What Does Research Tell Us about Fostering Algebraic Reasoning in School Algebra?

Decades ago, Freudenthal (1977) characterized school algebra as including not only the solving of linear and quadratic equations but also algebraic thinking, which entails the ability to describe in a general way the procedures used in solving problems as well as the mathematical relations that underpin algebraic objects. His characterization remains timely today because it captures not only the symbolic aspects of algebraic activity but also the kinds of relational thinking that underlie algebraic reasoning and that distinguish it from arithmetic activity, which is typically computational in nature. This research brief focuses on two specific areas of school algebra: the solving of word problems and the activity of conjecturing and proving. In dealing with the research-supported ways in which algebraic reasoning can be fostered in these two areas, specific attention is given to the role of teacher questioning. Past research has shown how critical the role of teacher questioning is to the development of mathematical reasoning across a variety of domains (e.g., Stigler & Hiebert, 1999; Herbel-Eisenmann & Cirillo, 2009). An additional aspect that cuts across various mathematical areas, and which is emphasized in the second part of the research brief on conjecturing and proving, is the need to support such activity with appropriate tasks where students engage in higher-level reasoning processes such as reflecting, explaining, and justifying (e.g., Henningsen & Stein, 1997; Kieran & Guzmán, 2010). Additional research-based examples of teacher questioning and task activities that have proved successful in encouraging students to reason algebraically in a variety of situations, including word-problem solving and conjecturing and proving, can be found in, for instance, Greenes and Rubenstein (2008), Kieran (1992, 2007), and in the NCTM (2010) resource, Focus in High School Mathematics: Reasoning and Sense Making in Algebra.

Promoting Algebraic Reasoning in Solving Word Problems

The use of problem-solving situations, including word problems, to give meaning to algebraic activity is widely accepted in the mathematics education community. However, research has provided ample evidence of students' preferences for arithmetic reasoning and their difficulties with the use of equations to represent algebra word problems (e.g., Stacey & MacGregor, 1999; Koedinger & Nathan, 2004). In fact, students typically believe that the whole aim of problem solving is to find the answer, even in the context of their algebra classes. The aspect of algebraic reasoning that involves representing the relationships between the givens and the unknowns often gets lost in the use of word problems as a vehicle for algebra learning.

Research on the use of word problems in algebra has found that a teacher's questions, if well conceived, can encourage students to make explicit their problem-solving approaches and to represent them in a general way. The report of a study of eighth graders by Smith (2004) begins by contrasting what could be called an ineffective approach to the teaching of algebra problem solving (Teacher 1), followed by a much more effective alternative (Teacher 2) that promotes algebraic reasoning.

Teacher 1 gave her students a problem like the one in figure 1. To help them get started, she drew three columns on the blackboard (labeled Day, Tom, and Freddy). While students were working on the problem, Teacher 1 encouraged those who had already found a solution to look for another way to solve the problem. When it was time to compare solutions and solution methods, one student came forward, filled the table on the blackboard with numerical values, and arrived at a correct answer. Teacher 1 then asked if anyone had used a different method. No one offered any and the "discussion" on the problem, as well as the lesson, ended. Teacher 1 was disappointed that she had not been able to get her students to think of any other methods.

Tom and his younger brother, Freddy, went to the candy store one day to buy some chewing gum. Tom bought 18 ten-piece packages of gum and Freddy bought 24 five-piece packages. Every day, each boy finishes one whole package of gum. One day, they looked at how much gum each boy had left. Tom noticed that his brother had more pieces of gum left than he had. How many days has it been since the boys bought the gum?

When Teacher 2 gave her students the Chewing Gum problem (see fig. 1), she first wrote it on the board and had a student read it aloud. Then she drew two rectangles, one for Tom and one for Freddy. The students counted out 18 circles and displayed them on Tom's rectangle to represent the 18

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Fig. 1. The Chewing Gum problem (adapted from Smith, 2004)

packages of gum, but counted them by tens to emphasize the number of pieces of gum that Tom had started with. A similar process was followed for Freddy's gum. Then Teacher 2 asked all the students to try to solve the problem. Notice that the teacher did not model a solving method for the students as had Teacher 1, but rather the problem situation itself.

