Problems to Ponder

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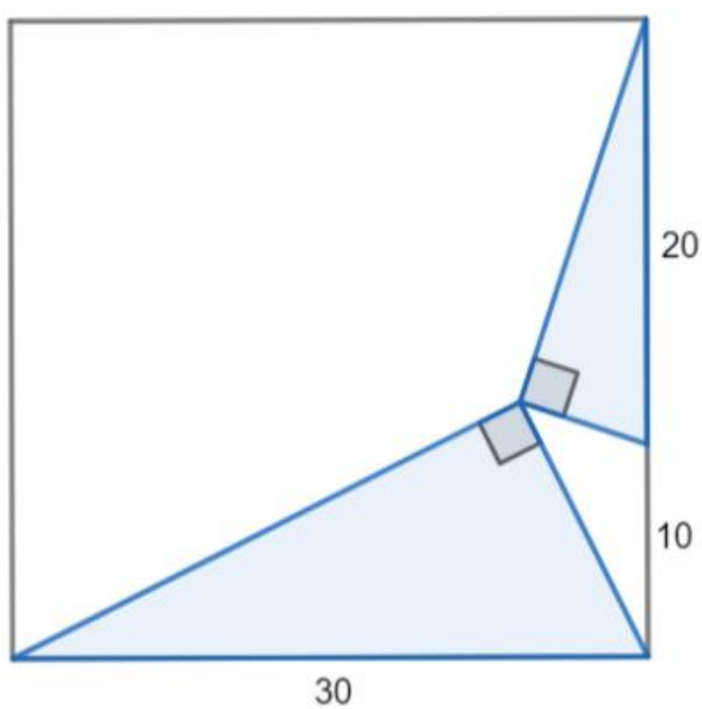
As shown below, a square of side length 30 has two right triangles positioned on its sides so that their right angles share a common vertex in the interior of the square and their hypotenuses are along the sides of the square. The hypotenuse of the larger triangle is 30. The hypotenuse of the smaller triangle is 20, and it is positioned along the right side of the square with one endpoint at the upper right vertex of the square. What fraction of the large square's area is covered by the two right triangles? Share your solution.

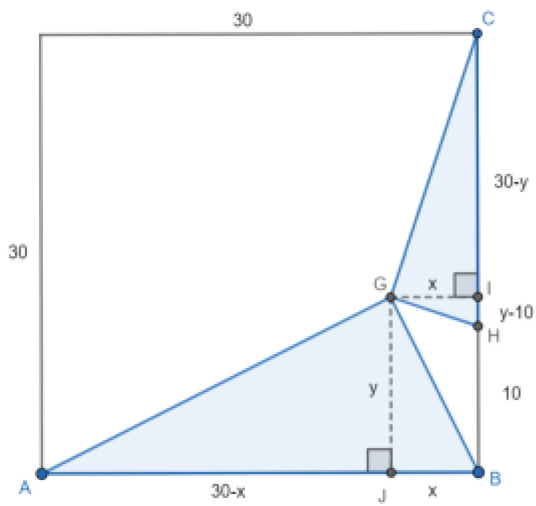


Solution:

The ratio of the sum of the areas of the two right triangles to the area of the square is 240 to 900; that is, $\frac{4}{15}$ of the large square is covered by the two triangles. There are a variety of solution methods. Below are some examples. If the hypotenuse of each triangle is considered its base, then calculating the altitude of each right triangle would be enough to calculate the triangle areas and answer the question.

Let x=the altitude of the hypotenuse-20 triangle, and y=the altitude of the hypotenuse-30 triangle. As shown below, because rectangle GIBJ has sides of length x and y,

- $AJ = 30 - x$
- $HI = y - 10$
- $CI = 30 - y$



By the geometric means theorem,

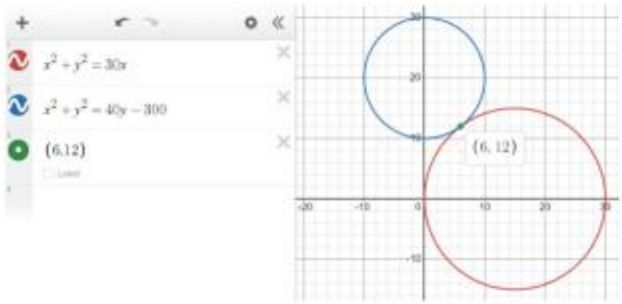
- from triangle AGB, $y^2 = x(30 - x)$, or $x^2 + y^2 = 30x$.
- from triangle CGH, $x^2 = (y - 10)(30 - y)$, or $x^2 + y^2 = 40y - 300$

The geometric means equations create a system of two equations in x and y:

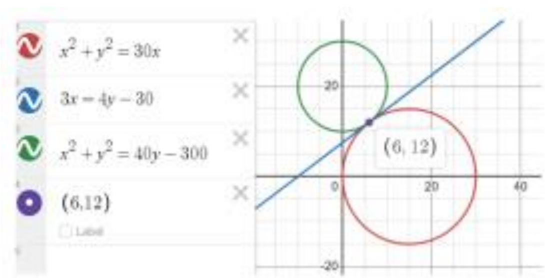
$$x^2 + y^2 = 30x$$

$$x^2 + y^2 = 40y - 300$$

This could be solved algebraically, but because both are circles, it may be easiest to solve via graphing.



The circles are mutually tangent at (6,12), so there is only one solution. Note that by substitution, the system gives $3x = 4y - 30$. If you graph both circles and this line, you see that this is the mutual tangent line between the circles!



Final calculations:

- $y = 12$ is the height of the base 30 triangle, so that area is $12 \times \frac{30}{2} = 180$.
- $x = 6$ is the height of the base 20 triangle, so that area is $6 \times \frac{20}{2} = 60$.
- The overall square has area $30 \times 30 = 900$.
- The fraction of the square that is shaded is $\frac{(180+60)}{900} = \frac{4}{15}$.