask the students to add 0.01_{three} to this quantity. *What is the numerical representation of this new quantity?*

[The new quantity is 121_{three}.] After the class has had time to find a solution, have the five students show the addition and the regroupings that must occur in the various place values. Tom needs to begin the process by adding 0.01 to the amount he already holds.

Ask them to relate what they just did to the solution process used when solving the problem with the standard algorithm for addition:

$$\begin{array}{r} 120.22_{\text{three}} \\ \underline{+\ 0.01_{\text{three}}} \end{array}$$

First, have the five students start over; they should each hold the correct number of straws to represent 120.22_{three}. Now hand Tom 0.01_{three}. This represents the first step of the algorithm—adding 2 measuring units of size 0.01_{three} to 1 measuring unit of size 0.01_{three}. The instructor should carry out the steps of the algorithm on the board and ask the students to explain and illustrate the steps with the straws.

Instructor: I added 2 measuring units of size 0.01_{three} to one measuring unit of size 0.01_{three}. What happens next?

Tom: I need to bundle the 3 measuring units and give them to Sam, who will exchange them for one measuring unit of size 0.1_{three}. Now I have no measuring units of size 0.01_{three}.

Instructor: How do I indicate that in the algorithm?

Tom: Write a 0 in the 0.01_{three} place of the answer and write a little 1 above the 0.1_{three} place to indicate the exchange of 3 measuring units of size 0.01_{three} for one measuring unit of size 0.1_{three}.

Instructor: OK, I did that. Now what?

Sam: I was already holding 2 measuring units of size 0.1_{three}, so after Tom hands me his, I now have 3 measuring units of size 0.1_{three}. But I can't hold three. So Anne takes them and exchanges them for 1 measuring unit of size 1_{three}.

Instructor: How do I show that in the algorithm?

Sam: Write a 0 in the 0.1_{three} place, because I no longer have any measuring units after the exchange. Write a little 1 above the ones place to indicate the exchange of 3 measuring units of size 0.1_{three} for 1 measuring unit of size 1_{three}.

Instructor: OK, I did that. Now what?

Anne: I had 0 measuring units of size 1_{three}, but I was handed 1 by Sam, so I now have 1 measuring unit of size 1_{three}. So in the algorithm, add 0 and 1 to get 1 in the ones place of the answer.

Stephen: Two measuring units of size 10_{three} and 0 measuring units of size 10_{three} is 2 measuring units of size 10_{three}, so write a 2 in the 10_{three} place.

Laura: One measuring unit of size 100_{three} plus 0 measuring units of size 100_{three} is 1 measuring unit of size 100_{three}, so write a 1 in the 100_{three} place of the answer.

Ask the students at the board to return to their seats.

Time: 50–75 min.
Activity Flow: Part 3; Based Place-Valued Numeration Systems and Measuring Units For Places to the Right of the Point

Rationale

In this activity, students are asked to solve problems in base three and then in base ten. By first working in an unfamiliar system (base three), they develop more explicit understandings about based place-valued numeration systems and measuring units smaller than the basic measuring unit. When the students subsequently work in base ten, they can make connections between the base ten system and what they learned in base three, and transfer the more explicit understandings to base ten.

Activity

In this activity, the students create sets of measuring units, including measuring units smaller than the basic measuring unit, and represent numerals with their measuring units.

In this activity we will apply what we just learned.

Distribute Handout 1.

Have the groups work on these two problems. All groups should put their solutions on the board while they are working so the instructor can monitor their understanding. After the activity, the instructor should choose a solution to go over, modeling appropriate language for the students.

Problem 1	
Student responses	**Suggested teacher responses**
The student is confused because the basic measuring unit is not a straw, and "has parts." That is, the basic measuring unit consists of nine boxes. The student's idea of "one" is one discrete unpartitioned object, and the basic measuring unit in this case does not match that description. The "nineness" is a perceptual distracter that interferes with the student's ability to view the quantity as the "one" and to apply the "3 times as big" relationship idea to this quantity. The student cannot generate the other measuring units.	Ask the student to construct a place-value chart with the measuring units. This helps the student focus on making the basic measuring unit 3 times as big. You should help the student understand that making a quantity 3 times as big always involves copying or repeating the quantity 3 times, and the way that it looks is irrelevant; it is the amount that matters, and this amount must be made 3 times as big.
The student extends vague understandings about base ten and base ten blocks to the problem. The student uses one box to represent the 0.1. *"Wouldn't 0.1 be one of these blocks* [in the rectangle with 9 blocks]*, since the basic measuring unit, or 1, is 9 blocks?"*	Remind the student of the relationship between the measuring units in a base *b* system. What base are you in? So what is the relationship between the measuring units? Remind the student of the construction of the base three measuring units completed earlier: What quantity is serving the same role as the straw?
Problem 2	
Student responses	**Suggested teacher responses**
The most common problem is that students are unable to identify the basic measuring unit.	Direct them to the "times as big/as small" relationship for the set of measuring units. Ask them to construct a place-value chart and to draw the given measuring unit in the chart. How can you find the other measuring units?
The students build the set of measuring units correctly, but their solution for 4.32 is inconsistent with their set of measuring units.	Ask them to construct a place-value chart that shows their measuring units. Then ask them what the digits in 4.32 mean.

