Mathematics Education in the United States 2020

A Capsule Summary Fact Book

Written for

The Fourteenth International Congress on Mathematical Education (ICME-14)

by

Sharon M. McCrone

Karen J. Graham

Anna E. Bargagliotti

Chris Rasmussen

John A. Dossey

under the Auspices of the

National Council of Teachers of Mathematics

and the

United States National Commission on Mathematics Instruction

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Mathematics Education in the United States 2020

A Capsule Summary Fact Book

Sharon M. McCrone

Karen J. Graham

Anna E. Bargagliotti

Chris Rasmussen

John A. Dossey

The preparation, production, and dissemination of this document were funded by National Science Foundation under grant number DRK-12; DRL 908084 *Fourteenth International Congress on Mathematical Education (ICME-14) Travel Grant* to the National Council of Teachers of Mathematics, Gail Burrill, Principal Investigator. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation, the National Council of Teachers of Mathematics, or the U.S. National Commission on Mathematics Instruction.

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The National Council of Teachers of Mathematics advocates for high-quality mathematics teaching and learning for each and every student.

Table of Contents

Preface . . . vii

Chapter 1: General Information on the U.S. Education System . . . **1**

- Overall Organization of Education in the United States . . . 1
  - Movement of U.S. Students through PK–12 Education . . . 1
  - Movement of U.S. Students through Postsecondary Education . . . 4
  - The U.S. PK–12 Education Enterprise . . . 4
  - The U.S. Postsecondary Education Enterprise . . . 6
    - Admission to Postsecondary Institutions . . . 6
- Governing Entities of U.S. Education Policy . . . 7
- Curriculum Standards and Curriculum Guidance . . . 8
- National Council of Teachers of Mathematics: A Brief Overview . . . 9
- Mathematical Association of America: A Brief Overview . . . 10
- American Statistical Association: A Brief Overview . . . 11

Chapter 2: The Intended Curriculum . . . **13**

- Evolution of PK–12 Standards: NCTM's *Principles and Standards for School Mathematics* to the Common Core State Standards for Mathematics . . . 13
- Creating Equitable Structures and Instructional Practices . . . 16
- States' Secondary School Graduation Requirements . . . 17
- Postsecondary Curricular Reports and Recommendations . . . 18

Chapter 3: The Implemented Curriculum . . . **21**

- The Impact of Standards and Educational Policy on the Implemented Curriculum . . . 21
- Textbooks and the Implemented Curriculum . . . 22
  - Mathematics Materials in Elementary Schools (K–Grade 5) . . . 23
  - Mathematics Materials in Middle or Junior High Schools (Grades 6–8) . . . 24
  - Mathematics Materials in Senior High Schools (Grades 9–12) . . . 24
- Coursework and Pedagogical Patterns in K–12 Mathematics . . . 25
- Mathematics Study at the Postsecondary Level . . . 28
- Changes in the Number of Baccalaureate Degrees in Mathematics . . . 31

Chapter 4: The Attained Curriculum . . . **32**

- The National Assessment of Educational Progress . . . 32
- Main and State NAEP: Changes in NAEP Frameworks for Grades 4 and 8 Mathematics . . . 33
- Executive Summary for NAEP 2019 Grade 4 and Grade 8 Results . . . 33
  - NAEP 2019 Grade 4 Results . . . 34
  - NAEP 2019 Grade 8 Results . . . 36
    - Achievement Levels . . . 36

Contents—*Continued*

NAEP 2019 Trial Urban District Assessment (TUDA) 38
NAEP 2012 Long-Term Trend Assessment Study 41
2019 State NAEP Results 43
Consortia for the Assessment of Common Core–Related Achievement 45
Partnership for Assessment of Readiness for College and Careers (PARCC) 45
Smarter Balanced Assessment Consortium (SBAC) 45
Other State Testing Programs 45
SAT and ACT 46
SAT Subject Tests 47
Advanced Placement Examinations 49
International Tests and Comparisons 50

Chapter 5: State Standards, Policy Documents, Curricular Trends, and New Directions 51
Status of State Standards 51
Catalyzing Change in High School Mathematics 52
Statistics Education Efforts of the American Statistical Association 55
The GAIMME Report 56
Emergence of Alternative High School Course Pathways 57
Emergence of New Lower Division Pathways 58

Chapter 6: Postsecondary Mathematics Education Professional Development 60
Teacher Preparation Programs at Postsecondary Institutions 60
Guides and Recommendations for Teacher Preparation 62
Teacher Education Requirements in Mathematics and Other Subjects 63
Teacher Preparation Trends and Programs in the U.S. 64
Professional Development for College Faculty of Mathematics, Mathematics Education, and Statistics 64
Professional Development Opportunities for Postsecondary Instructors 66
Current Faculty Members 66
New Faculty Members 67
Future Faculty Members 68

Chapter 7: Postsecondary Calculus 70
The MAA Characteristics of Successful Programs in College Calculus Project 71
Calculus 1 Student Career Goals by Demographics 71
High School Preparation for College Calculus 73
Instructional Experience 74
Persistence in the Calculus Sequence 76
Case Studies of Relatively More Successful Calculus 1 Programs 77
The Factors Influencing College Success in Mathematics Project 77
The MAA Progress through Calculus Project 78

Contents—*Continued*

Chapter 8: Programs for Special Populations of Students 80
Advanced Placement Programs 80
Advanced Placement Programs in Calculus 81
Advanced Placement Program in Statistics 82
Advanced Placement Program in Computer Science 83
Special Schools and Programs for Students in Mathematics 83
Programs for K–12 Students 84
Programs for Undergraduate and Graduate Students 85
Mathematics Competitions for K–12 Students 85
MATHCOUNTS 85
American Mathematics Competitions (AMC) 85
The American Statistical Association and National Council of Teachers of Mathematics Poster Competition and Project Competition 86
The Math League 86
The American Regions Mathematics League (ARML) 86
High School Mathematical Contest in Modeling (HiMCM) 87
The MathWorks Math Modeling Challenge (M3C) 87
Modeling the Future Challenge 87
Mathematics Competitions for Undergraduate Students 87
Student Mathematics League (SML) 87
Mathematical Contest in Modeling (MCM) and the Interdisciplinary Contest in Modeling (ICM) 88
Undergraduate Statistics Project Competition (USPROC; Consortium for the Advancement of Undergraduate Statistics [CAUSE]) 88
William Lowell Putnam Mathematical Competition 88

Chapter 9: Professional Organizations Supporting the Mathematical Sciences 89
Closed-Membership Organizations 89
Conference Board of the Mathematical Sciences 89
United States National Commission on Mathematics Instruction (USNC/MI) 89
Open-Membership Organizations 90
Kindergarten–Grade 12 90
Postsecondary 90
Special Groups in Mathematics Education 90

References 92

Preface

The population of the United States is approximately 327,200,000. Approximately 23 percent of these individuals are formally enrolled in schools and colleges. In the entire U.S. population, about 89 percent of individuals that were 25 years or older completed high school or its equivalent, and about 33 percent had at least a bachelor's degree (Snyder, Brey, and Dillow 2018).

No single government agency controls public education in grades PK–12 in the United States. Rather, authority for most educational decisions lies with education agencies in the 50 individual states, which in turn share decision making with the individual school districts within their borders. In the 2015–16 academic year, U.S. public schools educated approximately 50.4 million students, private elementary and secondary schools encompassed another 5.7 million students, and homeschooling accounted for another approximately 2 million students (Snyder, Brey, and Dillow 2018).

Similarly, both public and private institutions exist at the college and university level, with authority for state institutions residing at a mixture of state and local levels for public institutions and at the institutional level for most private institutions. A total of 4,360 accredited degree-granting institutions offering associate level (two-year) or higher degrees were in operation in 2016 in the U.S. Of these, 1,623 were public institutions, 1,682 were private nonprofit institutions, and 1,055 were private for-profit institutions. Four-year institutions totaled 2,832 and two-year institutions totaled 1,528 (Snyder, Brey, and Dillow 2018).

Determining what is happening in such a large and complex country as the United States is quite difficult, even for those in the United States and others who are familiar with U.S. education. Many attendees of conferences of the International Congress on Mathematical Education (ICME) are unfamiliar with education in the United States. Consequently, in 1999, the U.S. National Commission on Mathematics Instruction recommended that the National Council of Teachers of Mathematics (NCTM) request funds from the National Science Foundation (NSF) to consolidate available data about mathematics education in the United States for a document to be distributed at the Ninth International Congress on Mathematical Education (ICME-9), held in 2000, to provide mathematics educators throughout the world with information about this complex system. This process was repeated for subsequent ICMEs, held in 2004, 2008, 2012, and 2016. The present publication now extends the series with information available as of the end of 2019.

This report begins with some general information about education in the United States. It then describes the three kinds of curricula identified in the Second International Mathematics Study—intended, implemented, and attained (McKnight et al. 1987). This report consists of nine chapters. A brief survey of their focus and content may help readers orient themselves and navigate through them.

Chapter 1 presents a general overview of public and private educational opportunities in the United States, including the movement of U.S. students through the PK–12 school years and onto the admission to postsecondary education. The chapter concludes with three reflections in honor of NCTM's and the Mathematics Association of America's (MAA) 100th anniversaries celebrated in 2020.

Chapter 2 gives an overview of the history and current status of the intended curriculum for school mathematics—its origins and goals. This portion of the report begins with a brief overview of development in the standards movement in U.S. PK–12 education from the publication of NCTM's *Principles and Standards* in 2000 to the 2010 release of the *Common Core State Standards.* This is followed by a discussion of the documents and movements that have influenced U.S. mathematics education at both the PK–12 and postsecondary levels throughout the past decade.

Chapter 3 examines what is known about the actual implemented PK–12 curriculum, instructional approaches, content emphases, and materials in use. This chapter also considers evolving enrollments in mathematics and statistics coursework at the postsecondary level.

Growing naturally out of chapter 3, chapter 4 addresses the attained curriculum. It examines the extant outcomes from national and international assessments of student achievements in mathematics. The national assessments survey state- and national-level performances on the National Assessment of Educational Progress (NAEP). In addition, it discusses U.S.

student achievement outcomes in international comparative studies—Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA). Such results give a glimpse of how the performance of U.S. students compares with that of their international peers and provide a basis for asking questions about the impact of various factors in education and social environments that may offer explanations for differences in international student achievement in mathematics. Chapter 4 ends with an examination of student performance on college entrance examinations. The remaining chapters of the book examine the following topics:

- Chapter 5: The status of state standards nearly ten years after the release of the *Common Core State Standards for Mathematics* (NGA Center and CCSSO 2010a, 2010b) provides an overview of recent recommendations from *Catalyzing Change in High School Mathematics* (NCTM 2018) and highlights the emergence of mathematics pathways at the high school and college level.
- Chapter 6: This chapter discusses mathematics teacher education and professional development programs and standards; new resources for teachers and professional development that focus on data and statistics and weaving them into school curricula; and professional development of faculty and graduate students at the postsecondary level.
- Chapter 7: The chapter covers the current state of calculus at the postsecondary level.
- Chapter 8: Special programs for accelerated students at the K–12 school and postsecondary levels are examined as well as national and international competitions in mathematics for students.
- Chapter 9: Professional organizations that support the mathematical sciences are presented.

One message that comes through repeatedly in this report and its descriptions is that the variety of education programs available in the United States is very great, and thus the possibility of characterizing them adequately in a brief document like this one is very small. Another message is that all levels of the U.S. educational system exhibit great flux, and even though we have attempted to provide the latest information available, we realize that the content that we present in this report will quickly become dated. By listing our sources, we hope to enable interested readers to obtain updated information.

Finally, we would like to acknowledge the efforts of Gail Burrill, who wrote the proposal for the grant under which the funding for this publication was obtained and her contributions to the reflection on NCTM at its centennial anniversary in chapter 1 as well as other areas of this publication; David Bressoud for his reflection on MAA at 100 years in chapter 1 and his review of this document; and Donna LaLonde, Christine Franklin, and Rebecca Nichols for their reflection on the education activities of the American Statistical Association (ASA).

We would also like to thank the following individuals for their insightful, constructive, and editorially valuable advice: Robert Q. Berry III, Ed Dickey, John W. Staley, April Strom, Daniel J. Teague, and Trena Wilkerson as well as the fine work of the NCTM editorial staff. We offer a special thank you to John Dossey, our co-author, whose document structure and continued input were invaluable for the creation of this book.

We have tried to be as accurate as possible and apologize for any errors.

Chapter 1: General Information on the U.S. Education System

Overall Organization of Education in the United States

In this chapter a general overview of public and private educational opportunities is provided as well as a description of federal education law in the United States in 2020. Subsequent chapters give a more detailed examination of mathematics and statistics education in the United States at this point in time.

Figure 1.1 presents a graphic overview of the structure of education in the United States. The system consists of four broadly defined levels:

1. Elementary school (PK–grade 5 or PK–grade 6, corresponding to ages 4 to 10 or 11)
2. Middle school or junior high school (grades 6–8 or 7–8, ages 11 to 13 or 12 to 13, respectively)
3. Senior high school (grades 9–12, ages 14 to 17)
4. Postsecondary, or tertiary, education (grades 13 and above, ages 18 and older)

The ending and beginning points of each of the levels may vary by state and local school system regulations and preferences; however, postsecondary education requires graduation from twelfth grade or an equivalent degree (Snyder, Brey, and Dillow 2018). Note that in current vernacular, nursery schools are commonly called preschools or schools for early-childhood education. Similarly, junior colleges are more frequently referred to as two-year colleges. Also, throughout this document the term *postsecondary institution* is used when speaking of tertiary education institutions in the United States.

The numbered scale up the left-hand margin of figure 1.1 indicates the median ages for students enrolled at the various levels of K–12 education. The numbered scale on the right-hand side indicates the corresponding levels from prekindergarten through grade 12 education and the years normally taken for a full-time student to progress through the varied levels of tertiary education. One can loosely interpret the width of the horizontal bars as representing the percentage of students enrolled in the various forms of education at the PK–12 levels.

Movement of U.S. Students through PK–12 Education

PK–12 students are legally required to start and maintain enrollment in formal education by state-mandated ages. The minimum compulsory school-starting ages range from 5 to 8 years: age 5 (10 states and the District of Columbia), age 6 (25 states), age 7 (13 states), and age 8 (2 states). Standards for the length of compulsory education also vary by state with minimum allowed school-leaving ages of 16 to 18: age 16 (15 states), age 17 (10 states), and age 18 (25 states and the District of Columbia) (see Snyder, Brey, and Dillow 2018, Table 234.10). In summary, 7 states require at least 9 years of formal education, 11 other states require 10 years, 11 more states require 11 years, 14 states require 12 years, and 7 states require a total of 13 years of formal education. While there are compulsory ages set for each state, state standards allow for variances in their regulations for school-starting and school-leaving ages for students who are employed, have a physical or mental condition that makes attendance infeasible, have passed eighth grade successfully, or have the permission of their parents, district court, or school board (Education Commission of the States [ECS] 2018). The variance in these regulations across the 50 states is mirrored by the diversity in laws with respect to when schooling should begin and what constitutes the minimum amount of schooling acceptable for students in a state. Another example of diversity in education across the states manifests itself in the variability of the National Assessment of Educational Progress (NAEP) achievement results reported in table 4.3 in chapter 4. These two examples reflect differences in state standards, state expectations for students, and the structure of state funding.

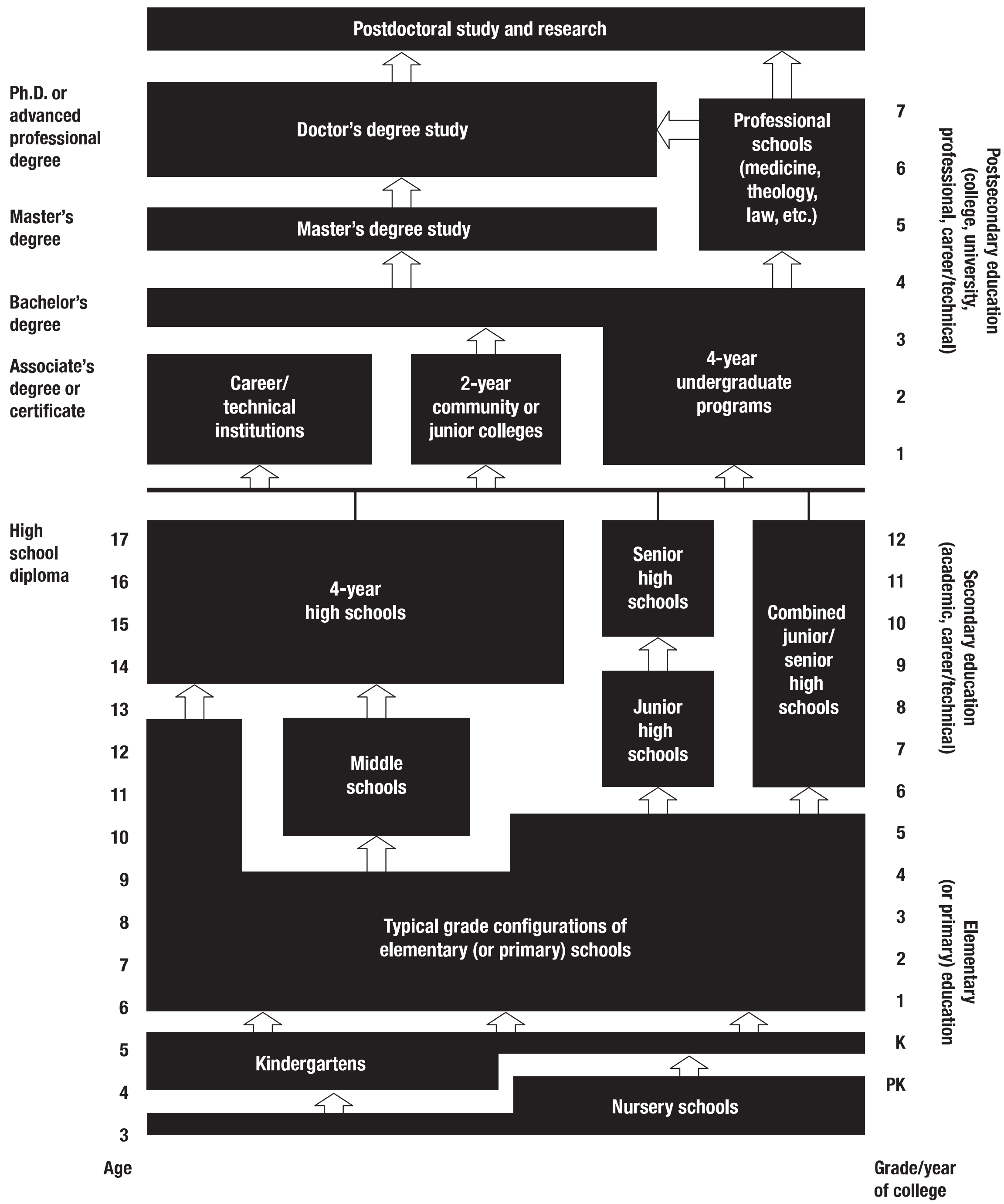


Fig. 1.1 The structure of education in the United States (Snyder, Brey, and Dillow 2018)

Not all students in the United States complete secondary education prior to leaving formal education. All states require compulsory education through the age of 16, while a few states require attendance until the ages of 17 or 18. Even so, students are not required to attend public schools; they may attend private schools or religion-based schools or may be homeschooled by their parents.