As the students worked on the problem, the teacher circulated around the classroom to see which methods they were using and encouraged them to explain their answers in a way that would allow others to understand what they had done. About halfway through the period, she asked a first group of students to come forward to present their method. They used a third rectangle and moved into it a "package of gum" from Tom's rectangle and a package from Freddy's rectangle, explaining that at the end of the first day Tom had 170 pieces of gum and Freddy had 115. They kept doing this until the younger brother had more pieces of gum left. Teacher 2 then summarized their approach: "You took one circle from each boy, counting down by tens for Tom and by fives for his brother until his brother had more gum. This is good, but it could take a long time when the numbers get bigger. Did anyone find an easier way than this?" (Smith, 2004, p. 101).

Another group came forward and drew a table of values on the board with three columns labeled: Day, Tom, and Freddy. The values they entered into this table showed that on the thirteenth day, the younger brother Freddy had more gum. However, Teacher 2 did not stop there. She continued with the following:

> *Now I wonder if any of you thought of a way to show how many pieces of gum each boy had every day. Many of you may not have thought of this way that we will do it, but that is okay, we will try it anyway. I would like you to add some columns to Group 2's table like this (headers: Day, Equation, Tom, Equation, Freddy) and think of an equation Group 2 might have used to find out how many pieces of gum each boy had. What would Day 1 look like?* (Smith, 2004, p. 102)

When one student from Group 2 responded that they took 10 away from 180 for Tom and 5 away from 120 for Freddy, the teacher filled this information on the first line of the table (180 – 10 = 170; 120 – 5 = 115) and asked the students to continue working on the task of completing the table. This table-of-values approach with its explicit documentation of the operations that produce those values is a significant and noteworthy aspect of Teacher 2's algebra teaching practice on word problems. When some students appeared to be confused, she asked them to stop and look at the two numbers for a given day and to decide what computation they needed to do to get each number. If they found one way to do this, she asked them to think about whether there might be easier ways to do it. As the students continued working, the teacher asked two students to put their work on the board (see fig. 2).

Day	Student 1	Student 2	Tom	Student 1	Student 2	Freddy
1	180 – 10 = 170	180 – 10 = 170	170	120 – 5 = 115	120 – 5 = 115	115
2	170 – 10 = 160	180 – 20 = 160	160	115 – 5 = 110	120 – 10 = 110	110
3	160 – 10 = 150	180 – 30 = 150	150	110 – 5 = 105	120 – 15 = 105	105
4	150 – 10 = 140	180 – 40 = 140	140	105 – 5 = 100	120 – 20 = 100	100
5	140 – 10 = 130	180 – 50 = 130	130	100 – 5 = 95	120 – 25 = 95	95
6	130 – 10 = 120	180 – 0 = 120	120	95 – 5 = 90	120 – 30 = 90	90
7	120 – 10 = 110	180 – 70 = 110	110	90 – 5 = 85	120 – 35 = 85	85
8	110 – 10 = 100	180 – 80 = 100	100	85 – 5 = 80	120 – 40 = 80	80
9	100 – 10 = 90	180 – 90 = 90	90	80 – 5 = 75	120 – 45 = 75	75
10	90 – 10 = 80	180 – 100 = 80	80	75 – 5 = 70	120 – 50 = 70	70
11	80 – 10 = 70	180 – 110 = 70	70	70 – 5 = 65	120 – 55 = 65	65
12	70 – 10 = 60	180 – 120 = 60	60	65 – 5 = 60	120 – 60 = 60	60
13	60 – 10 = 50	180 – 130 = 50	50	60 – 5 = 55	120 – 65 = 55	55

Fig. 2. The arithmetic equations produced by two students to yield the values in the Tom and Freddy columns (adapted from Smith, 2004)

The students of the class were then asked which of the equation-types (that of Student 1 or that of Student 2) would be more helpful if the number of Days got really large. They decided that Student 2's equation was the better of the two because it was more generalizable. They remarked: "All you need to know is how many days so you can multiply it by how many pieces of gum are in each package, ten or five" (Smith, 2004, p. 102). Because they then ran out of class time, the teacher concluded the lesson by asking the students to think about a more general way of writing the equation that would give them the number of pieces of gum each boy had on whatever day. One general formulation that they could possibly have generated is: (starting number of pieces) – (number of days) × (number of pieces chewed per day) = (number of pieces remaining).