Time: 75 min.
Activity Flow: Part 4; Conclusion and Homework

Rationale

Students need to have experiences with many representations of basic measuring units, different bases, and many types of problems (e.g., finding the basic measuring unit when they are given the quantity that is represented by 0.1, and vice versa) to become flexible in their ability to interpret the meanings of decimals. This homework assignment provides these experiences. The students need time to independently grapple with these situations. They also need to work on a number of challenging, nonstandard problems that will help them recognize what they do and do not understand about decimals. (This assignment may be hard for the students.)

Activity

Explain that we will shift our focus to the four basic operations in the next lesson. Therefore, the homework assignment offers more practice with constructing sets of measuring units for whole-number and decimal place values, but it also prepares them for the next topic by asking them to model some addition problems.

Hand out Lesson 5 Homework and Handout 2.

Lesson 4 Homework

1. Make a place-value chart for a base five numeration system. To do this, first choose a basic measuring unit. Now, in the first row of the place-value chart, use your basic measuring unit to draw the measuring units for the 1_{five} place, the 10_{five} place, and the 100_{five} place. These measuring units should be shown as pictures of amounts of stuff, not as numerals. Now show the multiplicative relationship between the measuring units by drawing arrows between the measuring units and labeling the arrows “× 5.”

2. Using your measuring units from #1, build the quantity represented by the base five numeral 203_{five}. To do this, first write the basic symbols 2, 0, and 3 in the second row of your place-value chart under the appropriate measuring units. The 2 should be written in the second row, under the drawing of the measuring unit of size 100_{five}.

   The 0 should be written in the second row, under the drawing of the measuring unit of size 10_{five}. The 3 should be written in the second row, under the drawing of the measuring unit of size 1_{five}. Now you can easily see that the numeral 203_{five} means 2 of the drawn measuring units of size 100_{five}, 0 measuring units of size 10_{five}, and 3 measuring units of size 1_{five}. So now draw the quantity represented by the numeral 203_{five}.

Place-value chart with pictures of the measuring units in the first row, arrows in the first row, and basic symbols in the second row:

Now draw the quantity represented by the numeral 203_{five}:

3. Using your measuring units from #1, build the quantity represented by the base five numeral 34_{five}. Use the same steps that were described in #2 above.

Place-value chart with pictures of the measuring units in the first row, arrows in the first row, and basic symbols in the second row:

Now draw the quantity represented by the numeral 34_{five}:

4. Count from 34_{five} to 203_{five}. That is, write the numerals from 34_{five} to 203_{five}. If this is hard for you, how can you use your place-value chart to help you figure out the next numeral?

5. Using the same basic measuring unit that you used in #1 for the base five numeration system, make a place-value chart for a base three numeration system. To do this, in the first row of the place-value chart, use your basic measuring unit to draw the measuring units for the 1_{three} place, the 10_{three} place, the 100_{three} place, and the 1000_{three} place. These measuring units should be shown as pictures of amounts of stuff, not as numerals. Now show the multiplicative relationship between the measuring units by drawing arrows between the measuring units and labeling the arrows “× 3.”

6. Using your measuring units from #5, build the quantity represented by the base three numeral 201_{three}. To do this, first write the basic symbols 2, 0, and 1 in the second row of your place-value chart under the appropriate measuring units. The 2 should be written in the second row, under the drawing of the measuring unit of size 100_{three}. The 0 should be written in the second row, under the drawing of the measuring unit of size 10_{three}. The 1 should be written in the second row, under the drawing of the measuring unit of size 1_{three}. Now you can easily see that the numeral 201_{three} means 2 of the drawn measuring units of size 100_{three}, 0 measuring units of size 10_{three}, and 1 measuring unit of size 1_{three}. So now draw the quantity represented by the numeral 201_{three}.