The percentage of students who complete a public school–education has been quantified in different ways in past years (Stetser and Stillwell 2014). For example, prior to 2010, the *average freshman graduation rate* (AFGR), which had been reported annually since 1990 on the basis of a congressionally mandated report, *The Condition of Education,* was used as a measure of graduation rates. The AFGR used the average number of eighth grade, ninth grade, and tenth grade students over a three-year period (sum divided by 3) to determine the average number of ninth graders in a reporting group and compared that average to the number of students graduating four years later. In the inaugural reporting year (1990), the AFGR was 74 percent. This statistic fluctuated a bit between 1994 and 2010, and then increased to 80 percent in 2010–11 before increasing again to the 82 percent for those graduating in 2012–13 (McFarland et al. 2019).

Another measure used is the *adjusted cohort graduation rate* (ACGR). This measure provides an estimate of the proportion of public high school students who graduate from high school four years after having entered ninth grade. The ACGR differs from the AFGR in that the ACGR identifies a cohort of students during the ninth-grade year and then adds in students who transfer into the cohort and subtracts out students who leave the school or district from the cohort. Coupled with the Common Core of Data compiled by the National Center of Educational Statistics (NCES), the ACGR estimate indicates that of the students who entered high school as ninth graders in the academic year 2013–14, 85 percent graduated during the 2016–17 school year. This was the highest completion rate in a four-year period since 2010. As the U.S. is a multiracial and multiethnic nation, it is often informative to consider trends in four-year graduation rates across racial and ethnic groups, recognizing that this does not fully explain differences in graduation rates. Among the 2016–17 graduates, the ACGR rates for various racial ethnic groups were as follows: Asian/Pacific Islanders (91 percent), White (89 percent), Hispanic (80 percent), Black (78 percent) and American Indian/Alaskan Native (72 percent) (Snyder, Brey, and Dillow 2018).

Students who do not complete high school with their class in four years may continue their enrollment until receiving their diplomas later or may opt to discontinue their education. The many students who discontinue their education may achieve the equivalent of a high school diploma through other means. The *status completion rate* (SCR), another measure used to track completion, provides the percentage of people by age ranges who are not attending a secondary school but have earned a high school diploma or have completed a high school equivalency program. In the 18- to 24-year-old age group, the SCR in 2016 was 93 percent compared with 90 percent in 2008, 86 percent in 2000, and 84 percent in 1980. Gender comparisons for 2016 showed that 94.3 percent of females and 91.6 percent of males had achieved a high school diploma or its equivalent by age 24. Similar differences exist among racial or ethnic subgroups: 94.5 percent for White non-Hispanics students, 92.2 percent for Black non-Hispanic students, and 89.1 percent for Hispanic students. This represents an upward trend in all ethnic groups when compared to SCR figures from previous years, with the largest increase seen in the Hispanic population, which showed an SCR below 70 percent in 2000 (McFarland, Cui, Rathbun and Holmes 2018; U.S. Department of Commerce 2020).

Movement of U.S. Students through Postsecondary Education

Students who graduate from high school may enter the workforce, attend a non-university postsecondary institution focusing on technical or vocational education, attend a two-year college, or attend a four-year college or university. At this level, the bars in figure 1.1 represent the flow of students still in the educational stream. Two-year colleges usually offer diverse selections of courses and programs, including those that overlap with the first two years of the curriculum at a four-year college, along with a number of courses that overlap with those found in the technical colleges and high schools. Many two-year colleges also have vocational streams of students who earn certification for a particular career, sometimes with and sometimes without a two-year degree.

In two-year colleges, an Associate of Arts (AA), an Associate of Science (AS), or an Associate of Applied Science (AAS) degree can usually be earned through the equivalent of two years of full-time study. One-year certificate programs are also offered in various technical fields. In addition, a number of vocational or trade schools offer programs in which students can focus on the knowledge and skills needed to perform a particular job. Vocational schools may be integrated with public schools as part of programs that facilitate the transition from school to work. In other instances, these schools are private schools, nonprofit or proprietary, operated outside the public school–system. The domains of these schools range from apprenticeship programs for trades to culinary arts.

U.S. four-year colleges and universities offer Bachelor of Science (BS) and Bachelor of Arts (BA) degrees that can typically be completed in four years of full-time study. In addition, many universities offer graduate programs leading to master's (MS, MA, or MEd) degrees and doctoral (PhD and EdD) degrees. Programs leading to professional degrees (law, medicine and health sciences, business, etc.) exist both in universities and at institutions that offer no other degree programs. The time needed to complete postbaccalaureate degrees varies with the field and institution.

The U.S. PK–12 Education Enterprise

Public schools in the U.S. are those schools that are funded and governed by local and state government agencies, with some limited federal funding. In the 2015–16 academic year, 98,456 public schools or agencies were in operation in the 50 states and the District of Columbia. This represented a slight decline from the previous five years, likely due to a trend in consolidating small schools, but their number has remained relatively stable if viewed over the past several decades. These approximately 98 thousand public schools provide a variety of educational services to an estimated 50 million PK–12 students enrolled in them. Most of the schools (89,644) focus on delivering the broad standard curriculum to their students. Another 1,419 provide targeted vocational or technical education, while 2,011 offer special education services. Another 5,382 offer some form of alternative education. Included in this number were 2,964 independent charter agencies (not including those already counted because they are imbedded directly in the curriculum of a public school–program). These operational schools were part of one of 18,328 operational public school–districts in the United States, ranging from 1,232 districts in Texas to 19 districts in Nevada and excluding Hawaii, which is a single-district state. In 2015–16, these districts employed the equivalent of 3.2 million full-time teachers (Glander 2017).

In addition to the public schools, other types of public, private, and home schools operate at the PK–12 levels in the United States. Charter schools are public schools that are funded through public and state support but are allowed to operate with freedom from many of the regulations that apply to traditional public schools. Magnet schools are public schools whose curricula address the standard requirements and regulations but provide targeted and advanced instruction in such areas as mathematics, science, or the arts. The provisions governing magnet schools also usually include a requirement that specific percentages of students come from particular cultural, ethnic, or racial groups in a school's or district's student body. Public and magnet schools' administrators are commonly responsible to a governing board elected by the public of the geographical area that the school serves. Charter schools' administrators are typically responsible to a board elected by the parents of the students, and they also are accountable to varied local and state regulations, depending on the laws of the state in which they are located.

In addition to these public- and state-funded schools, nonreligious and religious private schools are common in the United States. Such schools are independent, nongovernmental, or nonstate schools. Religious private schools consist of those that are supported by a particular religious group or denomination. These schools add instruction in religion to the curriculum and may modify instruction in the regular content to highlight particular aspects of the religious group's or denomination's history or beliefs. All private schools are funded by tuition, religious denominations, community foundations, or other donors. Annual tuition for private school ranges from nothing at schools whose tuition is covered by an endowment or a special program to nearly $50,000 a year at some of the most exclusive college preparatory schools in the U.S. In the 2015–16 school year, approximately 34,576 private schools were in operation, adding to the numbers of public schools stated above (Snyder, Brey, and Dillow 2018). Private schools' administration is typically a council or board, often established by the parents of the students attending.

A final form of schooling is homeschooling, in which parents and caregivers assume direct responsibility for the education of their children. States' laws regarding notification of homeschooling, curriculum requirements, assessment requirements, and parent educational minimums vary greatly across the U.S., but all 50 states have allowed some form of homeschooling since 1993 (Home School Legal Defense Association (HSLDA) 2020; Ray 2019).

The academic year 2015–16 is the latest year for which complete data on student enrollment has been reported in the entire educational enterprise in the United States. Federal education data in the U.S. usually lag two to three years behind the date of their release making it necessary in reporting to speak of past data and speak about projections ahead. Table 1.1 shows the enrollment projections through 2027 of the total number of students in K–12 public schools and postsecondary education.

Table 1.1

School and Postsecondary Enrollments and Projections over Time (in millions) (Snyder, Brey, and Dillow 2018, table 105.20)

	1990	**2000**	**2010**	**2015**	**2020**	**2027**
K–12 Public	41.2	47.2	49.4	50.4	50.9*	52*
K–12 Private	5.6	6.1	5.3	5.7	5.9*	6.1*
Postsecondary	13.8	15.3	21	19	20*	20.4*

*Indicates this is projected enrollment.

Total public and private elementary and secondary school enrollment reached 56 million in 2015, representing a 20 percent increase since 1990. In 2015–16, U.S.

K–12 public schools accounted for more than 50,000,000 students. Private elementary and secondary schools added another 5,700,000 students. Thus, slightly more than 56,000,000 students were involved in K–12 educational programs in the United States in the 2015–16 school year. Projections for the fall of the 2020 school year suggest that the number of students involved in K–12 public and private education would be around 56,800,000 students, composed of 50,900,000 public school students and approximately 5,900,000 students in private schools. These numbers do not include estimates of the homeschooled youth in K–12, who added approximately 2,000,000 additional students for the 2014–15 school year and a projected number of more than 2,400,000 homeschooled students for the fall of 2020 (Ray 2019).

The U.S. Postsecondary Education Enterprise

Postsecondary education in the United States includes two-year, four-year, and postbaccalaureate programs and institutions. A total of 4,360 accredited degree-granting institutions offering associate level (two-year) or higher degrees were in operation in 2016 in the U.S. Of these, 1,623 were public institutions; 1,682 were private nonprofit institutions; and 1,055 were private for-profit institutions. There were totals of 2,832 four-year institutions and 1,528 two-year institutions (Snyder, Brey, and Dillow 2018).

As seen in table 1.1, in 2015, 19.9 million students were enrolled in postsecondary education. Of these, 14.5 million were at public institutions and 5.4 million were at private institutions. The private institution enrollment is broken down further into approximately 4.1 million at nonprofit institutions and 1.3 million at for-profit institutions. The total number of students enrolled in postsecondary education in 2015 can also be further broken down by those pursuing a baccalaureate degree (undergraduate students) and those pursuing a postbaccalaureate degree. Approximately 17 million students were undergraduate students and 2.9 million were postbaccalaureate students. Of the undergraduate students, 6.5 million were at two-year institutions and 10.5 million were at four-year institutions. Table 1.1 also shows the projections of postsecondary enrollment, with a large increase in the decade from 2000 to 2010 and then remaining consistently about 20 million from then on.

Admission to Postsecondary Institutions

Graduates of public or private senior high schools may matriculate to the nation's colleges and universities, but they must apply to the individual schools to be considered for admission. Most state-supported, two-year colleges are considered "open access" because they accept any secondary school graduate from the geographic area that they serve or local residents who have earned a General Education Diploma (GED), equivalent to a high school diploma but earned through passing a series of five tests in writing, reading, social studies, mathematics, and science. Other two-year colleges and most four-year colleges require applicants for admission to have completed a specified number of courses in English, mathematics, science, social studies, and foreign language and to have a high school diploma. Admission to such colleges are based on multiple factors such as an applicant's intended field of study, secondary school course grades, percentile rank in secondary school graduating class, scores on college entrance examinations, letters of recommendation, participation in sports and other extracurricular activities, and other information supplied by a student's high school. However, one change that appears to be a growing trend is for four-year colleges and universities to waive student requirements to take and submit standardized test scores. Various sources report that as many as 1,000 four-year postsecondary institutions are now "test optional," meaning that students applying for admission to the school can choose whether or not to share entrance examination scores with the school (Fairtest 2019).

The mean costs of college undergraduate attendance, including tuition, fees, room, and food for in-state students at four-year public and private nonprofit colleges in 2019–20 were $21,950 and $49,870, respectively (College Board 2019c; Snyder, Brey, and Dillow 2018). These totals increased by approximately 22 percent over the decade beginning in 2009. The mean cost of attendance for students at public two-year colleges in 2019–2020 was $12,720, a 3 percent increase from the previous year and a little less than a 22 percent increase from 2009 (College Board 2019c).

Although many undergraduate students receive scholarships and other types of financial aid from various sources, including the college that they attend, government programs, or private foundations, the costs of attending a college and university are increasingly beyond the reach of many students and their families (College Board 2019d). The College Board estimates that full-time undergraduate students received an average of $15,210 in grant aid and federal tax benefits in 2018–19 to pay for a year of schooling at the postsecondary level. This represents a more than 50 percent increase over the 2008–09 academic year average aid and benefits.

Governing Entities of U.S. Education Policy

Because the Constitution of the United States does not claim education as a responsibility of the federal government, individual states have considerable leeway in structuring the education of their students at the PK–12 levels. State laws define the boundaries for the compulsory education of students; outline the general framework for required studies in reading, writing, mathematics, science, social science, physical education, and other subjects; define the minimum number of days of school attendance per year; and define the standards for teacher certification and professional development. State laws also provide the mechanisms by which local PK–12 schools are recognized by the state government and provide statutes for the founding and accreditation of private schools. In like manner, states have considerable leeway in developing regulations for and monitoring charter schools. These schools receive public funds but are not necessarily responsible for meeting all the regulations required of other public schools in the state or district. In addition, many states also have laws that stipulate regulations and monitoring for homeschooling (McFarland et al. 2019).

Although the educational curriculum is primarily determined at state or local levels, the U.S. Department of Education does set requirements and provides federal funding for special programs, such as school lunch programs for students in families of low social-economic status and compensatory programs for students needing special educational assistance. Beyond these assistance programs, the U.S. federal government provides overarching policies and guidance regarding the right of every student to have the opportunity to receive a public education.

The role of the federal government in education increased markedly with the establishment of the No Child Left Behind Act (NCLB), passed by Congress in 2001. NCLB authorized the U.S. Department of Education to manage a program that provided financial incentives for schools with good performance profiles and penalties for schools with poor performance records (U.S. Department of Education 2008). Updated policies have since relaxed some of the penalties for poor performance in favor of focusing on helping all students succeed. These new federal policies went into effect in 2015 with the Every

Student Succeeds Act (ESSA) and are due for revision or reauthorization in 2020 (U.S. Department of Education 2019). Highlights of the ESSA include retaining protections for disadvantaged high-needs students, continuing annual statewide assessments for measuring progress and effecting positive change where needed, supporting local initiatives, and increasing access to high-quality preschools.

As part of the public education enterprise in the U.S., 43 states and the District of Columbia maintain policies for establishing charter schools. Although charter schools are often provided with more autonomy than typical public schools with regard to state and local regulations, charter schools must meet the accountability standards set forth in their charters in order to maintain their status. Charter schools represent approximately 7 percent of public schools across the 43 states in which they exist, an increase from 2 percent of public schools in the year 2000. Similarly, the percentage of students attending charter public schools increased from 1 percent in 2000 to nearly 6 percent in 2016 (McFarland et al. 2019).

Just as charter schools and magnet schools allow families to seek alternatives to a traditional public school, state and local governments may provide additional opportunities for school choice, including using public funds to partially or completely pay for private schooling. As of the start of the 2016–17 school year, 24 states and the District of Columbia offered school choice programs for select student populations. One such option comes from school vouchers, which are state-funded scholarships that can be used toward payment for students to attend private schools. Other programs use tax credits or tax deductions to help families send students to private schools. In nearly all states that offer school choice programs, these programs exist for targeted subgroups of students, such as students from low-income households or from chronically low performing schools (as determined by specified threshold levels and state assessment measures), students with some form of disability, and students from rural communities that do not have public schools nearby. A few states open up school choice funding programs to all public school–students, such as through education savings accounts (ESAs) that allow families to set aside money in special accounts that can then be used to fund private schools, private tutors, or homeschooling materials (Cunningham 2018).

Curriculum Standards and Curriculum Guidance

Although the U.S. government provides control of public school curriculum to states and even local governments, there exist many documents developed by national professional societies of expected or suggested educational standards to guide school curriculum at all grade levels and in all school subjects (cf. NGSS Lead States 2013; National Council for the Social Studies 2010; National Council of Teachers of Mathematics 2000). Note that the U.S. government does not govern or provide oversight for these national professional societies, rather they are all member driven. Language arts (including reading and writing) and mathematics have long been the focus of the "standards movement" in the U.S. as these are expected competencies for graduation and postsecondary entrance examinations (U.S. Department of Education 2008). Assessments in language arts and mathematics are also part of the compulsory assessments administered by all states at grades 3, 8, and 11 as required by ESSA (U.S. Department of Education 2019). Subsequent chapters provide more details about the standards movement in mathematics, beginning in the 1980's, which includes the ongoing development, adoption, and modification of standards and curriculum guides.

Professional societies have long played a role in leadership, policy recommendations, and oversight for curriculum and teaching of all subject areas. Specific to mathematics teaching and learning, several professional societies have been in existence, providing guidance through the collective expertise of its membership for more than 100 years. With the 2020 advent of the 100-year anniversaries of the National Council of Teachers and Mathematics (NCTM) and the Mathematical Association of America (MAA), in particular, distinguished members of these entities as well as the American Statistical Association (ASA) have provided historical overviews of these societies and their roles in mathematics education for PK–12 and postsecondary education.