It is noted that Teacher 2 specifically probed students to give more detailed and connected explanations. She also helped them to construct another solution method. In Teacher 2's class, solution methods were analyzed and compared—something that did not happen in Teacher 1's class because only one solution method was presented. Thus, in Teacher 2's class, students were able to develop mathematical connections across solution methods. Last but not least, this research-based example has shown that, in algebra problem-solving situations, it is not merely getting the answer that counts; even more important, these situations are about relationships among solving methods and finding ways to represent these relationships and methods with equations that are as generalizable as possible. Related research that bears upon these issues can also be found in Boaler and Humphreys (2005).

Reasoning, Conjecturing, and Proving in Algebra

Reasoning, conjecturing, and proving are at the heart of all mathematical activity. However, too few algebra classes are the sites of such activity. Classrooms where conjectures are generated and then tested by finding evidence that goes beyond numerical cases and that is based on mathematical relations and properties are rare indeed. Research shows that students need to be taught how to do this and that such activity needs to be valued in the algebra classroom (Healy & Hoyles, 2000). However, research also indicates that teaching practice that fosters such activity needs to be supported by tasks where students are encouraged to reflect, conjecture, explain, and justify. Engaging in these processes allows students to go deeper into the mathematical relations that underpin algebraic objects.

The following research-based example involved tenth-grade students with tasks that focused on explaining, predicting, comparing, conjecturing, and proving (Kieran & Drijvers, 2006). The core of the task set was factoring $x^n - 1$, for integral values of n. The first part of the activity (or task set) was aimed at developing awareness of the $(x - 1)$ factor (see fig. 3). Notice the content of the task questions: anticipating/predicting a result (Q2 and Q6); comparing expressions for similarities and differences (Q4); and explaining the reason for a particular kind of result (Q5 and Q7).

The next part of the task set required confronting and reconciling students' paper-and-pencil factorizations of $x^n - 1$, for integral values of x from 2 to 13, with the factorizations produced by the digital tool at their disposal. Following this was the conjecturing task (see fig. 4), which required explaining the reasons for the produced conjectures. The last part of the task set involved trying to prove one of the conjectures that had just been generated (see fig. 5).

1. Perform the indicated operations: $(x - 1)(x + 1)$; $(x - 1)(x^2 + x + 1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product: $(x - 1)(x^3 + x^2 + x + 1)$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ? $(x - 1)(x + 1)$; $(x - 1)(x^2 + x + 1)$;, and $(x - 1)(x^3 + x^2 + x + 1)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. On the basis of the expressions we have found so far, predict a factorization of the expression $x^5 - 1$.
7. Explain why the product $(x -1)(x^{15} + x^{14} + x^{13} + \ldots + x^2 + x + 1)$ gives the result $x^{16} - 1$.

Fig. 3. Some of the initial tasks of the activity involving predicting, comparing, and explaining

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?
ii) contain more than two factors?

iii) include $(x + 1)$ as a factor?
Please explain.

Fig. 4. Task involving conjecturing and explaining

Prove that $(x + 1)$ is always a factor of $x^n - 1$ for even values of n.

Fig. 5. The proving task

The students struggled for about fifteen minutes on the proving part of the task set, having had little prior experience with such tasks. During this time, the teacher circulated so as to see what kinds of proofs the small groups of students were generating. One of his pivotal comments while circulating around the class was:

> *Teacher:* "You need more than just examples. You need to think about where the $(x + 1)$ comes from."

When the teacher sensed that most of the students had arrived at some kind of "proof," he opened the floor for whole class discussion, during which time students would share their approaches. He invited selected students, one at a time, to come to the board, to write down their proof, and to explain it to the rest of the class. Student 1 offered the following:

> *Student 1:* My theory is that whenever $x^n - 1$ has an even value for n, if it's greater or equal to 2, that, one of the factors of that would be $x^2 - 1$, and since $x^2 - 1$ is always a factor of one of those, a factor of $x^2 - 1$ is $(x + 1)$, so then $(x + 1)$ is always a factor.