Place-value chart with pictures of the measuring units in the first row, arrows in the first row, and basic symbols in the second row:

Now draw the quantity represented by the numeral 201_{three}:

7. Explain why 34_{five} and 201_{three} represent the same amount (the same quantity or amount of stuff; compare your answers to #3 and #6).

8. Count from 201_{three} to 2201_{three}.

9. Suppose the length below represents the basic measuring unit: ___________

   a. Using this basic measuring unit, make a place-value chart for a base four numeration system. In the first row of the place-value chart, use your basic measuring unit to draw the measuring units for the 1_{four} place, the 10_{four} place, and the 100_{four} place. These measuring units should be shown as pictures of amounts of stuff, not as numerals. Now show the multiplicative relationship between the measuring units by drawing arrows between the measuring units and labeling the arrows "× 4."

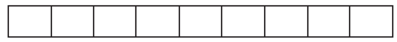
   b. Now using your place-value chart, draw the quantity that is represented by the base four numeral 102_{four}.

   c. Now convert 102_{four} to a base seven numeral by drawing pictures. (Hint: Rebundle your quantity in part (b).)

(*Return to p. 85*)

Handout 1

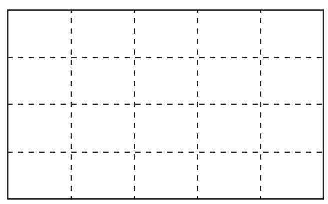
Suppose the basic measuring unit in a base three system is this area:



a. Build a set of measuring units for the two places to the left of, and two places to the right of, the "tricimal point." In other words, the quantity that is represented by 1_{three} is given, and you need to build the quantities that are represented by the numerals 10_{three}, 0.1_{three}, and 0.01_{three}. Show the measuring units in a place-value chart and use an arrow between the measuring units to show the multiplicative relationship between the measuring units.

b. Use the set of measuring units that you constructed in part (a) to build the quantity that is represented by the numeral 12.12_{three}. First, write these basic symbols in the second row of the place-value chart in part (a) under the appropriate measuring units. Then, in the space below, draw the quantity that is represented by 12.12_{three}.

2. Suppose that in a base ten system, the measuring unit that is represented by the numeral 10 is this area:



a. Build a set of measuring units for the two places to the left of, and two places to the right of, the "decimal" point. In other words, the quantity that is represented by 10 is given, and you need to build the quantities that are represented by the numerals 1, 0.1, and 0.01. Show the measuring units in a place-value chart and use an arrow between the measuring units to show the multiplicative relationship between the measuring units.

b. Use the set of measuring units that you constructed in part (a) to build the quantity that is represented by the numeral 4.32. First, write these basic symbols in the second row of the place-value chart in part (a) under the appropriate measuring units. Then, in the space below, draw the quantity that is represented by 4.32.

(*Return to p. 85*)

Handout 2

Modern Based Place-Valued Numeration Systems, Basic Measuring Units, and Sets of Measuring Units

Developing a based, place-valued numeration system:

First, some quantity (a straw, a dot, a sheep) is chosen (often because it is convenient) and called the *basic measuring unit* [*BMU*].

A fixed number is chosen that corresponds to the number of basic symbols that will be used in the system. This number is called the *base* [*b*].

Each *place* in a numeral is associated with a *measuring unit [MU]*, which is its *place value*. The *place* directly to the left of the *point* is associated with the *BMU* and is called the *ones place*. In any base, the *BMU* is a particular quantity that is considered "one."

Every other *place* is associated with a quantity that is constructed from the *BMU*. This quantity is called a *measuring unit* [*MU*].

In base *b*, orienting oneself at the *ones place* and the *BMU*, the next largest *measuring unit* [*MU*] is constructed by grouping together *b copies* of the *BMU*. Then, this new *MU* has a magnitude that is *b* times as big as the *BMU*. Each successive *place* to the left of the *ones place* corresponds to the next largest *MU*. Its *place value* is *b* times as big as the *place* to its immediate right.

Again, orienting oneself at the *ones place* and the *BMU*, the next smallest *measuring unit* is constructed by partitioning the *BMU* into the fixed amount (*b*) equal parts and choosing one of these parts. Then this new *MU* has a magnitude that is *b* times as small as the *BMU*. Each successive *place* to the right of the *ones place* corresponds to the next smallest *MU*. Its *place value* is *b* times as small as the *place* to its immediate left.