National Council of Teachers of Mathematics: A Brief Overview

Founded in 1920, the National Council of Teachers of Mathematics, the world's largest professional organization for teachers of mathematics from PK–grade 12 with more than 40,000 individual and institutional members, provides broad national leadership in matters related to mathematics education. NCTM carries out this role through a variety of avenues, including annual national and regional conferences for teachers and the publication of three journals. *Mathematics Teacher: Learning and Teaching PK–12*, which debuted in January 2020 replacing the three grade-band specific journals (*Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and the *Mathematics Teacher*), reflects the current and potential future practices of mathematics education from preschool to twelfth grade and is available in a digital version. *Mathematics Teacher Educator*, co-published with the Association of Mathematics Teacher Educators (AMTE), provides a professional knowledge base focused on strengthening practitioner knowledge for mathematics teacher educators in connection to the preparation and support of teachers of mathematics. The *Journal for Research in Mathematics Education* is the premier research journal in mathematics education and serves as a forum for inquiry and dissemination of emerging understandings related to the teaching and learning of mathematics.

NCTM has long been a leader among other educational professional societies in the development and publication of curriculum standards. This movement and the various documents are described in more detail in chapter 2. In addition, the Council publishes classroom resources and position papers, sponsors professional development workshops, and provides other resources on its website, https://www.nctm.org. Some of these resources are focused on content with activities and lesson plans while others, such as *5 Practices for Orchestrating Productive Mathematics Discussion,* 2nd edition (Smith and Stein 2018), provide research-based models and support for instruction in mathematics classrooms.

The NCTM mission statement advocates for high-quality mathematics teaching and learning for each and every student. The Council's advocacy focuses on both raising awareness among decision makers and the public related to issues concerning high-quality mathematics teaching and learning and on advancing a culture of equity in schools and classrooms in which all students have access to high-quality mathematics and mathematics teaching. To this end, NCTM publications, such as *Principles to Actions: Ensuring Mathematical Success for All* (2014), provide guidance and resources for the implementation of research-informed, high-quality teaching that supports the learning of each and every student in equitable environments. NCTM has more than 200 affiliates throughout the United States, its territories, and Canada. Most are organized by geographic area, but some, such as the National Council of Supervisors of Mathematics, TODOS: Mathematics for All, the Benjamin Banneker Society, and Women and Mathematics Education, are organized around a specific topic in mathematics education.

The Council is celebrating its centennial in 2020. It is fitting that in its centennial year, NCTM is being honored at the Fourteenth International Congress for Mathematical Education in Shanghai, China, by receiving the International Commission on Mathematics Instruction's Emma Castelnuovo Award in "recognition of 100 years of development and implementation of exceptionally excellent and influential work in the practice of mathematics education." The award announcement cited NCTM's standards work as foundations for policy and practice across the world, its professional journals, its contributions to providing leadership and resources for professional development, legislative and policy leadership, and international collaboration.

Mathematical Association of America: A Brief Overview

The Mathematical Association of America (maa.org) was founded in 1915 as an organization of mathematicians, students, and enthusiasts of mathematics seeking to advance the understanding of mathematics and its impact on our world. In the 1950s, in recognition of the fact that the undergraduate mathematics curriculum in the United States was broadly in disarray, MAA began its work on shaping undergraduate mathematics instruction. Information on its work, reports, and recommendations into the 1990s can be found at tiny.cc/First40Years.

MAA continues to be one of the foremost sources for information on best practices in the curriculum and pedagogy of undergraduate mathematics education. For example, the MAA has published both curriculum guides and guides for the instruction of mathematics and to make recommendations to and provide support mathematics for instructors at the postsecondary level (MAA 2004, 2015, 2018). MAA has also taken a leadership position among the professional societies in advocating for improvements in classroom practice. The *Common Vision Report* published by MAA—a joint venture with the American Mathematical Association of Two-Year Colleges, the American Mathematical Society, the American Statistical Association, and the Society for Industrial and Applied Mathematics—presents consensus views of how to improve undergraduate mathematics instruction. MAA also took the lead on the creation of the position statement on *Active Learning in Postsecondary Mathematics Education*, endorsed by the presidents of all of the professional societies in the mathematical sciences that are concerned with academic issues.

In addition to the efforts listed above, the association runs a number of programs designed to improve instruction. Project NExT, started in 1994, works over a full year with 80 to 100 new career mathematicians, making them aware of resources and best practices as well as introducing them to the many aspects of their new career. College Mathematics Instructor Development Source (CoMInDS) provides resources for instructor development. Preparing for Industrial Careers in Mathematical Sciences (PIC-Math) trains faculty in how to better prepare students for industrial careers. StatPREP runs workshops to help faculty learn how to teach statistics with modern methods of data analytics. More details on these programs are provided in chapter 6.

MAA has been engaged in research via its special interest group on Research in Undergraduate Mathematics Education and, more explicitly, through its studies of best practices in precalculus through single variable calculus: Characteristics of Successful Programs in College Calculus and Progress through Calculus (information on both programs available at maa.org/PtC). More information on these and related projects are provided in chapter 7.

American Statistical Association: A Brief Overview

The American Statistical Association (ASA) was formed in November 1839 in Boston. Notable figures who joined the society included U.S. President Martin Van Buren and Minister to France Lewis Cass. Florence Nightingale also became a member and was recognized for using statistical analysis techniques in her data-collection efforts benefitting public health and welfare. The organization began publishing the *Journal of the American Statistical Association (JASA)* in 1888, when its mission was solidified and membership was growing. The number of journals published or co-published by ASA is 16, including the *Journal of Statistics Education (JSE)*, which was added to its collection of publications in 1993. JSE is an open access peer-reviewed journal focused on improving statistics education at all levels. In collaboration with NCTM, the ASA also publishes the peer-reviewed online journal *Statistics Teacher (ST)*. ST supports the teaching and learning of statistics through education articles, lesson plans, announcements, professional development opportunities, technology, assessment, and classroom resources.

The ASA strategic plan identifies the support of educational initiatives as a focus area for the association. ASA has several initiatives and committees dedicated to this strategic focus, including the ASA/NCTM Joint Committee on Curriculum in Statistics and Probability and the ASA/MAA Joint Committee on Undergraduate Statistics Education. ASA also supports education by providing resources for K–12 teachers and teacher educators as well as developing national guidelines for statistics education and supporting projects related to education. These include the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A Pre-K–12 Curriculum Framework* (American Statistical Association 2007, 2020), which provides recommendations and a curriculum framework with examples for teaching statistics in the PK–12 years, the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) College Report* (American Statistical Association 2016), and the *Statistical Education of Teachers (SET)* (Franklin et al. 2015) report, which outlines the content and conceptual understanding teachers need to know to assist their students' development of statistical reasoning skills. *Preparing K–12 Teachers of Statistics: A Joint Position Statement of the American Statistical Association and National Council of Teachers of Mathematics* details the preparation and support teachers need to successfully support students' learning of statistics in the PK–12 curriculum (available for PDF download: https://www.nctm.org/Standards-and-Positions/Position-Statements/Preparing-Pre-K-12-Teachers-of-Statistics/).

Recently, ASA made a further commitment to education by creating the ASA K–12 Statistical Ambassador position. This position, held by Christine Franklin (lead author of the *GAISE Pre-K–12 Report* and the *SET Report*) was created to emphasize the commitment of the organization to providing leadership in the creation and presentation of professional development materials for teacher educators and teachers. Other regularly offered professional development opportunities include K–12 Statistics Education Webinars, the Meeting Within a Meeting (MWM) Statistics Workshop for Mathematics and Science Teachers, and the Beyond AP Statistics Workshop.

Two more resources to be published in 2020 from the Joint ASA/NCTM committee: (1) *Focus on Statistics—Investigations for the Integration of Statistics into High School Mathematics Classrooms,* a collection of datacentric investigations for high school students (these investigations are also ideal for integrating into an introductory statistics course); (2) a targeted book, *Statistics and Data Analysis for Teachers* (Franklin and

Bargagliotti forthcoming), of statistical investigations written as a resource for teacher educators preparing classroom teachers.

Throughout all the growth and changes, ASA's vision affirms its commitment to education by imagining "a world that relies on data and statistical thinking to drive discovery and inform decisions." More details and updates on ASA's efforts are provided in chapters 2 and 5 of this document.

Chapter 2: The Intended Curriculum

The intended curriculum refers to the set of goals and objectives described in policy documents and recommendations such as national and state standards. This chapter explores the evolution and current state of PK–12 standards and accompanying assessment efforts, recommendations for creating equitable structures and instructional practices, as well as an overview of recommendations at the postsecondary level.

Evolution of PK–12 Standards: NCTM's *Principles and Standards for School Mathematics* to the Common Core State Standards for Mathematics

The release of National Council of Teachers of Mathematics' (NCTM's) *An Agenda for Action* in April 1980 set the stage for a proactive era of professional input for the reform of mathematics education in the United States that continues to this day. *An Agenda for Action* provided a framework for a structured discussion of the direction in which K–12 mathematics education should go in the coming years. The points were widely circulated and discussed, and they brought significant attention to the improvement of K–12 mathematics at state meetings of mathematics teachers and among faculty at colleges and universities working with teachers of mathematics.

In March 1986, the NCTM Board of Directors passed a motion to begin "the development and implementation of professional standards for mathematics education in grades K–12." The first set of standards was limited to the content of the K–12 curriculum and statements about related changes necessary in the evaluation of learning environments and instructional materials. NCTM immediately began work on the development of *Standards for Teaching Mathematics* (1991) and *Assessment Standards for School Mathematics* (1995). These two new Standards documents completed the picture of school mathematics while teachers were immersed in a period of adoption, refinement, professional development, and the development of curricular materials in response to *Curriculum and Evaluation Standards for School Mathematics* (McLeod et al. 1996).

Following the release of the *Curriculum and Evaluation Standards for School Mathematics*, state standards underwent a significant shift as their curricular content recommendations moved toward alignment with those espoused by the NCTM Standards. But while the content taught at a particular grade level in schools within a state might exhibit little variance, what was taught at the same grade level across states exhibited far greater variability. At the same time, strides made at the two-year/community college level brought more coherence to these institutions' curricula and focused more attention on their instruction while tying the outcomes to career- and college-readiness outcomes beyond secondary school. These efforts continued throughout the 1990s and into the middle of the next decade, with initiatives from related organizations playing major roles in providing information on how to meet the needs of students and chronicling the positive changes in student achievement.

NCTM revised the 1989 Standards in 2000 to reflect the growth of knowledge about learning and teaching over the intervening period of time. In addition, this revision and update, published as *Principles and Standards for School Mathematics* (PSSM; NCTM 2000), sharpened the original 1989 recommendations by unifying content, the teaching of content, and assessment into a single picture of three mutually supportive domains that make a mathematics program whole. The release of *Principles and Standards* reignited the movement to pull state standards for mathematics closer to the curriculum content espoused by the NCTM Standards. But, as mentioned earlier, although there might have been little variability in the content being taught at the same grade levels in schools within a state, there was far greater variance in what was being taught at the same grade levels between states. This gap did not close.

In 2006 NCTM published its *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*, which provides grade-level recommendations for the placement of topics and key concepts within the teaching of topics at those grade levels. Although some believed that the focal points verged on the development of a national curriculum model, which NCTM had avoided in 1989, this NCTM initiative can be viewed as a response to the Council's recognition of the need to provide curricular coherence and clarity for teachers and students. *Curriculum Focal Points* focuses on creating coherence in the study of mathematics by a careful building of key concepts over time, interlinked so that each key concept builds on and extends the previous one, which paved the way to it. This linking of topics over grades provides the structural frame that supports understanding mathematics as a discipline.

The appearance in 2009 of NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* rounded out NCTM's "focus" initiative with a publication examining a path for secondary school mathematics in the 21st century. Instead of focusing on content directly, as *Curriculum Focal Points* had done, this publication sought to bring focus to high school mathematics by preparing students to use their mathematical understanding in reasoning about and making sense of mathematical situations. Building on the earlier *Principles and Standards* articulated by NCTM, this publication addressed the components of reasoning and illustrated how they apply in each of the content domains addressed in *Principles and Standards*.

By 2007, all the U.S. states had some form of state standards, but these exhibited significant inconsistencies at the grade levels in which specific topics were introduced and in the time allotments authorized for students to develop fluency with procedural skills. In some cases, grade levels for given mathematical topics were never specifically addressed, leaving the decisions to local districts or even schools (Reys 2006). These conditions moved the National Governors Association Center for Best Practices (NGA) and the Council of Chief State School Officers (CCSSO) to launch the Common Core State Standards Initiative in 2008 with the appointment of a writing group. The Common Core State Standards for Mathematics (CCSSM; NGA and CCSSO 2010a, 2010b) was released in 2010, providing a path for unifying the existing state standards and grade-level expectations for students into a suggested common set of goals for all states. CCSSM elaborates grade-by-grade standards and related expectations for K–grade 8 as well as formulates high school standards organized by conceptual categories that define the mathematics that students need to know to be college or career ready. The secondary school recommendations provide guidance for curricula delivered by the traditional topical design as well as curricula delivered in an integrated format (NGA and CCSSO 2010b).

CCSSM was intended to move the nation to a common understanding of what children need to know and when, even if the standards were not identically the same from state to state. That is, the goal was to increase the ability of the system to deliver high school graduates who were college and career ready in mathematics, according to an explicit description of what this means. A goal of CCSSM was to ensure that students understand mathematics as structure, reasoning, and procedures that make sense (rather than as a long list of rules to follow) and that students acquire the ability to reason with and use mathematics. As such, CCSSM was envisioned as a set of standards that would be adopted or adapted by the states, allowing all students to have equal access to similar instructional activities based on materials created to reflect a similar vision of school mathematics.

States began adopting the CCSSM in the 2010–11 school year. Initially, 45 states, the District of Columbia, and some of the U.S. territories adopted CCSSM as their official curriculum guide for mathematics. The states of Alaska, Minnesota, Nebraska, Texas, and Virginia chose not to participate in CCSSM, although Minnesota adopted the English Language Arts portion of the Common Core State Standards. As of 2019, thirty-nine states continue to implement CCSSM or a revised version, one state continues to implement while reviewing, one state is developing new standards to replace the CCSSM and six states that had formerly adopted the Common Core have implemented alternative statewide standards (see https://www.ccrslegislation.info).

In July 2009, a year before the release of CCSSM in June 2010, the U.S. Department of Education announced a competitive program, Race to the Top, which would award grants to states and consortia to adopt "internationally benchmarked standards and assessments that prepare students for success in college and the workplace." Although the criteria did not explicitly name CCSSM or require its adoption, grant applications from adopting states received extra points in the grant approval process if they had adopted CCSSM by August 2, 2010. This action on the part of the Department of Education, and hence the U.S. government, was viewed in many quarters as a highjacking of the Common Core State Standards movement initiated by the states and without federal funding. This perception was strengthened by the funding of two state-based consortia for the development of assessments of Common Core curricula at state levels.

The assessment consortia, commonly referred to as PARCC (Partnership for Assessment of Readiness for College and Careers 2015) and SBAC (Smarter Balanced Assessment Consortium 2015), had their full-scale field-testing in 2014–15 and served as yearly statewide assessments in several of the participating states in the 2015–16 school year. When the assessments were developed, 45 states reported plans to use them, however, results from *Education Week's* latest survey (April 2019) indicate that this number has continued to drop, with only 15 states and the District of Columbia reporting that they are still using PARCC or SBAC as their statewide assessments during the 2018–2019 school year. Factors influencing this decline include the length and costs of the assessments and a backlash against what has been perceived as the involvement of the federal government in what students should learn. Thirty-two states use assessments that they have designed or bought, and three use a hybrid of questions from either PARCC or SBAC and questions they have developed. The goal of these consortia to develop tests that better measured learning and that would allow policymakers to compare students across states has not been realized. (https://www.edweek.org/ew/section/multimedia/states-using-parcc-or-smarter-balanced.html)

A key component of CCSSM is the Standards for Mathematical Practice, which complement the content domains and standards for K–8 and the subject-area conceptual categories for high school by describing processes and practices in which students should engage in "doing" mathematics and the expectations and outcomes that programs should hold for students. These Standards for Mathematical Practice are based on the NCTM Process Standards (NCTM 2000) and the levels of mathematical proficiency described in the National Research Council's *Adding It Up* (NRC 2001). The Standards for Mathematical Practice are as follows:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure
- Look for and express regularity in repeated reasoning.

Like the NCTM Process Standards on which they are based, the Standards for Mathematical Practice address the intellectual habits that a program needs to model for students so that the associated behavioral and cognitive capabilities are inculcated in the students. The standards go across grade levels and specific content recommendations. They continue to resonate with the community as important aspects of the school mathematics curriculum.

Several other themes characterize the mathematics education activities of the decade since the release of CCSSM. One was the recognition of the need for a more unified, integrated curriculum under a STEM (science, technology. engineering, and mathematics) umbrella. National reports such as *Successful K–12 STEM Education: Identifying Effective Approaches in Science, Technology, Engineering, and Mathematics* (NRC 2011) and *Monitoring Progress toward Successful K–12 STEM Education: A Nation Advancing?* (NRC 2013b) call for an increase in the number of STEM majors graduating from undergraduate programs and the development of teachers prepared to staff courses in mathematics and science in grades 7–12 across the country. Another activity was the complementary need for increased attention within the PK–12 curriculum to the quantitative and data analysis skills required of an educated citizen in the 21st Century. In 2007, the American Statistical Association (ASA) released a report, *Guidelines for Assessment and Instruction in Statistics Education* (*GAISE Report*): *A Pre-K–12 Curriculum Framework.* The GAISE report was used to inform the statistics and probability standards in CCSSM. Beginning in 1989, the NCTM standards documents had always included Probability and Statistics as a strand. The ASA GAISE report further unpacks the Probability and Statistics strand by elaborating in detail on statistics as a process. An updated GAISE report addressing current changes in access to data and statistical processes over the last decade was released in April 2020. The GAISE II report is jointly published by ASA and NCTM. The Society for Industrial and Applied Mathematics (SIAM) released a similar report with recommendations for mathematical modeling in 2016, *Guidelines for Assessment and Instruction in Mathematics Modeling Education.* More details about the report and the educational activities of these professional organizations are in chapter 5.