The teacher then asked the class: "Is everyone willing to accept his explanation?" One student volunteered what he considered to be a counterexample, $x^{12} - 1$, maintaining that this could be factored down to include a sum and difference of cubes, thereby yielding $(x + 1)$, but without passing through the $x^2 - 1$ factor. However, other students argued that $x^{12} - 1$ could produce $x^2 - 1$ if it were factored differently. Then the teacher stepped in with a question of his own: "Just out of interest, what would happen if this was $x^{14} - 1$? Where does that leave your proof?" According to Student 1's proof, this binomial with its even exponent would lead to $(x^7 + 1)(x^7 - 1)$, but not to $x^2 - 1$. The teacher then remarked that this "proof" had a gap in it.

The next student who was asked to come forward offered a generic proof that had been constructed by her and her partner and which involved factoring by grouping (note that a generic proof involves the use of a generic example; although a particular case might seem to be the focus, it is not used as a particular case, but rather as an example of a more general class of objects):

> *Student 2:* When n is an even number
>
> *Teacher:* Write it on the board, show it on the board [he says to Student 2].
>
> *Student 2:* [She writes "$x^8 - 1$" and below it: $(x - 1)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$—the general rule that had emerged earlier from their work within the task set].
>
> *Teacher:* OK, listen, 'cause this is interesting [addressed to the rest of the class]; it's a completely different way of looking at it, to what most of you guys did. OK, so explain it [to Student 2].
>
> *Student 2:* When n is an even number [she points to the 8 in the $x^8 - 1$ that she has written], the number of terms in this bracket is even, which means they can be grouped and a factor is always $(x + 1)$.
>
> *Teacher:* Can you show that?
>
> *Student 2:* [She groups the second factor as follows: $x^6(x + 1) + x^4(x + 1) + x^2(x + 1) + 1(x + 1)$, which the class could now see would yield $(x + 1)(x^6 + x^4 + x^2 + 1)$ with the required $(x + 1)$ factor].

Student 2 argued, as she presented her proof at the board using $x^8 - 1$ as an example, that it would work for any even n.

Notice how the teacher, while circulating, had encouraged the students to move toward a proof that would be structurally based rather than merely numerical. Notice too how the presentation of the students' work proceeded from an incomplete approach to a more complete and acceptable strategy. The generic proof of Student 2 was one that was not only a valid proof but also one that explained (in the sense of Hanna, 1995)—one that the teacher had sensed would be accessible to the students with their limited experience in proving, which indeed it was. Research suggests that generic proofs merit much greater attention in algebra teaching practice than they currently receive.

Notice too that the teacher did not hesitate to point out where the gap was in Student 1's proof by offering a counterexample ($x^{14} - 1$) to that approach. Clearly, teacher intervention is often needed, at appropriate moments, to help students understand the weaknesses of their proving approaches so that they might evolve in this kind of mathematical activity. In fact, one student in the class was so intrigued by the $x^{14} - 1$ counterexample (with its factor of $x^7 + 1$) that he went on to generate a novel proof that would work for such examples—a proof involving the factoring of $x^n + 1$ for odd values of n (see Kieran & Guzmán, 2010, for further details). Lastly, note that the proving part of the task set had been preceded by a considerable amount of work on developing the related reasoning processes of predicting, comparing, explaining, and conjecturing. Without such process-related activity and without appropriate teacher intervention, students risk being inadequately prepared to participate in meaningful discussions about mathematical proofs.

Further research involving conjecturing and proving in algebra has been carried out by, for example, Martinez and Castro Superfine (2012), with tasks that they elaborated from the basic Calendar Algebra problem of Bell (1995). Other kinds of problems that have been found appropriate for conjecturing and proving activity with high school students include number-theoretic tasks such as, "Show that the addition of any three consecutive integers will always give a multiple of 3 and that the sum will always be the triple of the second number" (e.g., Arcavi, 1994; Kieran, 1997).

This research brief has reported on the ways in which algebraic reasoning can be fostered in two specific areas of school algebra. The first area, in algebra problem solving, involved a focus on the relationships among solving methods and on finding ways to represent these relationships and methods with equations that are as generalizable as possible. A key aspect of this part of the research brief was the role of teacher questioning. The second area that was reported on involved the activity of conjecturing and proving. Here, the research brief emphasized the development of students' reasoning processes of predicting, comparing, explaining, and conjecturing—supported by tasks and teacher intervention that aim at encouraging such reasoning.

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