A set of *MUs* is formed that consists of the *BMU* and the *MUs* constructed as above, and is such that each new *MU* has a magnitude that is *b* times as big or as small as the adjacent *MUs* (on its right and left, respectively).

Suppose one chooses a *BMU* and constructs sets of *MU*s for based place-valued numeration systems with different bases. Once the *BMU* has been identified, then it is the quantity that represents "1" or "one." Its size is completely independent of the chosen base. However, the different based place-valued numeration systems are distinguished by how the corresponding set of *MUs* is constructed. The quantity of the groupings required to move to the next bigger or next smaller *place* is different. That is, the size of the *MU*s associated with each of the *place values* changes among different based systems.

In base *b*, the numeral 10 and the words "one-zero" mean 1 group of the quantity that is *b* times as big as the *BMU* and zero groups of the *BMU*. The numeral 0.1 and the words "zero point one" mean 1 group of the quantity that is *b* times as small as the *BMU* and zero groups of the *BMU*.

Examples: Let a straw be the *BMU*. Then the straw is "one" and is represented by 1_b.

Example 1: Consider a set of MUs for base ten.

- The *BMU* is a straw, so a straw is represented by "1_{ten}" or "one."
- The next larger *place* is associated with a *MU* that is the quantity of 10 straws, and is represented by "10_{ten}" or the "one-zeros" place. Continuing with this construction, then the next largest *place* is associated with the quantity of 10 groups of 10 straws, or 1 group of 100 straws. It is represented by "100_{ten}" or the "one-zero-zeros" place, etc.

- The next smallest *place* is associated with the quantity that is one tenth the size of one straw or 10 times as small as one straw. It is represented by “0.1_{ten}” or the “zero-point-one” place. Continuing with this construction, the next smallest place is associated with an *MU* equal to the quantity that is one tenth of one tenth of one straw, or one hundredth of one straw, or one hundred times as small as one straw. It is represented by “0.01_{ten}” or the “zero-point-zero-ones place.”

Example 2: Consider a set of MUs for base three.

- The *BMU* is a straw, so a straw is represented by “1_{three}” or “one.”
- The next larger *place* is associated with a *MU* that is the quantity of three straws, and is represented by “10_{three}” or the “one-zero base-three” place. Continuing with this construction, then the next largest *place* is associated with the quantity of three groups of three straws or one group of nine straws. It is represented by “100_{three}” or the “one-zero-zero base-three” place, etc.
- The next smallest *place* is associated with the quantity that is one third the size of one straw or three times as small as one straw. It is represented by “0.1_{three}” or the “zero-point-one base-three” place. Continuing with this construction, the next smallest place is associated with an *MU* equal to the quantity that is one third of one third of one straw, or one ninth of one straw, or nine times as small as one straw. It is represented by “0.01_{three}” or the “zero-point-zero-one base-three” place.

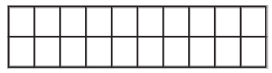
Example 3: Consider a set of MUs for base seven.

- The *BMU* is a straw, so a straw is represented by “1_{seven}” or “one.”
- The next larger *place* is associated with a *MU* that is the quantity of seven straws, and is represented by “10_{seven}” or the “one-zero base-seven” place. Continuing with this construction, then the next largest *place* is associated with the quantity of seven groups of seven straws or one group of 49 straws. It is represented by “100_{seven}” or the “one-zero-zero base-seven” place, etc.
- The next smallest *place* is associated with the quantity that is one seventh the size of one straw or seven times as small as one straw. It is represented by “0.1_{seven}” or the “zero-point-one base-seven” place. Continuing with this construction, the next smallest place is associated with an *MU* equal to the quantity that is one seventh of one seventh of one straw, or one forty-ninth of one straw, or forty-nine times as small as one straw. It is represented by “0.01_{seven}” or the “zero-point-zero-one base-seven” place.

(*Return to p. 85*)

Lesson 5 Homework

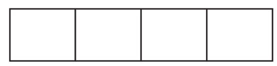
1. Suppose the basic measuring unit is given as this area:



   a. Build the quantity that is represented by 4.2 in a *base ten* system. Make a place-value chart first.

   b. Build the quantity that is represented by 4.2 in a *base five* system. Make a place-value chart first.