Creating Equitable Structures and Instructional Practices

One of the goals of the Common Core State Standards was to ensure that all students would emerge from secondary education ready for college and career. In the years following their release and the revisions made by the states, important questions continue to be raised by many in the mathematics education community. Do we have mechanisms in place to ensure that all students will receive the equal opportunities that the standards promise? Do all students have access to knowledgeable teachers who are effective in using instructional modes that accommodate students who learn in different ways? Can we give assurances that the barriers that in the past impeded equitable access to help, information, and the tools of learning in a modern mathematical setting have been dismantled? Are students and teachers assigned to classes in a fashion that ensures that, as much as possible, competence and experience are shared equally across all students? Has the language in our schools changed from students' "abilities"—terminology that assumes that students arrive already ranked—to students' "capabilities"—terminology that focuses on and emphasizes what students can do and where programs can take them? Is the curriculum, together with the supporting materials, equally available across all schools in all locations?

To address these issues, NCTM published *Principles to Actions: Ensuring Mathematical Success for All* in 2014. This publication focuses on what success should look like in mathematics education and outlines a set of eight research-based effective teaching practices that cut across grade level and mathematics content and are important no matter what standards have been adopted. The text elucidates the principles that we must identify and adhere to if we are to eradicate the barriers and ensure that all students have valid opportunities to learn mathematics. *Principles to Actions* continues to resonate with teachers and district leaders.

In 2016, NCTM organized a task force with the charge to identify, describe, and document the range of problems and challenges that are faced in ensuring that grades 9–12 mathematics works effectively for each and every student. The rationale for the Council's decision to focus initially on high school mathematics was twofold: first, change in high school level mathematics achievement has been relatively flat over the past thirty years relative to the progress made at the elementary and middle school levels, and second, many educators perceived that the high school Common Core standards as compared to the elementary and middle levels were unwieldy—there were too many standards and they lacked coherence. *Catalyzing Change in High School Mathematics: Initiating Critical Conversations* was released at the NCTM Annual Meeting and Exposition held in Washington, D.C. in April 2018. The document makes key recommendations focused on expanding the purpose of mathematics beyond college and career readiness, creating equitable structures and equitable instructional practices, and advancing a set of essential concepts in the key content areas of number, algebra and functions, geometry and measurement, and statistics and probability that all students should learn and understand at a deep level before they graduate from high school regardless of their future plans. In addition, it provides examples of course pathways for students as they complete four years of high school mathematics that match student needs and interests. Two additional Catalyzing Change publications were developed, one focused on recommendations for the early childhood and elementary level and the other focused on middle school mathematics. These two documents were released in April 2020. They demonstrate NCTM's commitment to a systemic approach to addressing the teaching and learning of mathematics across PK–12. More detail on the recommendations of the Catalyzing Change publications will be discussed in chapter 5.

States' Secondary School Graduation Requirements

The release and state adoption of CCSSM and the emphasis on mathematics for college and career readiness raised the matter of the mathematical preparation that was required for all students. States were forced to reexamine their legislated high school graduation requirements. Graduation requirements in the U.S. are measured by "units of study with each unit requiring one school year (or 36 weeks) of instruction." Achieve, a nonprofit education reform organization that works with states to improve the college and career readiness of all students, has maintained data on each state's graduation requirements for most of its 23-year history. According to Achieve, "this information comes from available public sources, includes state websites with guidance to schools and families and state laws and regulations" (*Methodology for Achieve "Graduating Ready" Data Explorer*, May 22, 2019). A review of the most recent requirements in place for the class of 2019 indicates that the number of mathematics units required ranges from two to four units with at least algebra 1 and geometry listed as specified requirements for 42 states and the District of Columbia. Twenty-five states offer different types of diplomas, with seven states offering a diploma that specifies a STEM endorsement. The graduation requirements for three states are determined by the local district and not mandated by the state. There continues to be a breadth of choices for students to meet graduation requirements within and across states. In the most recent analysis, Achieve classified each graduation option as to whether or not the option requires students to complete one unit of a college- and career-ready (CCR) course that is not necessarily a mathematics course.

- Nine states have at least one diploma classified as Mandatory CCR—students are required to complete a CCR course in order to graduate.
- Eleven states have at least one diploma classified as Default CCR—students are expected to complete a CCR course of study but could opt out and complete a less demanding set of requirements.
- Eighteen states have at least one diploma classified as Opt-in CCR—students select a course of study from a menu of options.
- Thirty-one states have at least one diploma classified as Minimum—students are required to take either a state-defined set of requirements that are below the CCR threshold or no specific course is prescribed.
- Three states have at least one diploma classified as Local Control—the state does not set graduation requirements. (https://highschool.achieve.org/graduation-requirements-data-explorer)

Postsecondary Curricular Reports and Recommendations

During the past decade, the focus in curricular documents related to collegiate mathematics shifted from a listing of course content to documents looking introspectively at content and appropriate instructional strategies specific to collegiate classrooms as well as at ways in which the collegiate mathematics curriculum might be linked to the applications of mathematics in new disciplines. Issues related to articulation between high school and collegiate level mathematics continue to be a focus of discussions at the state and national level. Noteworthy publications include the following:

- *A Common Vision for Undergraduate Mathematical Sciences Programs in 2025* (Saxe and Braddy 2015)
- *Guidelines for the Assessment and Instruction in Statistics Education: College Report 2016* (ASA 2016)
- *The Statistical Education of Teachers* (Franklin et al. 2015)
- *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* (MAA 2015; updated in 2017)

- *Guidelines for Assessment and Instruction in Mathematics Modeling Education* (SIAM and COMAP 2016)
- *Guide to Evidence-Based Instruction Practices in Undergraduate Mathematics* (MAA 2018)
- *IMPACT: Improving Mathematical Prowess and College Teaching* (AMATYC 2018)

Details of these reports will be described later in this document.

An increasing number of students begin their postsecondary mathematics coursework at a two-year college (Conference Board of Mathematical Sciences (CBMS) 2015) either through dual-credit programs or by attending a two-year college as a matriculated student following high school graduation. Leading the way in developing standards for content and pedagogy at the postsecondary level, the American Mathematical Association of Two-Year Colleges published *Beyond Crossroads: Implementing Standards in the First Two Years of College* in 2016. Building on this report, in 2018 they released *IMPACT: Improving Mathematical Prowess and College Teaching,* an update that focused on the following four primary pillars: proficiency, ownership, engagement, and student success. The IMPACT guide also included a chapter on research in mathematics education at the two-year college level as an effort to drive much needed future research investigations in mathematics at this level. Echoing some of the same themes in the AMATYC reports, the MAA Committee on the Undergraduate Program in Mathematics (CUPM) developed and published its own curriculum guide. The *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* clearly expresses the systemic links among cognitive processes, mathematical practices, mathematics content, and the development of mathematical ways of knowing and applying the mathematical sciences in areas beyond the classroom. It also emphasizes the importance of developing model syllabi for courses interactively with colleagues within and outside one's own department or campus as well as elaborates how courses fit together to provide programs of study matching students' goals for the future. The CUPM work extends the role of curriculum as it plays out in professional development, assessment, technology, undergraduate research, and a number of other facets of a successful program. Focusing on the interactive nature of curriculum in making choices, the CUPM work shows the diversity of the mathematical programs to which the committee's recommendations apply.

The 2015 CUPM report refers to two special publications of the National Research Council (NRC) that provide assistance to a department when developing undergraduate programs of instruction in mathematics. *Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century* (NRC 2012) and *The Mathematical Sciences in 2025* (NRC 2013a) both point to the role that mathematics plays in the modern world. They focus on issues that departments of mathematical sciences should be thinking about as they seek to build capabilities within their faculty and to alert students to current and future applications and opportunities. Two updates to the 2015 CUPM guide include *Guidelines for Collegiate Faculty to Teach Mathematics to Blind or Visually Impaired Students* by Dr. A. Maneki, Senior STEAM Advisor for the National Federation of the Blind Jernigan Institute, and a 2017 Addendum on Actuarial Mathematics (https://www.maa.org/node/790342).

Alerting students to opportunities is key to recruiting and retaining them in STEM subjects as they move through our classes and programs. The President's Council of Advisors on Science and Technology (PCAST 2012) presented recommendations in its report *Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics* with an eye to assisting secondary schools, colleges, and universities in recruiting and retaining capable students in fields in these areas. Such actions require shifting instructional programs to use validated instructional techniques and having the financial resources to make significant changes in classroom environments. These changes might include moving from lecture formats to settings that invite students to engage in group work, moving from inspecting tables and graphs in a book to using technology to represent and analyze data and interpret results, and considering community problems from a mathematical modeling standpoint as students prepare for careers in STEM fields or in mathematics and science teaching. Such changes also require concomitant changes in the instructional patterns in postsecondary institutions, especially at the undergraduate level, before these students are lost to other majors.

Changing pedagogy to better align with evidenced-based instructional practices is daunting, but it is a move that professional societies are strongly calling for. For example, in 2016 the Conference Board of the Mathematical Sciences (CBMS), which is an umbrella organization for 18 different professional societies in the mathematical sciences, put out a position statement on active learning and called on "institutions of higher education, mathematics departments and the mathematics faculty, public policy-makers, and funding agencies to invest time and resources to ensure that effective active learning is incorporated into postsecondary mathematics classrooms." Following up on this call and the need to better support a shift to more engaged classrooms, the MAA published the *Instructional Practices Guide*, which provides an overview of postsecondary mathematics education professional development.

Further supporting postsecondary mathematics and statistics department change in both curriculum and instruction is the Transforming Postsecondary Education in Mathematics (TPSE Math) effort. Started in 2014, TPSE Math convenes several national level meetings every year focusing on four priorities: lower division pathways, upper division pathways, graduate education, and teaching strategies and practices.

From 1995 to the present, the work done at the two- and four-year college levels, as well as at the university level, supports and extends the work started by the MAA's Committee on the Undergraduate Program in Mathematics in the mid-1960s and continues through its new undergraduate guidelines of 2015, its updates in 2017, and guidelines for instructional practices in 2018. These reports support the perspective among undergraduate mathematics and statistics departments that high-quality content alone is not sufficient to produce high-quality learning. A major portion of what students need to learn and be able to use resides in the interaction of that content with the ways in which the students have learned it. Departments from the community college level through graduate schools have realized that the goals must change to include high-quality curricula and programs grounded in learning and realistic, data-filled problems. Further, national scientific organizations, state boards of higher education, and professional groups are stepping forward to ensure that resources are available to provide both ongoing professional development for collegiate faculty and support for continued curricular development work. More detail is presented in chapters 6 and 7.

Chapter 3: The Implemented Curriculum

The implemented curriculum refers to the actual activities and experiences of students in the school classroom toward achieving curricular goals of the intended curriculum. Historically, the implemented curriculum in school mathematics has been guided by a mixture of individual state expectations for topics to be taught within mathematics classes and in some states or districts, by associated student abilities and expected outcomes, as described in chapter 2. The instructional materials selected by the state, the school district, or the individual schools have also set bounds on these expectations and often determine the implemented curriculum. Finally, the content and representations are influenced by the individual classroom teacher's mathematical knowledge and pedagogical knowledge.

The various mathematical standards documents (cf., Franklin et al. 2015; NCTM 2000, 2009, 2018) and prior legislative efforts (cf., NGA and CCSSO 2010a; U.S. Department of Education 2019) outlined in the previous two chapters of this report were developed precisely to address this mixture of expectations and delivery to help in the development of common criteria across the states. In addition, standards-based guides for teachers' preservice mathematics education and continued professional development in mathematical content knowledge, mathematical practices, and related pedagogical methods were created (Association of Mathematics Teacher Educators [AMTE] 2017; Conference Board of the Mathematical Sciences [CBMS] 2012; NCTM 2014). The overall goal of these guides is to enhance teachers' understanding of the intended standards-based curriculum and provide a more coherent delivery of that curriculum and, hence, improved student outcomes.

The Impact of Standards and Educational Policy on the Implemented Curriculum

The Every Student Succeeds Act (ESSA) contains the requirement that states adopt and implement assessments to measure mathematical learning progress at designated ages or years in school. However, it is up to each state to determine the curriculum standards on which to assess mathematical progress (U.S. Department of Education 2019). As previously described in chapter 2, the Common Core State Standards for Mathematics (CCSSM) provide a strong framework for states to adopt as a curriculum or assessment guide or to revise as needed.

Whether states choose to follow the recommendations of CCSSM or develop their own curriculum standards in mathematics, the implemented mathematics curriculum in U.S. schools remains dictated to a large degree by the contents of textbooks and other commercially available instructional materials and the sequencing of the topics found in those materials. The assessments used locally and by state governments also dictate the mathematics content and practices students encounter in school. Textbooks have been described as an author's or collaborative authors' interpretation of a given curriculum that offers both content and pedagogical opportunities for teachers and learners. The implemented curriculum, although driven by the text and materials, is dependent on the classroom teacher's interpretation of the textbook being used (van den Ham and Heinze 2018).

Research has shown that the overall alignment of instructional materials and assessments with curriculum standards can positively impact students' mathematics learning (Polikoff 2015). With the advent of the Common Core State Standards, there appears to be greater consistency across textbooks and on expectations of what students are to learn as schools nationwide are beginning to work toward helping their students attain essentially the same set of learning outcomes. Over the past decade, many K–12 mathematics textbooks have been written with an eye toward guiding the curriculum as recommended by the Common Core State Standards for Mathematics. However, despite claims of addressing the standards, it is somewhat unclear how much CCSSM has actually influenced relevant changes to the curriculum of a decade ago or how well aligned the curricula are to CCSSM (Koedel and Polikoff 2017; Polikoff 2015).

Textbooks and the Implemented Curriculum

Textbooks and materials published in the U.S. include traditional print textbooks, online textbooks (electronic version of print materials), online practice modules, and a wide range of supplemental documents such as assessments, activities, technology applications, and virtual and physical manipulatives. In addition to published textbooks, some schools or districts develop their own "teacher made" curriculum materials in an effort to align the school curriculum with standards or with the needs of particular students when published materials are not seen as adequate for particular student populations.

Textbooks in digital form and a wide range of supplemental materials have been on the increase over the past decade. In addition, online tutoring resources, such as Khan Academy, are often recommended by teachers and used by students. To provide flexibility and to meet the needs of a wide audience across the U.S., many publishers offer the option of obtaining the print or digital versions of their materials. Even so, it has been found that print media is still preferred by teachers and students (Meany and Mickey 2019).

Government sponsored educational surveys do not collect data on textbook alignment or adoption. However, some data are available from publishers, assessment organizations, education research groups, and isolated research studies. One independent research firm, Simba Information (formerly known as Education Market Research), has conducted surveys for several years on curriculum materials being used in U.S. classrooms. Their results, found in the *K–12 Mathematics Market Survey Report 2019,* are used to inform the discussion of the K–12 U.S. mathematics curriculum that follows (Meany and Mickey 2019).

At all levels of mathematics education, digital platforms are being developed and expanded to address the diverse markets across states and districts. Administrators and teachers cite the desire for products that attend to building both mathematical and problem-solving skills while adjusting to the needs of individual students. Some Artificial Intelligence (AI) curriculum developers offer specific products such as intelligent tutors to supplement a traditional mathematics curriculum while others offer a complete digital curriculum across multiple grade levels. Carnegie Learning has been in the digital curriculum business for several decades. However, many newer companies have come on board in this market over the past decade, such as Illustrative Mathematics (IM) and Imagine Learning, have outpaced Carnegie Learning in sales. The IM curriculum is unique in this market; it started as an open resource through a nonprofit company, but it has now partnered with publishers to offer a more complete package to schools, including digital learning tools, tutorials, assessments, professional development, and some print materials as well. Even so, some elements of the IM curriculum continue to be available for free.

Given the range of materials available, it is not surprising that teachers who were surveyed report using alternative materials along with a core mathematics curriculum. It was found that educators who work at the elementary level tend to rely more heavily on the core mathematics instructional materials purchased by their school district.

Teachers in grades K–5 have less time to create lessons for mathematics because they are also responsible for lessons in reading, science, and social studies. Across all grade levels, 52 percent of mathematics teachers surveyed in 2018 reported following a core mathematics program closely, 39 percent reported having a core program that they could pick and choose from as desired, and 9 percent reported not using any core mathematics program (Meany and Mickey 2019).

When asked which core mathematics programs were being used in schools, three publishers who offer a variety of mathematics programs were named most frequently. These included Houghton Mifflin Harcourt (named by 23 percent of respondents), Pearson (named by 18 percent) and McGraw-Hill (named by 15 percent). The 56 percent market dominance of these three parent companies has decreased significantly from prior surveys that reported 73 percent of core mathematics programs being provided by these same companies (Resnick and Sanislo 2015), demonstrating the growing variety of offerings.

Another research firm, EdReports, conducts reviews of textbooks and education materials and makes the results available for comparison purposes (EdReports 2020). EdReports claims to review up to 90 percent of currently published textbook materials for all grade levels in English language arts and mathematics, and for grades 6–8 in Science. Mathematics textbooks and supplementary materials are rated on (1) focus and coherence, (2) rigor and mathematical practices, (3) alignment with grade-level standards, and (4) usability. These curriculum materials are judged as meeting expectations, partially meeting expectations, or not meeting expectations. The materials are said to meet expectations for focus and coherence if a minimum of 65 percent of the materials are focused on grade-level content standards, and coherence is seen among related topics and across grade levels. Rigor is reviewed by determining whether lessons include conceptual understanding, fluency, and procedures as well as applications and the use of mathematical practices to enrich the lessons, such as the Standards for Mathematical Practice described in CCSSM. Alignment with standards for college and career readiness (whether CCSSM or other state-designed standards) refers to overall consistency with grade-level standards in combination with the overall ratings in focus and rigor. Finally, usability refers to the design and intended use of the materials, including planning and support materials for teachers as well as materials to guide assessment, differentiation, and technology use. The sections below provide a brief overview of the textbooks used at the various grade levels that meet expectations for all four categories described here.