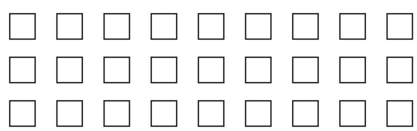
2. Suppose the measuring unit that represents the quantity 0.1 in a *base ten* system is this area:



   a. Construct the measuring units for the ones place, the tens place, and the hundredths place.

   b. Draw the quantity that is assigned the numeral 1.

3. Suppose the measuring unit that represents the quantity 10_{six} in a *base six* system is this area:

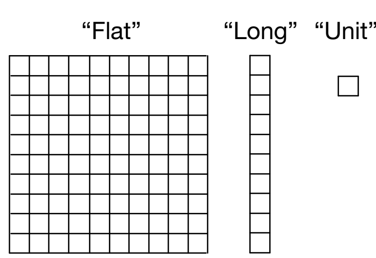


   a. Find and show the basic measuring unit.

   b. Build the quantity that is represented by 13.3_{six}. Make a place-value chart first.

4. Why do the numerals 0.6 and 0.60 represent the same amount?

5. Teachers use *base ten* blocks, shown below, to help children understand the relationship between the measuring units in our system.



a. Suppose the basic measuring unit is a "unit" (which is how the blocks are typically used in the lower grades). How many flats and units would you use to represent the quantity [1210] if you could only use those measuring units?

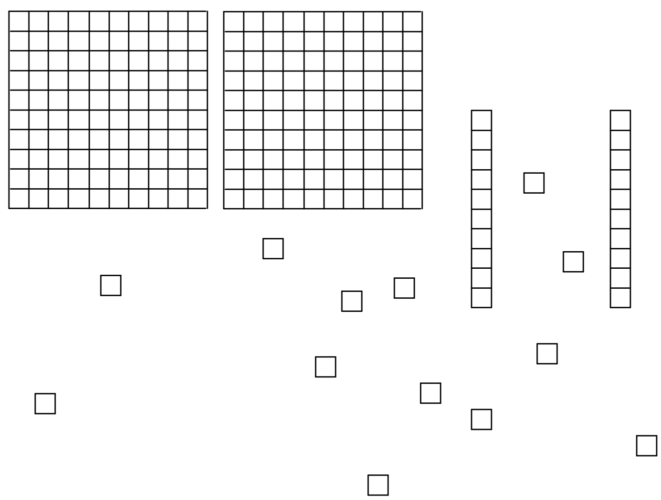
Flats ________________ Units ________________

b. Suppose the basic measuring unit is a "flat." How many "units" would you use to represent the quantity [0.18] if you could only use "units"?

Units ________________

Why is 0.18 called "eighteen hundredths"?

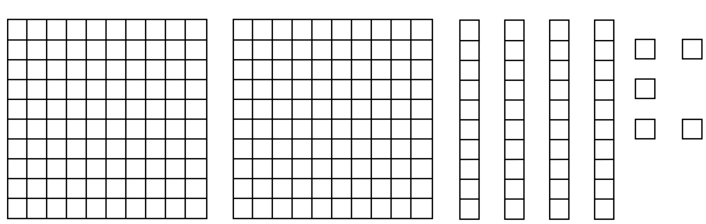
c. Suppose the basic measuring unit is a long. What is the numerical representation of the quantity below? ___________________________________



d. Suppose the basic measuring unit is a flat. What is the numerical representation of the quantity in part (c)? ______________________________

e. Suppose the basic measuring unit is the next-sized block, the "large cube," which is equal to 1000 "units." What is the numerical representation of the quantity in part (c)? ______________________________

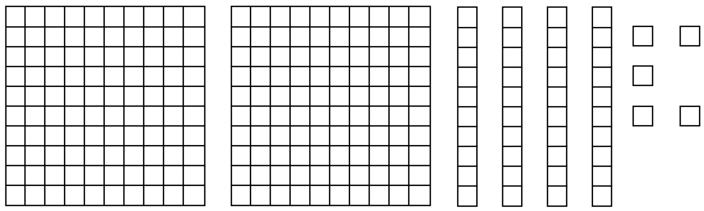
6. Consider the following quantity.



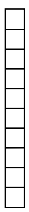
a. Let the area of the long be the basic measuring unit in base ten.