Mathematics Materials in Elementary Schools (K–Grade 5)

Seven textbook series for K–grade 6, K–grade 5 or grades 1–5 were chosen as meeting expectations for focus and coherence, rigor and mathematical practices, alignment with grade-level standards, and usability.

- *Bridges in Mathematics* (2015), published by The Math Learning Center
- *enVision Florida Mathematics* and *enVision Mathematics Common Core* (2020), published by Pearson
- *Eureka Math* (2015), published by Great Minds
- *Into Math Florida* (2019), published by Houghton Mifflin Harcourt
- *Math Expressions* (2018), published by Houghton Mifflin Harcourt
- *Ready* (2017), published by Curriculum Associates
- *Zearn* (2016), published by Zearn

The enVision, Into Math, and Ready textbook series all showed no significant weaknesses in the reviews. In addition, the Ready Series stood out as having a well-designed and effective lesson structure (EdReports 2020). Eureka Math also was designated as being strong in design and structure despite some weaknesses in connecting mathematical practices to content. The Bridges, Math Expressions and Zearn textbook series all met expectations but were cited for weaknesses in mathematical practices such as poorly identified practices or weakly conceived materials not attending to the full meaning of the practices.

Mathematics Materials in Middle or Junior High Schools (Grades 6–8)

Similar to the K–5 textbook series, the enVision, Into Math and Ready Math series met expectations on all criteria for grades 6–8. However, at these middle school levels, some of the texts were cited as having some weaknesses in the area of connecting mathematical practices to content. In addition to these three series, another seven textbook series met expectations on all criteria.

- *Agile Mind Middle School Mathematics* (2016), published by Agile Mind
- *Carnegie Learning Math Solution* (2018), published by Carnegie Learning
- *EdGems Math* (2018), published by EdGems Math LLC
- *Illustrative Math 6–8* (2018), published by LearnZillion
- *Open Up Resources Mathematics* (2017), published by Open Up Resources
- *Reveal Math, Common Core Edition* (2019), published by McGraw-Hill Education
- *The Utah Middle School Math Project* (2017), published by University of Utah Middle School Math Project

Three sets of textbooks, Carnegie Learning Math Solution, Illustrative Math 6–*8* and Open Up Resources Math stood out as having no weaknesses in focus and coherence, rigor and mathematical practices, or alignment with grade-level standards and strength in usability. In addition, they are noted for having tools to collect data on student progress on specified standards. No weaknesses were cited for the EdGems Math and Reveal Math curriculum materials (EdReports 2020).

Mathematics Materials in Senior High Schools (Grades 9–12)

High school mathematics programs and curriculum materials reviewed by EdReports were not as up-to-date; some of the most highly rated books were published in 2014 and 2015. This is likely because textbooks tend to be on a seven- to eight-year rotation before schools consider material replacement (Meany and Mickey 2019). At the high school level, the mainstream core curriculum currently found in U.S. secondary school classrooms is built around a sequence of three full-year courses: algebra 1, geometry, and algebra 2 or algebra 1, algebra 2, and geometry, followed by precalculus, usually giving strong attention to functions and trigonometry. Since the mid-1950s, an increasing percentage of students have completed a year of calculus at the high school level. This latter course, especially when it is an Advanced Placement (AP) Calculus course (curriculum and assessments developed and administered by the College Board), usually covers the content ordinarily found in the first one or two semesters of university-level calculus. In most school districts where students participate in AP Calculus courses, algebra 1 is taught in the eighth grade. Another common AP course at the secondary school level is AP Statistics. This course has become a popular alternative to students who do not wish to pursue a pathway that would require calculus.

Six high school textbook series reviewed by EdReports were chosen as meeting expectations for focus and coherence, rigor and mathematical practices, alignment with grade-level standards, and usability.

- *Agile Mind High School Mathematics* (2016) and *Integrated Mathematics* (2019), published by Agile Mind
- *Carnegie Learning Math Solution* (Integrated and Traditional; 2018), published by Carnegie Learning
- *Core-Plus Mathematics* (2015), published by McGraw-Hill Education
- *College Preparatory Mathematics Core Connections* (Integrated and Traditional; 2015), published by CPM Educational Program
- *Discovering Mathematics*: *Algebra* (2014), *Geometry* (2015), and *Advanced Algebra* (2017), published by Kendall Hunt
- *enVision Algebra 1, Geometry,* and *Algebra 2* (2018), published by Pearson

The Carnegie Learning Math Solution and CPM series are highlighted for their coherence and consistency with high school standards for college and career readiness. In addition, they contain materials that are seen as helpful for students in meeting rigorous expectations of the standards and for making meaningful connections between mathematical content and the Common Core standards for mathematical practice (EdReports 2020; NGA and CCSSO 2010a). Of the high school textbook series listed, Discovering Mathematics stands out for its strengths in addressing the CCSSM content standards and mathematical modeling, connecting with content of grades 6–8, and support for the intentional development of mathematical reasoning, seeing mathematical structure, and generalizing (EdReports 2020).

Coursework and Pedagogical Patterns in K–12 Mathematics

Over the past twenty-five years, high school graduation requirements and college admission requirements have become more rigorous while the percentage of four-year colleges and universities now requiring two years of algebra and a year of geometry for admission has increased. Also, spurred by concern about U.S. students' lackluster mathematical performances in international studies, the public appears to have become more aware of the role that mathematics can play in the future lives and careers of secondary school students. This is evidenced by recent documents focusing on mathematics content and practices for career and college readiness, for example (NCTM 2018; NGA and CCSSO 2010a). Furthermore, teachers of eighth graders report placing a heavier emphasis on algebra and functions in 2015 (91 percent) when data from that National Assessment of Educational Progress (NAEP) questionnaire is compared to responses from the 2009 (84 percent) and 2013 (86 percent) NAEP teacher questionnaires (NAEP Data Explorer 2015a). This trend of increasing emphasis on algebra has, in turn, contributed to a steady and significant increase over time in the percentage of students topping out in twelfth grade at a higher level in the mathematics curriculum than had been reached by students in previous years.

More information on the NAEP assessments is provided in chapter 4. For most of the information reported from NAEP assessments, the NAEP Data Explorer was used.

From this point forward, when NAEP assessments are referenced and when data, graphs, tables and comparisons are provided from NAEP, it is assumed that the data are from the NAEP Data Explorer (NDE; National Center for Education Statistics 2015).

The data in table 3.1, taken from the NAEP mathematics assessment student questionnaires across various years, reflect a transition in the highest level of mathematics taken by high school students toward higher level courses such as precalculus and calculus. So, when we see a decline in enrollment in high school courses at the level of algebra 1 and geometry, we are seeing an increase in the percentage of secondary school students enrolling in algebra 2, trigonometry, or precalculus as their most advanced mathematics course taken in high school. In particular, table 3.1 shows statistically significant increases in the percentage of high school students taking algebra 2 or precalculus in 2015 when compared to data from 2005.

Table 3.2 also contains data from the NAEP 2015 *Student Questionnaire Results* related to twelfth graders highest level of mathematics taken. The data here show varying results when the information is split by students who report being accepted to a four-year college and those who did not report being accepted to a four-year college. Sixty-eight percent of twelfth graders who reported they were accepted into a four-year college also reported having taken either precalculus or calculus as their highest level of mathematics in secondary school. In striking contrast, of those students who did not report being accepted into a four-year college, only 28 percent reported having taken either precalculus or calculus in secondary school.

Table 3.1

Percentage of Grade 12 Students Assessed in NAEP Mathematics According to Highest Level Mathematics Course Taken in High School (NDE 2015a)

High School Course / Year	**2005**	**2009**	**2013**	**2015**
Algebra 1 or lower	8%*	5%*	3%	3%
Geometry	12%*	10%*	8%	7%
Algebra 2/Trigonometry	41%*	42%	45%	44%
Precalculus	21%*	24%*	26%	27%
Calculus	18%	18%	18%	19%

*Significantly different (p<0.05) from 2015.

Table 3.2

Percentages of Students Assessed in NAEP Mathematics Reporting Acceptance into a Four-Year College and Most Advanced Mathematics Course Taken (NDE 2015a)

	Calculus	**Precalculus**	**Algebra 2 or Trigonometry**	**Geometry**	**Algebra 1 or Lower**
Reported college acceptance	33%	35%	30%	1%	1%
No response	7%	20%	56%	12%	5%

In the most recent NAEP assessments, data were collected on computer or tablet and internet availability at home. Eighty-one percent of fourth graders who were assessed through NAEP mathematics in 2019 reported having access to a computer as well as internet at home. At the eighth-grade level, 89 percent of students who took part in the NAEP mathematics assessment reported at-home availability of computers as well as internet access. As might be expected, the percentage of students reporting availability of a computer and/or internet access at home was dependent on a range of factors, with the biggest contributor being socioeconomic status. For fourth graders, the percentage of public school–students reporting having both internet and computer access at home ranged across states from 70 percent to 87 percent, and the range for eighth-graders across states was 80 percent to 97 percent.

In 2009 through 2015, fourth-grade and eighth-grade teachers were asked about computer-based activities used in the classroom for mathematics learning. Mathematics review and practice and extending mathematics learning were the two activities for which teachers reported heavier and more frequent student computer use in the classroom. Table 4 contains the responses in terms of the percentage of grade-4 and grade-8 students whose teachers reported using computers at least twice a week for mathematics review and practice or for extending mathematics learning. The results from prior year questionnaires show slight, but statistically significant, increases in computer use for these activities at both grade levels when compared to 2015 results.

Table 3.3

Percentages of Fourth- and Eighth-Grade Students reported by Their Teachers as Using Computers in the Classroom at Least Twice a Week (Main NAEP Survey Questionnaires [NDE 2015b])

Computer Usage at Least Twice a Week		2009	2011	2013	2015
Fourth graders	Practice and review	41*%	52*%	60%	66%
	Extending learning	32*%	41*%	45*%	52%
Eighth graders	Practice and review	15*%	18*%	22*%	31%
	Extending learning	10*%	13*%	14*%	20%

*Significantly different (p<0.05) from 2015.

A curricular trend at the high school level that appears to be growing in momentum is dual enrollment. High school dual enrollment programs with postsecondary institutions allow high school students to take college-level courses that will offer credit toward their high school diploma as well as to college credit. In some cases, high school teachers are trained by and overseen by college faculty. In such cases, students may enroll in college-level classes and attend these classes at their high school facility. In other instances, students will enroll in classes at local postsecondary institutions (including two-year and four-year institutions) and will attend classes at these institutions during a portion of the regular high school day or through online coursework. Although current data is not available, information reported in 2003 and 2011 indicate at least a 50 percent increase in dual enrollments, from nearly 700,000 high school students who took courses in dual enrollment programs in 2002–03 to 1.2 million students enrolled in such programs during the 2010–11 academic year (NCES 2005, 2013). Enrollment in mathematics courses at postsecondary institutions during high school allows students the opportunity to move more quickly through the college curriculum, to pursue a two-year college degree while still in high school, or to better prepare for the four-year college experience (Studypoint 2019).

Mathematics Study at the Postsecondary Level

At the postsecondary level, students have a wide variety of options for studying mathematics and statistics. Coursework is available through community colleges, universities, and a variety of vocational schools, work-based educational programs, and commercial outlets. The data collected every five years by the Conference Board of the Mathematical Sciences (CBMS) provide the best trend data for curricular programs and enrollments in two- and four-year colleges. The most recent report comes from the 2015 survey (Blair, Kirkman, and Maxwell 2018) and the *Digest of Education Statistics 2017* (Snyder, Brey, and Dillow 2019).

Mathematics courses at the types of postsecondary institutions mentioned above range from arithmetic and prealgebra to differential equations, linear algebra, and statistics at vocational and two-year colleges; and from intermediate algebra, precalculus, and introductory statistics through advanced graduate courses at four-year institutions and comprehensive universities. Tables 3.4 and 3.5 demonstrate this wide range as well as the change in mathematics enrollments from 1985 to 2015 at two- and four-year colleges, respectively. In these tables, precollege courses include arithmetic, prealgebra, and elementary and intermediate algebra. Note that precollege course enrollment at two- and four-year institutions is correlated with a lower probability of graduation.

Precalculus courses include college algebra and trigonometry as well as finite mathematics, non-calculus-based business mathematics, mathematics for prospective elementary school teachers, and other courses for non-science majors. Calculus includes both mainstream and nonmainstream courses (e.g., calculus courses tailored to students in other majors, such as life sciences or business). These tables do not include mathematics or statistics courses taught outside mathematics and statistics departments. Enrollments are for the fall quarter or semester of the 2015—16 academic year (Blair, Kirkman, and Maxwell 2018).

Two-year college enrollments increased and were rising from the nearly 6.18 million enrolled in 2005 to slightly more 7.22 million in the fall of 2010, an increase of about 16.7 percent (Snyder and Dillow 2015). However, since 2010, enrollment trends in two-year colleges have been decreasing, with 2015 seeing just over 6.2 million, a decrease of more than 14 percent over the five-year period. An examination of the data in table 3.5 shows that this same period saw a decrease of about 5.2 percent in the number of students enrolled in mathematics at two-year institutions.

Although enrollments decreased overall, the only category that saw a decrease was the set of precollege mathematics courses. There were areas of increase worth noting in the two-year college offerings. Introductory mathematics and precalculus enrollments were up by 20.9 percent over 2010 enrollments; calculus enrollments by 10.1 percent; statistics enrollments by more than 100 percent; and enrollments in other courses (liberal arts, math for elementary teachers, and so on), by 12.1 percent. This pattern contrasts with four-year college data over the same period as the data in tables 3.4 and 3.5 show.

Table 3.4 shows that from 1985 to 2010, more than half of the mathematics enrollments in two-year colleges have been at the precollege level. This has changed somewhat with the 2015 data showing larger enrollment increases in precalculus and statistics. The overall decrease in the number of mathematics courses in two-year colleges is partially a function of the overall decrease in enrollments at these institutions. Increases in

Table 3.4

Estimated Enrollment (in thousands) in Mathematics Courses in Two-Year Colleges

Course	Year						
	1985	**1990**	**1995**	**2000**	**2005***	**2010***	**2015***
Precollege	482	724	800	763	964	1,150	782
Introductory/ Precalculus	188	245	295	274	321	368	445
Calculus	97	128	129	106	108	138	152
Statistics	36	54	72	74	117	137	280
Other	133	144	160	130	187	231	259
Total	936	1,295	1,456	1,347	1,697	2,024	1,918

*Data in 2005 and forward are reported by sections by average size rather than by percentage of total students calculations. (From Blair, Kirkman, and Maxwell 2018).

some mathematics subjects may be partially explained by changing recommendations for mathematical and statistical pathways for career readiness (NCTM 2018; NGA and CCSSO 2010a).

Four-year college enrollments increased slightly over this same period of time from approximately 13.3 million enrolled in the fall of 2010 to slightly more than 13.7 million enrolled in the fall of 2015, an increase of only 3.0 percent (Snyder, Brey, and Dillow 2019). An examination of the data in table 3.5 shows that this same period saw an increase of about 12.3 percent in the number of four-year college students enrolled in mathematics courses.

Not only did enrollments increase overall between 2010 and 2015 but the increase also occurred across the full range of the four-year college offerings. Precollege enrollments were up by 21 percent over 2010 enrollments but only 15.5 percent over enrollments in 2000. Precalculus enrollments were up by 15.9 percent, calculus enrollments were up by 5.5 percent, and statistics enrollments were up by 19 percent. Enrollments in advanced mathematics courses were up less than 3 percent. Although increases in advanced mathematics classes stayed relatively steady over the past five years, this area did see some of the greatest increases in the previous three survey years.

Table 3.5

Estimated Enrollment (in thousands) in Undergraduate Mathematics and Statistics Courses in Four-Year Colleges

Course	Year						
	1985	**1990**	**1995**	**2000**	**2005***	**2010***	**2015***
Precollege	251	261	222	219	201	209	253
Precalculus	593	592	613	723	706	863	1,000
Calculus	637	647	538	570	587	765	807
Statistics	dna	125	143	171	182	263	313
Other	138	119	96	102	112	150	154
Total	1,619	1,744	1,612	1,785	1,788	2,250	2,527

*Data in 2005 and forward reported by sections by average size rather than by percentage of total students calculations. (From Blair, Kirkman, and Maxwell 2018).

The graphs in figures 3.1 and 3.2 show the changes in growth (and decline) in enrollment for both two-year and four-year colleges and their contributions to the total number of undergraduate students enrolled in mathematics and statistics at U.S. two-year colleges and four-year colleges and universities. Although differences occur in the rates of growth of individual subareas within each subdivision of postsecondary education, one can see the overall increasing percentage of the total contributed by the two-year college enrollments in mathematics over time.