Make a place-value chart that shows your measuring units:

Now in the picture below, show how the quantity would be bundled with your measuring units:

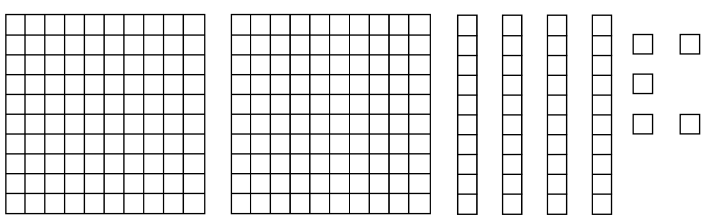


Now write the base ten numeral for this quantity: ______________________________

b. Now represent the same amount of stuff with a base four numeral. **Use the same basic measuring unit from part a.** That is,  is still your basic measuring unit.

Make a place-value chart that shows your measuring units:

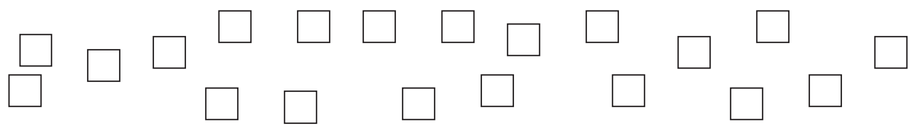
Now in the picture below, show how the quantity would be bundled in the base four system with your measuring units:



Now write the base four numeral for this quantity: ______________________________

7. As we saw in #5, the quantity that you choose for the basic measuring unit (or "one") affects the numerical representation of all other quantities.

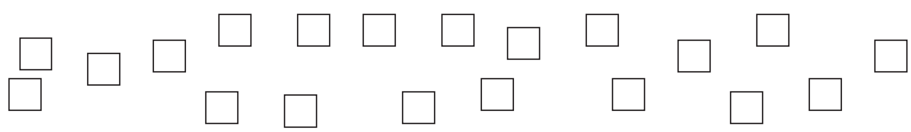
   a. Let □ be the basic measuring unit. Represent the quantity below with a numeral in **base three**. First make a place-value chart that shows your measuring units for base three. Then bundle the quantity using your measuring units. Then write the base three numeral for the quantity.



   Place-value chart that shows your measuring units for base three:

   What is the base three numeral for this quantity?: ___________

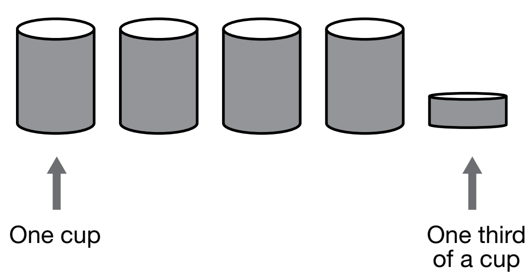
   b. Now let □ □ □ be the basic measuring unit. Again, represent the quantity below with a numeral in **base three** (same quantity as in part a). First make a place-value chart that shows your measuring units for base three. Then bundle the quantity using your measuring units. Then write the base three numeral for the quantity.



   Place-value chart that shows your measuring units for base three:

   What is the base three numeral for this quantity?: ___________

8. Suppose you are trying to represent this quantity of liquid (below) in base ten. You let the basic measuring unit be one cup. Make a place-value chart that shows all of the measuring units that you need as you try to find a base ten numeral for this quantity, and explain how you found a base ten numeral for this quantity. What is the base ten numeral for this quantity?



From your work on this problem, explain why we get repeating decimals.

9. In the next lesson, we will start looking at the meaning of addition and subtraction. Imagine you are a six-year-old. How would you solve the following story problems with objects (e.g., building blocks, pennies, lengths of ribbons, base ten blocks, lengths on a number line, volumes of water)?

For each problem, answer the following questions:

1. Describe the actions that you performed on the objects.

2. What did you use for your basic measuring unit in each problem?

   a. Josh had six cookies. His mom gave him five more. How many cookies does Josh have altogether?

   b. Five cows are in a field. Three are standing and the rest are lying down. How many cows are lying down?

   c. Dave had thirteen gumdrops. He gave four to Cheryl. How many gumdrops does he have left?

   d. Megan had some markers. She gave six to Janet. Now she has nine left. How many markers did she have to start with?

   e. There are four SUVs, two pickup trucks, and six cars in a parking lot. How many vehicles are there in the lot in all?

   f. There were four SUVs in a parking lot. Two pickup trucks and six cars pull into the parking lot. How many vehicles are now in the lot?

   g. Darnell has some red and green grapes. Two pounds are green and three pounds are red. How many pounds of grapes does he have in all?

   h. Joe has six balloons. His sister Connie has nine balloons. How many more balloons does Connie have than Joe?

   i. Sean has four more pennies than Meg. Meg has eight pennies. How many pennies does Sean have?

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