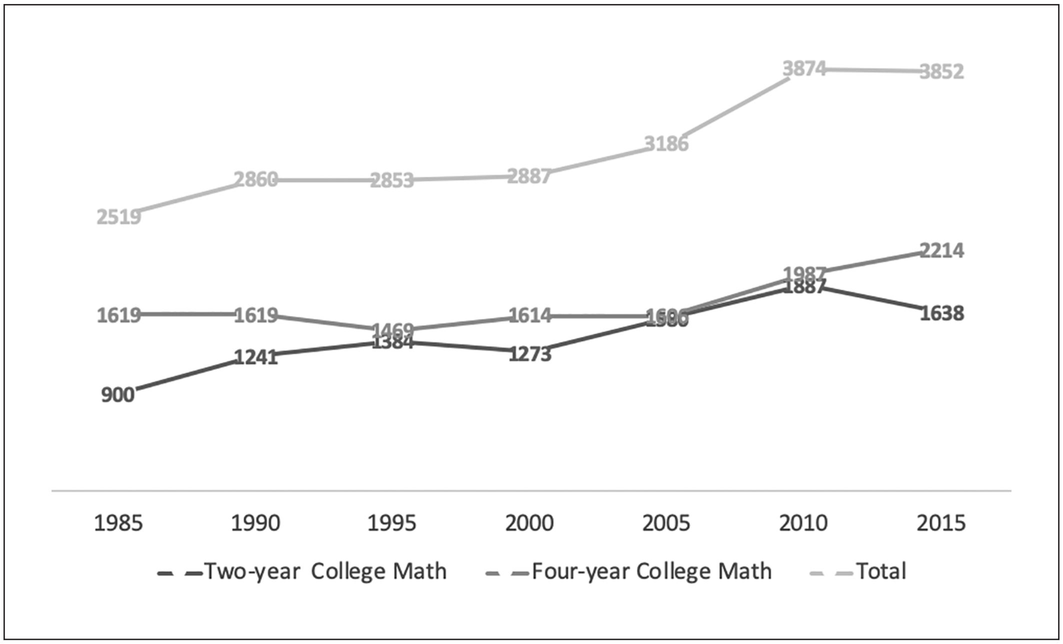


Fig. 3.1 Undergraduate enrollments (in thousands) in mathematics 1985–2015
(Data from Blair, Kirkman, and Maxwell 2018)

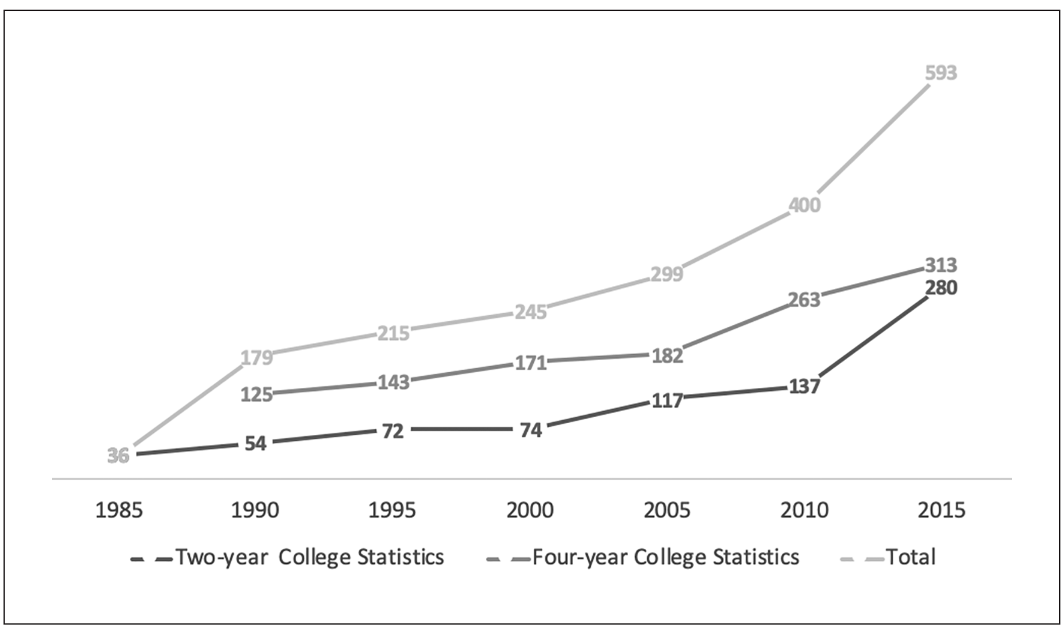


Fig. 3.2 Undergraduate enrollments (in thousands) in statistics 1985–2015
(Data from Blair, Kirkman, and Maxwell 2018)

Changes in the Number of Baccalaureate Degrees in Mathematics

Data from the CBMS 2015 Survey shows trends across time of the number of bachelor's degrees in mathematics and statistics awarded from four-year programs in U.S. institutions. The CBMS categories for mathematics majors that were collected in 2015 includes mathematics, mathematics education, statistics, actuarial mathematics, and majors that are a combination of these or combined with other disciplines such as computer science or business. The totals of students majoring in any of these degree programs has fluctuated slightly over the past two decades, with increases and decreases between reporting periods of roughly 3.0 percent (Blair, Kirkman, and Maxwell 2018).

Data from the American Freshman study indicate the changes by year in the percentage of freshmen entering baccalaureate granting institutions with an intention to major in mathematics or statistics: 0.9 percent (2010), 0.9 percent (2011), 0.9 percent (2012), 1.0 percent (2013), and 1.1 percent (2014) (Higher Education Research Institute 2010, 2011, 2012, 2013, and 2014; National Science Board 2014). Although the number of bachelor's degrees in mathematics is lowest among the areas listed in the sciences by the National Center for Science and Engineering Statistics, the mathematics requirements of students majoring in disciplines outside the physical sciences have increased significantly in the same period. Some of the mathematics needed to fulfill these requirements is taught outside departments of mathematics and statistics, but the increases in these requirements are a major factor in the overall increase in the number of courses taken in departments of mathematics and statistics (National Science Board 2014). At the same time, the piecemeal taking of courses to fulfill such demands does not imply a ready long-term supply of mathematically trained individuals to meet the nation's needs, which can be a matter for concern.

Chapter 4: The Attained Curriculum

Central to the measure of the success of a curriculum is the academic attainment of the students who have participated in the instructional experiences associated with it. Several measures of attainment are computed in the United States. At the national level, there is test-item data from the mathematics tests administered through the National Assessment of Educational Progress (NAEP). Outside of NAEP, the national and state-level data from the two major college entrance examinations—the ACT and the SAT programs—also provide other stable assessments of student achievement outcomes. In addition, to measure the attained curriculum at the state-level, each state administers a test aligned with their state standards. A total of 15 states plus Washington D.C. use either the PARCC or SBAC assessments, two tests aligned with the Common Core State Standards for Mathematics. Thirty-two states use their own designed tests and another three states use a test that mixes PARCC and SBAC questions with their own. As a result of these various measures, we examine the attained curriculum through data from NAEP, ACT, and SAT as well as state-level tests.

The National Assessment of Educational Progress

The U.S. government, through the Department of Education's National Center for Education Statistics (NCES) and with guidance from the National Assessment Governing Board (NAGB), administers a large-scale assessment program under the title of the National Assessment of Educational Progress (NAEP). This program's mathematics assessments began in 1973 and continue to be administered periodically to assess student knowledge of and their opportunity to learn mathematics by surveying random samples of American youth.

When people in the United States mention NAEP, they are usually referring to what is known as Main NAEP. This program focuses its assessment on a random sample of students from grades 4, 8, and 12, according to a schedule of recurring assessments beginning in 1973. The current program for Main NAEP assessments is every two years on the odd years for grades 4 and 8 and every four years for grade 12 on the odd years beginning in 2017 (NAGB 2015). In this report, we focus on the results of the last NAEP exam administered in 2019. In 2019, fourth-grade and eighth-grade students were assessed (grade 12 was not assessed).

In addition to each Main NAEP assessment, since 1990, a random sample of students from each state has also been assessed to provide each state with the same data that Main NAEP provides to the nation. This program is called State NAEP.

A third NAEP program, known as TUDA NAEP, is the Trial Urban District Assessment, which began in 2002. This program provides a NAEP assessment profile to the nation's largest urban districts at grades 4 and 8 through expanded samples of students drawn in conjunction with the State NAEP assessments. This program is limited to large urban school districts with a minimum of roughly 20,000 students in grades K–12 and is given in the same years as Main NAEP.

The final NAEP program is the NAEP Long-Term Trend Assessment (LTTA). This assessment differs from the other NAEP assessments in that it selects random samples of 9-, 13-, and 17-year-olds, thus maintaining the practice followed over time in this trend analysis. The last LTTA assessment was conducted in 2019. This most recent mathematics assessment was administered to approximately 149,500 students in grade 4 and 147,000 students in grade 8.

Main and State NAEP: Changes in NAEP Frameworks for Grades 4 and 8 Mathematics

The formats and expectations of Main and State NAEP Assessments are released a few years ahead of a given assessment in a NAEP Mathematics Framework. Frameworks for recent and present programs can be obtained from the NAGB website (NAGB 2019). The assessments are developed by a committee consisting of teachers from the grade levels assessed, collegiate professionals, and test and measurement experts from NAGB and the firms contracted to carry out the assessments in the field. The assessment itself is presented in a balanced incomplete block design, starting with blocks of items assembled into test booklets, each consisting of a fixed number of blocks of items, as well as a set of student questions concerning academic experiences and demographics. Over time, individual blocks of items are released, and new blocks of items are inserted to keep the test focused on the content targets articulated in the current NAEP framework.

The NAEP assessment in mathematics is focused on measuring the implemented curriculum, not on research into what curricular experts are thinking might be appropriate for a given grade level. It is a test of what is currently being taught in U.S. classrooms, not what might be taught. The content frameworks for the NAEP mathematics assessments span five broad content areas: (1) number properties and operations, (2) measurement, (3) geometry, (4) data analysis, statistics, and probability, and (5) algebra. In 2019, the grade 4 assessment had 38 percent of items related to number properties and operations, 17 percent related to measurement, 15 percent related to geometry, 13 percent related to data analysis, statistics, and probability, and 17 percent related to algebra. At the grade 8 level, a total of 24 percent of items were related to number properties and operations, 15 percent were measurement, 17 percent were geometry, 15 percent were data analysis, statistics, and probability, and 29 percent were algebra. NAEP items are also classified by mathematical complexity: low, moderate, and high. The complexity provides a measure of the cognitive demands of the item. Low-level items focus on recall, moderate-level items focus on making connections, and high-level complexity items focus on modeling.

Executive Summary for NAEP 2019 Grade 4 and Grade 8 Results

The two sections that follow provide an overview of the NAEP 2019 results for grade 4 and for grade 8. The overviews are somewhat repetitive as they cover the assessments at the two grades, each in some detail. However, to meet the needs of individual readers, both grade levels are covered in the same degree of detail.

In 2019, about 7,810 schools (7,230 public and 380 private) and 139,900 students (134,700 public and 2,400 private) were involved in the grade 4 mathematics assessment, and 6,150 schools (5,670 public and 340 private) and 136,900 students (132,500 public and 2,300 private) participated in the grade 8 mathematics portion of the Main NAEP assessment. The fact that the public and private school numbers do not always sum to the totals within a category is due to rounding that occurs in processing and in the weightings assigned to the scores to meet the sampling design specifications. The students were randomly selected according to a complex sampling design to form the basis from which results at both national and state levels could be developed and compared statistically. The performance of students from public and private schools in the 50 states, as well as students from the District of Columbia and the Department of Defense Education Activity (DoDEA) schools, was selected as the basis of the reports (NCES 2016).

NAEP 2019 Grade 4 Results

The national fourth-grade mean scale score in mathematics in 2019 was 241, up one point from the 2017 average. The 2019 average score was statistically significantly different from the 2017 average score. Overall, figure 4.1 illustrates that the grade 4 national average score increased by 27 points since 1990.

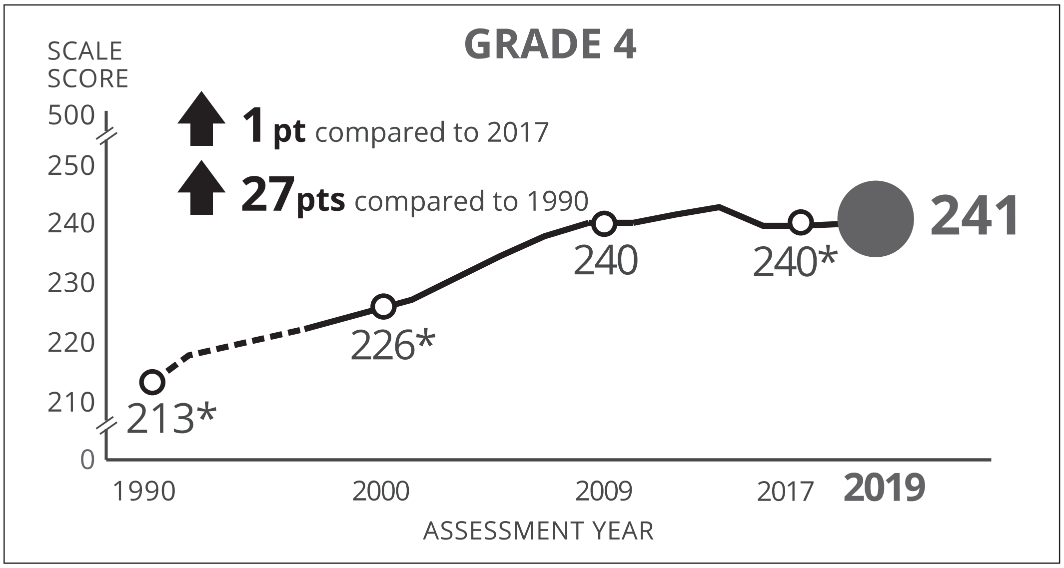


Fig. 4.1 Trend in fourth-grade NAEP mathematics mean scale scores (Graphic from the National Center for Education Statistics, *NAEP Report Card: 2019 NAEP Mathematics Assessment*. Washington, DC: U.S. Department of Education, National Center for Education Statistics 2019. Used with permission.)

The solid line on the figure indicates that the test allowed for accommodations for students needing them. The dotted line reflects a time when accommodations were not permitted for qualifying students. Reporting in NAEP is limited to reporting at the nearest tenth of a score point. However, statistical tests for significant differences are calculated by using unrounded average scores.

Because of the large sample sizes involved in the NAEP assessment for a given grade level, a small change in the mean performance may be judged as statistically significant, even when the actual difference in average performance is less than getting one more item correct on a NAEP assessment. However, when looking at the mean performances for fourth graders over the set of assessments since 1990, what one sees is impressive steady growth in student performance.

A total of 41 percent of fourth-grade students taking the NAEP exam in 2019 were classified as at or above proficient, 32 percent were proficient, and 9 percent were advanced. A total of 81 percent of students met the NAEP Basic baseline or above. Nineteen percent of students performed below the NAEP Basic level in 2019.

The comparison of these values across the years in the decade from 1990 to 2019 is shown in figure 4.2. The figure gives the relative percentage of the nation's fourth graders within each of the achievement levels. It also shows a general shift upward toward fewer students below basic and more proficient and advanced students. Though there was an increase in the advanced and basic category from 2017 to 2019, the differences were not significant.

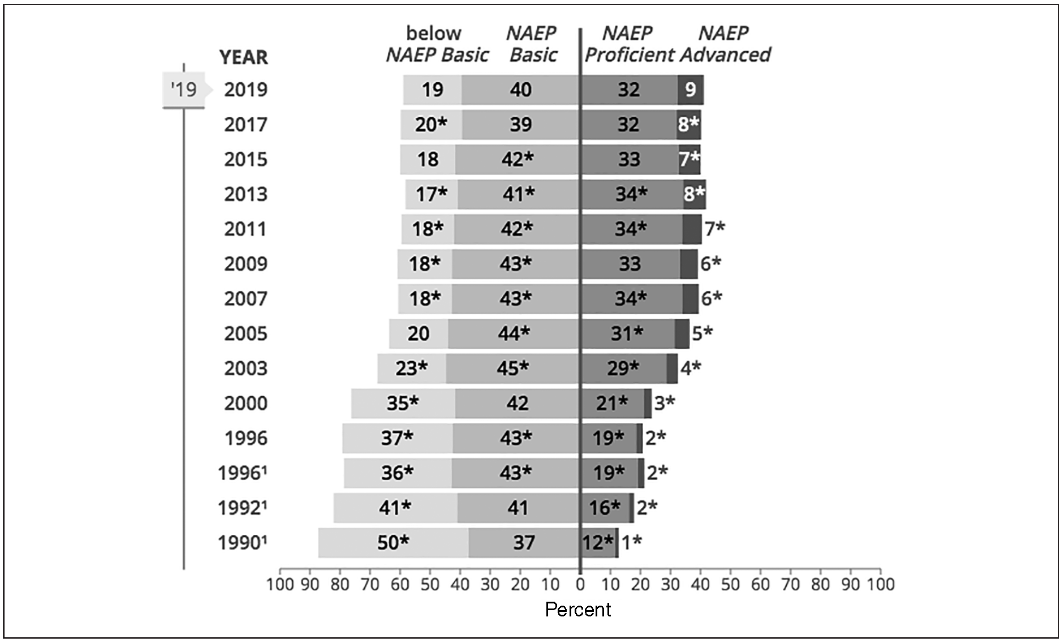


Fig. 4.2 Percentage of grade 4 students in each NAEP achievement level by year (Graphic from the National Center for Education Statistics. *NAEP Report Card: 2019 NAEP Mathematics Assessment*. Washington, D.C.: U.S. Department of Education, National Center for Education Statistics 2019. Used with permission.)

Overall, a total of 44 percent of males performed at or above Proficient. Male students had a mean score of 242, one point above the mean. A total of 38 percent of females also performed at or above Proficient. The mean score for female students was 239. A six-category race breakdown was given in the 2019 NAEP assessment. Students were classified as White, Black, Hispanic, Asian/Pacific Islander, Asian, Native Hawaiian/Other Pacific Islander, American Indian/Alaska Native, or being two or more races. Table 4.1 shows the average NAEP scores for each of these groups as well as the percentage of students meeting Proficient or above NAEP categories in 2019.

Table 4.1

Average Scores by Racial and Ethnic Background

Race/Ethnic Background	Average NAEP Score 2019	% of Students Meeting Proficient or above NAEP Benchmark
White	249	52%
Black	224	20%
Hispanic	231	28%
Asian/Pacific Islander	260	66%
Asian	263	69%
Native Hawaiian or Other Pacific Islander	226	28%
American Indian or Alaska Native	227	24%
Two or more races	244	44%

White student performance began the decade with a mathematics assessment mean scale score of 248 for 2009 and ended the decade a point higher at 249, which was eight points above the mean score of 241 for all students. Overall, white students have maintained scores between 248–250 throughout the decade. In 2011 and 2013, the grade 4 performance of White students was significantly different from the 2019 average score. All other increases and decreases throughout the decade were not significant. A total of 48 percent of the students taking the NAEP exam in 2019 were White. In 2009, the overall percentage of students that were White was 57 percent.

Black fourth graders started the decade in 2009 with a score of 222 and ended the decade two points higher at 224. Their 2009 score was significantly different from their 2019 score. Since then, while the scores increased, the increases were not statistically significant. A total of 15 percent of the students taking the NAEP exam in 2019 were Black. In 2009, the overall percentage of students that were Black was 16 percent.

Hispanic fourth graders began the decade at a mean of 227, a level that marked a statistically significant difference from their average of 231 in 2019. In addition, 2011 also marked a statistically significant increase in mean test scores compared to 2019 for Hispanic students. Overall, a total of 27 percent of the students taking the NAEP exam in 2019 were Hispanic. In 2009, the overall percentage of students that were Hispanic was 21 percent. The large percentage difference of Hispanic students taking the exam over the course of the decade represents the shifting demographics within the overall U.S. population.

A total of 6 percent of the test takers in 2019 were Asian/Pacific Islander students compared to 5 percent in 2009. The average test scores for this group of students went from 255 in 2009 to 260 in 2019. The difference is statistically significant.

The remaining three categories represented a very small percentage of the population of students taking the NAEP exam. Throughout the decade, 1 percent of the students taking the exam each testing year were Native American/Alaska Native and less than 1 percent were Native Hawaiian/Other Pacific Islander. Five percent of students were classified as Asian—a category that was named beginning in 2011; only 4 percent of students were categorized with an ethnicity of two or more races, increasing from only 2 percent at the beginning of the decade. Students identified as having two or more races ethnicity had an average score of 243 in 2009 and an average score of 244 in 2019. None of the average scores were significantly different throughout the decade.

NAEP 2019 Grade 8 Results

Figure 4.2 shows the national trend in eighth-grade student performance on the Main NAEP 2019 mathematics assessment. The eighth-grade mean score in the 2019 assessment was 282 points. This indicates a statistically significant decrease from the 2017 average test score of 283 points. While the average score decreased from the 2017 score, the overall trend of average test scores for eighth-grade students on the NAEP exam has been increasing since 1990. The average test score is up 19 points since the 1990 test.

Achievement Levels

In 2019, 31 percent of eight graders performed at the Below Basic level, and 35 percent of eighth-grade students performed at asic level. That is to say, in 2019, 66 percent of the nation's eighth-grade students were performing below the Proficient level. The remaining 34 percent of students were performing at or above the Proficient level. This group splits into 24 percent at the Proficient level and 10 percent at the Advanced level. Performance

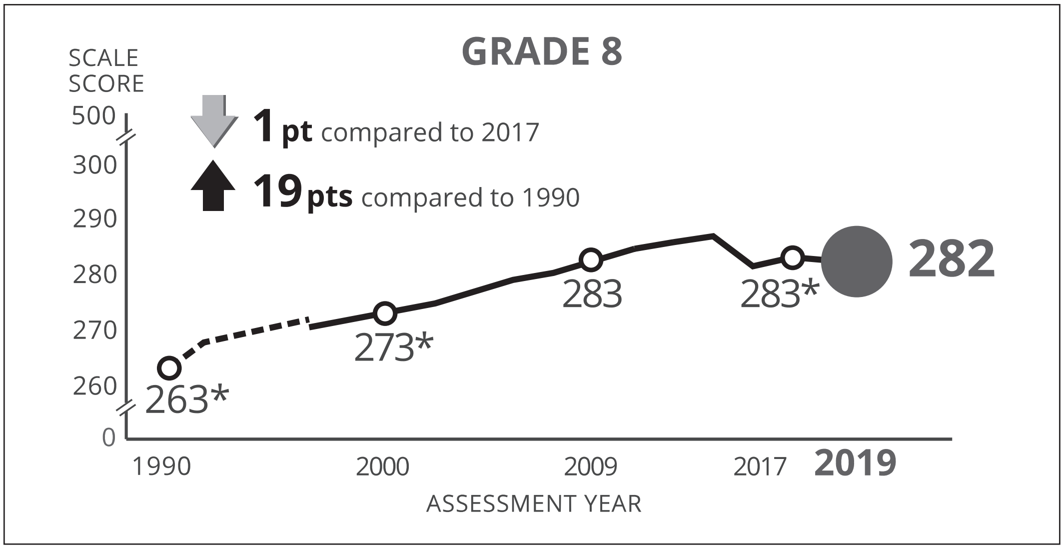


Fig. 4.3 Trend in Eighth-grade NAEP mathematics mean scale scores. (Graphic from the National Center for Education Statistics, *NAEP Report Card: 2019 NAEP Mathematics Assessment.* Washington, D.C.: U.S. Department of Education, National Center for Education Statistics, 2019. Used with permission.)

at or above the Proficient level on NAEP assessments demonstrates that students are solid, both conceptually and procedurally. The percentage of eighth-grade students performing at or above the Proficient level in 2019 has stayed the same as 2009, however, the breakdown of proficient and advanced has shifted. While in 2009, 26 percent of students scored proficient and only 8 percent scored advanced, in 2019, the proficient students dropped to 24 percent while the advanced students increased to 10 percent. These shifts resulted as statistically significant. Similar shifts occurred in the lower categories as well. In 2009, 27 percent of students categorized as below basic while in 2019, there were 31 percent. In the basic category, the percentages changed from 39 percent in 2009 to 35 percent in 2019. These shifts were also deemed to be statistically significant. Figure 4.3 shows the percentages of eighth graders at the various levels of mathematics achievement on NAEP assessments between 1990 and 2019.

Both male and female students had a mean score of 282, the same as the overall mean. Data exist that provide a basis to talk about trends across the past decade, 2009–2019, for students of White, Black, Hispanic, Asian/Pacific Islander, Asian, American Indian/Alaska Native, and students with two of more racial ethnic backgrounds. The groups' percentages of the 2019 student eighth-grade population were 49 percent, 14 percent, 26 percent, 6 percent, 6 percent, 1 percent and 3 percent respectively.

White students scored an average of 292 in 2019. This score was not significantly different than their mean scale score at the beginning of the decade in 2009 (the mean scale score in 2009 was 293). Black/African American students had a mean scale score of 260 in 2019. Again, this was not significantly different than the mean scale score at the beginning of the decade. Hispanic/Latino students had a 2019 mean scale score of 268, and students with two or more races had a mean score of 286. Students of Asian/Pacific Islander descent (310) had an increased mean scale score of 9 points since the beginning of the decade. This was a significant increase from 2009 to 2019. The inverse was true of American Indian/Alaska Native students. Their mean scale score in 2019 of 262 represented a significant drop of 6 points from 2017.

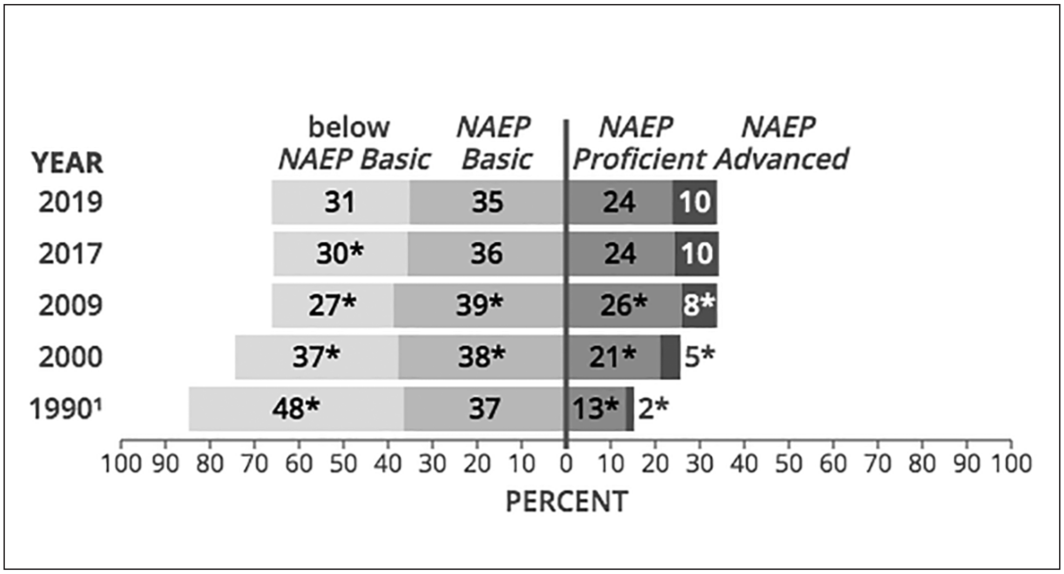


Fig. 4.4 Percentage of grade-8 students at each NAEP achievement level by assessment year. (Graphic from the National Center for Education Statistics, *NAEP Report Card: 2019 NAEP Mathematics Assessment*. Washington, D.C.: U.S. Department of Education, National Center for Education Statistics 2019. Used with permission.)

NAEP 2019 Trial Urban District Assessment (TUDA)

Starting in 2003 and continuing every two years since then, the National Assessment has conducted a Trial Urban District Assessment (TUDA), involving some of the largest school districts in the nation. In light of the fact that one-quarter of the nation's youth live in urban areas, the success of urban students (living in cities with more than 250,000 people) in preparing for postsecondary study in STEM fields is critical to the quality of the nation's workforce in coming decades. Hence, special monitoring of mathematical learning opportunities for these youths is crucial to ensure their progress as well as continued U.S. economic progress (https://www.nagb.gov/content/nagb/assets/documents/naep/naep-2019-tuda-one-pager.pdf).

The results of TUDA 2019 are reassuring in that they showed significant progress in some of the largest districts that have been involved in TUDA assessments and have implemented innovative programs directed toward the goals of STEM program improvement. Table 4.2—shown in two parts, (a) and (b)—contains trend data in mean mathematics scores for grades 4 and 8 from the 2019 TUDA assessment for the nation, large cities, and twenty-seven participating urban districts. The TUDA 2019 results are included in the Nation's Report Card (https://www.nationsreportcard.gov/mathematics?grade=4). Overall, average mathematics mean scores across both grades were relatively stable across the 27 districts participating in TUDA compared to prior assessments. Compared to 2017, five districts scored higher, one district scored lower, and 21 districts showed no significant change in mean scores. These results are encouraging but indicate that additional ground still needs to be gained to achieve the necessary STEM levels in urban settings. In fact, the large majority of the TUDA districts had average scale scores that were lower than the national average reported from the regular NAEP exam (https://www.nationsreportcard.gov/profiles/districtprofile?chort=1&sub=MAT&sj=XQ&sfj=NL&st=MN&year=2019R3).

Table 4.2

Trend in Mean NAEP Mathematics Scores for TUDA Districts: Grades 4 and 8, 2009–2019

a. Grade 4

	2009	2011	2013	2015	2017	2019	% ≥ Prof 2019
Large Cities	231	233	235	234	232	235	34
Albuquerque	NA	235	235	231	230	230	30
Atlanta	225	228	233	228	231	232	31
Austin	240	245	245	246	243	243	45
Baltimore City	222	226	223	215	215	216	15
Boston	236	237	237	236	233	234	32
Charlotte	245	247	47	248	244	246	49
Chicago	222	224	231	232	232	232	31
Clark County (NV)	NA	NA	NA	NA	230	235	33
Cleveland	213	216	216	219	214	218	13
Dallas	NA	233	234	238	234	235	32
Denver	NA	NA	NA	NA	229	235	35
Detroit	200	203	204	205	200	205	6
District of Columbia (DCPS)	220	222	229	232	231	235	36
Duval County (FL)	NA	NA	NA	243	248	244	45
Fort Worth (TX)	NA	NA	NA	NA	230	233	29
Fresno	219	218	220	218	221	224	18
Guilfort County (NC)	NA	NA	NA	NA	240	236	35
Hillsborough County (FL)	NA	243	243	244	245	242	41
Houston	236	237	236	239	235	235	31
Jefferson County (KY)	233	236	234	236	233	232	33
Los Angeles	222	223	228	224	223	224	20
Miami-Dade	236	236	237	242	245	246	47
Milwaukee	NA	NA	NA	NA	216	215	17
New York City	237	234	236	231	229	231	32
Philadelphia	222	225	223	217	214	217	18
San Diego	236	239	241	233	237	240	42
Shelby County (TN)	NA	NA	NA	NA	225	228	25

(Continued)

b. Grade 8, *continued*

	2009	2011	2013	2015	2017	2019	% ≥ Prof 2019
Large Cities	271	274	276	274	274	274	27
Albuquerque	NA	275	274	271	270	267	20
Atlanta	259	266	267	266	265	268	21
Austin	287	287	285	284	283	282	36
Baltimore City	257	261	260	255	255	254	10
Boston	279	282	283	281	280	279	35
Charlotte	283	285	289	286	287	288	41
Chicago	264	270	269	275	276	275	27
Clark County (NV)	NA	NA	NA	NA	272	272	24
Cleveland	256	256	253	254	257	253	8
Dallas	NA	256	253	254	268	264	15
Denver	NA	NA	NA	NA	272	275	29
Detroit	238	246	240	244	246	244	5
District of Columbia (DCPS)	251	255	260	258	262	269	24
Duval County (FL)	NA	NA	NA	275	275	274	25
Fort Worth (TX)	NA	NA	NA	NA	269	265	18
Fresno	258	256	260	257	255	254	11
Guilfort County (NC)	NA	NA	NA	NA	276	280	33
Hillsborough County (FL)	NA	282	284	276	277	276	29
Houston	277	279	280	276	273	274	25
Jefferson County (KY)	271	274	273	272	271	273	26
Los Angeles	258	261	264	263	267	261	16
Miami-Dade	273	272	274	274	274	276	27
Milwaukee	NA	NA	NA	NA	254	252	11
New York City	273	272	274	275	275	273	27
Philadelphia	265	265	266	267	260	256	16
San Diego	280	278	277	280	283	283	35
Shelby County (TN)	NA	NA	NA	NA	257	265	16

Notes:
[1] % ≥ "Prof" is the percentage of students in public schools at or above Proficient in 2019.
[2] "Large cities" includes students from all cities in the nation with populations of 250,000 or more, including the participating TUDA districts.
[3] NA is data not available because the district did not participate in TUDA that year.
[4] (Data from National Center for Education Statistics, https://www.nationsreportcard.gov/mathematics/districts/scores?grade=4)

NAEP 2012 Long-Term Trend Assessment Study

In conjunction with the main NAEP assessment, the National Center for Education Statistics also administers an additional NAEP assessment, the NAEP Long-Term Trend Assessment (LTTA), to a nationally representative sample of students. The LTTA administers the same assessment over time under the same conditions. Approximately 26,000 students took the LTTA in mathematics in 2012, the latest administration of the test. Initiated in 1973, the main goal of the LTTA is to measure student progress over time. In mathematics, the LTTA measures knowledge of mathematical facts, ability to carry out computations using paper and pencil, knowledge of basic formulas, and ability to apply mathematics to daily living skills.

The test for the Long-Term Trend Assessment (LTTA) in mathematics has remained relatively unchanged over time since its first administration in 1973. Consequently, the LTTA results provide a barometer for measuring today's students' achievement against the expected performance of previous generations—that is, these data help to answer the question, can students today still do what their parents were taught and expected to do? Minor changes have been made to the LTTA over its lifespan, but in each case statistical and validity studies have been carried out to ensure that the goals of the assessment and the reporting levels do not require a break in the trend reporting (NAGB 2015). The LTTA draws samples of 9-, 13-, and 17-year-olds. Data from the NAEP long-term trend assessments in mathematics from 1973 to 2012 are shown in figure 4.4. These results were the same results reported in the 2016 ICME U.S. Fact Book report.

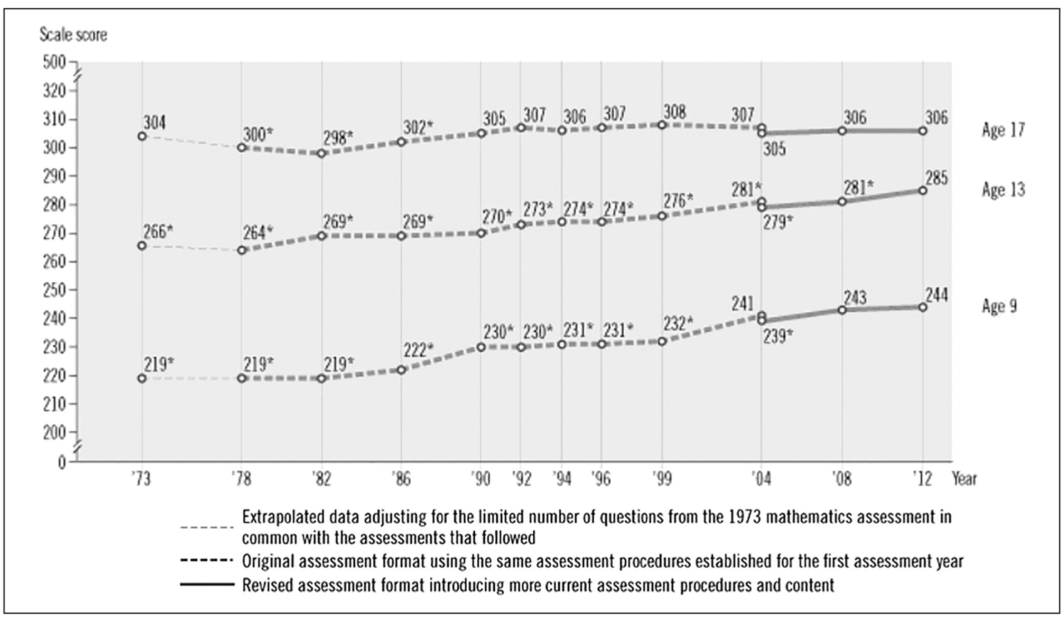


Fig. 4.5 Trends in *NAEP Mathematics Average Scores for 9-, 13-, and 17-Year-Old Students from 1973–2012* (https://nces.ed.gov/nationsreportcard/pubs/main2012/2013456.aspx)

LTTA analyses over time have to be considered carefully, given the changes that were implemented with the 2004 assessment. Two statistical bridge studies were conducted, and the comparable content between the two has allowed the extension of the age-based trend lines as shown in figure 4.5, with the solid shading representing comparable levels of performance. A dotted line reflects data for the period from 1978 to the changes that went into effect with the 2004 LTTA administration, and the solid line represents data from 2004 to the 2012 assessment. A comparison from 1973 to 1978 was made, but too many factors changed in that period to incorporate them as part of the first trend line. Thus, it is possible to make statistically meaningful comparisons over time from 1978 to the present, but one must be careful from a validity standpoint when making comparisons that involve the content areas previous to 1978.

In 2004, 2008, and 2012, the LTTA for mathematics was conducted by using the new long-term trend assessment test forms. The new assessment can change gradually over time, like Main NAEP, contrary to the invariant assessment that was used from 1973 through 1999. A bridge assessment has indicated that the continuation of the trend line between the old and new assessments is appropriate (Perie, Moran, and Lutkas 2005). Figure 4.5 contains the data for the performances of students at each of the three age ranges in the NAEP LTTA. The 2008 level of performance for both 9- and 13-year-old groups is statistically higher than that of the same age groups at every testing period from 1999 or back to 1978. The trend line for 17-year-olds shows a pattern of insignificant variation from 1990 to 2008 (Rampey, Dion, and Donahue 2009). These findings indicate that, on average, elementary and middle school students in 2008 had a better command of the fundamental concepts and skills deemed important in 1978 than their age-related peers across the history of the assessment. The 17-year-old group, with the slight exception observed in the 1978–1986 period, showed no appreciable growth or decline in their command of these basic concepts and skills over the period constituting the history of the assessment.

The LTTA was repeated again in 2012. The data from that cycle provides additional support for the interpretation of the trends. For a full analysis, one needs to read the reports of the 2004, 2008, and 2012 LTTA studies in mathematics (Perie, Moran, and Lutkus 2005; Rampey, Dion, and Donahue 2009; NCES 2013). The three data points for age-group performance provide a third data point for each of the new trend lines. At the age-9 level, the 2012 performance scale score was judged to be significantly higher than the one for 2004 but not significantly different from 2008. The short trend line shows a modest increase over the eight-year span from 2004 to 2012. At the age-13 level, the 2012 performance scale score was judged to be significantly higher than the 2004 and 2008 scale score levels. This indicates a slightly stronger trend of improvement since 2004. At the grade-12 level, there were no significant differences in scale score performance from 2012 over performance for grade-12 students in either 2004 or 2008.

When one looks back to the trend lines from the 1978 assessment to the 2012 assessment, one sees the long-term changes in student performance in mathematics over a relatively constant test. At the age-9 level, the 2012 data point was judged significantly higher than that of any other data points for that age level for each assessment from 2004 back to 1978. There was no significant difference between the 2012 and 2008 assessment. This is a strong indication that U.S. students at age 9 perform, on average, better than their parents and grandparents did on the “basics” in mathematics.

At the age-13 level, the evidence is even stronger. Under the metrics for both trend lines, the mean score scale value for 2012 for age-13 students was judged significantly higher than the mean performance for all of the assessments from 2008 back to 1978. This again affirms that in 2012, 13-year-old U.S. students, on average, performed better on the “basics” than their counterparts in their parents’ and grandparents’ generations did.

The LTTA results provide a different profile for U.S. students at age 17. At this level, the outcomes are much the same as those seen in the grade 12 Main NAEP assessments, and they paint a picture of little change over time. Here, the short LTTA trend line shows no difference in mean scale scores for 2004, 2008, and 2012.

The grade-12 results aside, the NAEP LTTA results confirm that at the age-9 and age-13 levels, today's U.S. schoolchildren are performing at a significantly higher level than their parents and grandparents did in mathematics. This speaks positively to progress in curricula and teaching at these levels. While the LTTA measure has been important to understand progress in curricula and teaching at these levels, the future of LTTA is unknown. The 2016 LTTA has been postponed two times already; once to 2020 and once to 2024 (Haertel 2016). The next LTTA is currently scheduled for 2024.

2019 State NAEP Results

Table 4.3 illustrates the vast differences among states' mean achievement scores and the percentages of students reaching the level identified as Proficient.

Table 4.3

Mean State Main NAEP Scores and Percentage Proficient for 2019

	Grade 4		Grade 8	
Jurisdiction	**Mean NAEP Scale Score—2019**	**Percent Proficient or about—2019**	**Mean NAEP Scale Score—2019**	**Percent Proficient or about—2019**
Alabama	230	28	269	21
Alaska	232	33	274	29
Arizona	238	37	280	31
Arkansas	233	33	274	27
California	235	4	276	29
Colorado	242	44	285	7
Connecticut	243	45	286	39
Delaware	239	39	277	29
Florida	246	48	279	31
Georgia	238	36	279	31
Hawaii	239	40	275	28
Idaho	242	43	286	37
Illinois	237	38	283	34
Indiana	245	47	286	37
Iowa	241	42	282	33
Kansas	239	40	282	33
Kentucky	239	40	278	29
Louisiana	231	29	272	3
Maine	241	42	282	34
Maryland	239	39	280	33
Massachusetts	247	50	294	47
Michigan	236	36	280	31
Minnesota	248	53	291	44
Mississippi	241	39	274	24

(Continued)

Table 4.3 *Continued*

Mean State Main NAEP Scores and Percentage Proficient for 2019

Jurisdiction	Grade 4		Grade 8	
	Mean NAEP Scale Score—2019	Percent Proficient or about—2019	Mean NAEP Scale Score—2019	Percent Proficient or about—2019
Missouri	238	39	281	32
Montana	241	43	284	36
Nebraska	244	45	285	37
Nevada	236	34	274	26
New Hampshire	245	46	287	38
New Jersey	246	48	292	44
New Mexico	231	29	269	21
New York	237	37	280	34
North Carolina	241	41	284	37
North Dakota	243	44	286	37
Ohio	241	41	286	38
Oklahoma	237	5	276	26
Oregon	236	37	280	31
Pennsylvania	244	47	285	39
Rhode Island	239	40	276	29
South Carolina	237	36	276	29
South Dakota	241	43	287	39
Tennessee	240	40	280	31
Texas	244	44	280	30
Utah	244	46	285	37
Vermont	239	39	287	38
Virginia	247	48	27	38
Washington	240	39	286	40
West Virginia	231	30	272	24
Wisconsin	242	45	289	41
Wyoming	246	48	286	37
Other Jurisdictions				
District of Columbia	235	34	269	41
DoDEA	250	54	292	31

The national average scale score was reported as 240, with 40 percent of students scoring at or above Proficient for the fourth-grade level. Compared to the national average scale score, the table reveals that 15 jurisdictions have scale scores significantly above the national average, 20 do not have a significant difference, and 17 jurisdictions performed significantly below the national average.

The 2019 eighth-grade student state results show that 22 jurisdictions performed significantly higher than the national average, 14 had no differences, and 16 performed significantly lower than the national average. The national average was 281 for eighth-grade students.

Consortia for the Assessment of Common Core–Related Achievement

At the elementary, middle school, and high school levels, states across the United States test students on mathematics to measure student achievement. As mentioned in chapter 2, the Partnership for Assessment of Readiness for College and Careers (PARCC) and the Smarter Balanced Assessment Consortium (SBAC) were two assessments consortia developed as a result of a competition that was part of the Race to the Top program of the Department of Education. Although initially linked to the Department of Education for initial funding, these consortia were then funded from the states that belong to them to provide the assessments online. Further, although the two consortia are not directly or fiscally linked to the Common Core State Standards program, the tests developed by the consortia are tied to the assessment of the CCSSM program.

Partnership for Assessment of Readiness for College and Careers (PARCC)

The first full-scale administration of the PARCC had 5 million students taking tests across grades 3–8 and in grade 11 during the 2014–15 school year. Eleven member states and the District of Columbia had students taking part in the 2014–15 school year's testing. In 2019, there were only three active members (District of Columbia, Louisiana, and Massachusetts) remaining. Due to the small number of states administering the assessments, the test results are not representative and thus not reported.

Smarter Balanced Assessment Consortium (SBAC)

The first full-scale administration of the SBAC assessment had more than 6 million students taking tests across grades 3–8 and in grade 11 during the spring summative assessments in the 2014–15 school year. Students were selected from 18 member states as well as from the Bureau of Indian Education and the U.S. Virgin Islands. In 2019, thirteen states, the Bureau of Indian Education, and the U.S. Virgin Islands were members. Similar to the PARCC assessments the test results are not representative of national trends; as such, the results are omitted from this report.

Other State Testing Programs

To comply with the federal Every Student Succeed Act–requirements, most U.S. states contract with a test development vendor, usually a for-profit company, to create tests for grades 3–8 and high school addressing state adopted mathematics standards. During the 2019–20 school year, companies such as Data Recognition Corporation, Pearson, and Measured Progress among several others help contract with states not participating in PARCC or SBAC. Some states use the SAT or the ACT (or both) to comply with the high school assessment requirement. Others use end-of-course tests in subject areas like algebra 1. A report on the varied approaches to assessment in different states is available at https://www.future-ed.org/wp-content/uploads/2019/09/REPORT_New-TestingLandscape-1.pdf.

SAT and ACT

Typically, a student in the United States applies for college in the twelfth grade, the last year of high school. The selectivity of colleges in the United States varies, from community colleges and postsecondary institutions that require no more than a high school diploma or its equivalent to selective colleges that may accept 10 percent or fewer of their applicants. The selectivity of an institution sometimes varies with the academic major that a student expresses an intent to pursue. Because college entrance examination scores provide a quantifiable and comparable measure for students coming from different high schools and different areas of the country, they are often used by college admissions. As a result, most students anticipating college in the United States take a college entrance examination during their junior or senior year of secondary school.

Two major and independent college admission examinations exist in the United States: the SAT test, administered by the College Board, and the ACT test, administered by ACT, Inc. The SAT assesses high school students' general capabilities in critical areas of reading, writing, and mathematics. A student's results for each of the two sections (1. Evidence-based Reading and Writing and 2. Mathematics) of the SAT are reported on a 200–800 scale. A student's overall college-readiness is reported by the sum of the individual scale scores on a 400–1600 scale. The mathematics test employs multiple-choice items and open-ended items for which the response is gridded into a special array of bubbles on an optically scanned answer sheet. The mathematics portion of the test covers number and operations; algebra and functions; geometry; and statistics, probability, and data analysis. The SAT focuses on three main topic areas that students are most likely to encounter in college—algebra (19 questions), problem solving and data analysis (17 questions), and mathematics that leads to advanced topics (16 questions)—and six questions testing understanding of additional topics in geometry (https://collegereadiness.collegeboard.org/about/key-features). The items use real-world contexts and focus on problem-solving skills. One portion of the SAT allows the use of a calculator and one does not.

Over 2.2 million students took the SAT in 2019. Of these, 52 percent were female and 48 percent were male. The racial/ethnic background of the SAT takers was 43 percent White, Black/African American 12 percent, Hispanic/Latino 25 percent, Asian 10 percent, American Indian/Alaska Native 1 percent, two or more races 4 percent, and no response was 5 percent. The mean SAT mathematics score was 528 (out of 800). College and career readiness benchmarks were established by examining the relationship between SAT scores and grades in related college courses at two- and four-year colleges. If a student meets the benchmark, then that student has a 75 percent likelihood of earning a C or better in the related college courses. In 2019, 48 percent of students taking the SAT met the mathematics benchmark score (530 or more). Males had a mean SAT score of 537 and females had a mean score of 519. A total of 45 percent of females met the benchmark for readiness and 51 percent of males met the benchmark. Breaking out the total number of students meeting the benchmark by race/ethnic groups, the percentages for 2019 were White students, 59 percent; Black/African American, 22 percent; Hispanic/Latino, 31 percent; and Asian, 80 percent.

The ACT assesses high school students' general subject-matter knowledge and college or workforce readiness in four skill areas: English, mathematics, reading, and science. The test is composed entirely of multiple-choice items, and each of the four skill areas is reported on a 1–36 scale. A general summary score on the same 1–36 scale is used to report a student's overall skill level (ACT 2015). Calculator use is allowed on the mathematics portion of the ACT. More than 1.78 million students took the ACT in 2019, equating to approximately 52 percent of the U.S. high school graduating class. The mean cumulative test score was 20.7. In mathematics, the mean score was 20.3. Similar to the SAT, the ACT also provides benchmarks for college readiness. The benchmark represents the level of achievement required for a student to have a 75 percent chance of obtaining a C or higher in corresponding first-year college courses. According to the benchmarks set, a total of 26 percent of students taking the ACT in 2019 met the benchmark overall in all four subjects tested. In mathematics specifically, a total of 39 percent of test takers met the benchmarks. Broken down by race/ethnicity 48 percent of White students taking the ACT met the benchmark, 12 percent of Black/African American, 25 percent of Hispanic/Latino students, and 68 percent of Asian students met the benchmark.

Overall mean test scores have remained relatively consistent over the past decade. At the beginning of the decade in 2009, the mean math test score for the SAT was 515 and the mean math test score for the ACT was 21.0. Over the decade, these scores have fluctuated up and down by 16 points on the SAT (low point of 508 in 2016 and a high point of 531 in 2018) and 1 point on the ACT (low point of 20.3 in 2019 and high point of 21.1 in 2011).

Graduating seniors' mean mathematics performance on both the SAT and ACT has shown substantial improvement over the past decade (see table 4.4).

Table 4.4

National Mean Grade 12–Scores for Mathematics on the SAT and ACT Exams

Year	Test	
	SAT Math	**ACT Math**
2009	515	21.0
2010	516	21.0
2011	514	21.1
2012	514	21.1
2013	514	20.9
2014	513	20.9
2015	511	21.0
2016	508	20.6
2017	527	20.7
2018	531	20.5
2019	528	20.3

SAT Subject Tests

In addition to the SAT examination itself, the College Board also administers subject area tests to allow students to gauge their progress as well as to have evidence to offer to colleges of their suitability for admission or higher course placement. Two of these course-level tests are the Mathematics Level 1 and Mathematics Level 2 examinations. Mathematics Level 1 assesses students' knowledge of the first three years of a college preparatory curriculum. More specifically, the test covers numbers and operations (10–14 percent of the questions); algebra and functions (38–42 percent of the questions); geometry and measurement (38–42 percent of the questions);, and data analysis, statistics, and probability (8–12 percent of the questions). Mathematics Level 2 covers this same material at an advanced level with the addition of the study of trigonometry and elementary precalculus. Both tests consist of 50 multiple-choice questions and allot 60 minutes of time for completion. Scores are on a 200–800 points scale. These two tests are described in more detail in College Board publications found on its website, along with sample examinations, at http://www.SATSubjectTests.org. Students are allowed to use most graphing calculators from a published list that includes clear information on the types of more advanced calculators that are not allowed. In 2019, there were 45,745 students taking the Mathematics Level 1 subject exam and approximately 141,951 students taking the Mathematics Level 2 exam. The mean scores were 651 and 721, respectively. The distribution of test scores is given in table 4.5.

Table 4.5

Number of Student Scoring Less Than 600, More Than 700, and More Than 750

	Number of Students Scoring < 600	**Number of Students Scoring > 700**	**Number of Students Scoring > 750**
Math Level 1	20,585	11,893	3,659
Math Level 2	28,390	78,073	56,780

An examination of student performance on these tests gives a picture of student performance on a pair of standardized tests covering common content over time. Table 4.6 provides a look at student performance on the SAT Mathematics Level I and SAT Mathematics Level II examinations for 2010 and 2015. The data illustrate the growth that took place from 2010 to 2015 for both tests. It is impossible to link back further because of some changes made in the calculator-use regulations as the examinations were changed from Math IC and Math IIC, which allowed the use of calculators. The newer tests have some items that demand that test takers use calculators in order to work the problems in a reasonable amount of time. At both Mathematics Level I and Mathematics Level II, one sees a movement into the upper three score intervals. The distribution of these percentages of scores within the intervals also plays a role in the difference of the performances for each test from 2010 to 2015.

Table 4.6

Distribution of Percentages of Student Performance on the SAT Mathematics Level I and Mathematics Level II Tests for the Years 2010 and 2015

SAT Mathematics Level I			**SAT Mathematics Level II**		
Math Level I Scores	**2015**	**2010**	**Math Level II Scores**	**2015**	**2010**
750–800	9	6	750–800	38	25
700–749	19	16	700–749	15	13
650–699	18	17	650–699	16	15
600–649	17	20	600–649	14	15
550–599	13	15	550–599	10	14
500–549	11	11	500–549	5	9
450–499	7	7	450–499	2	6
400–449	4	5	400–449	1	2
350–399	2	3	350–399	0	0

Table 4.6 *Continued*

Distribution of Percentages of Student Performance on the SAT Mathematics Level I and Mathematics Level II Tests for the Years 2010 and 2015

SAT Mathematics Level I			SAT Mathematics Level II		
Math Level I Scores	**2015**	**2010**	**Math Level II Scores**	**2015**	**2010**
300–349	1	1	300–349	0	0
250–299	0	0	250–299	0	0
200–249	0	0	200–249	0	0
Mean	619	605	Mean	690	649
Total tests	65,319	85,109	Total tests	144,772	163,713

Data from College Board SAT Mathematics Test Archives (http://research.collegeboard.org/programs/ap/data)

Advanced Placement Examinations

As is discussed further in chapter 8, the College Board offers a set of Advanced Placement (AP) examinations for students who study college-level courses considered "advanced" for high school level study. The examinations allow for placement of students in mathematics courses beyond introductory courses, following a syllabus provided by the College Board. This syllabus is developed by a committee of subject-matter experts for the content of the course at the undergraduate level. Additional subject-matter experts vet the content on a regular basis for concurrence and pacing relative to courses at the freshman or sophomore level in college. Students study the material for a year under the leadership of a teacher who is knowledgeable about the syllabus and the manner in which students' understanding of the content of the course will be tested. The test is administered in April, near the end of the school year, and is then graded by a select group of teachers and university faculty who use a common rubric for the open-ended items to assign a final score of 1, 2, 3, 4, or 5. Scores of 3 or better are considered meeting college-level expectations. The three examinations for the yearlong AP courses that are relevant to this report are—

- Calculus AB, covering topics in differential and integral calculus that are roughly equivalent to those in a first-semester college calculus course;
- Calculus BC, which explores the concepts, methods, and applications of differential and integral calculus, including topics such as parametric, polar, and vector functions, and series; and
- Statistics, covering major concepts and tools for collecting, analyzing, and drawing conclusions from data that are the focus of a one-semester, introductory, non-calculus-based college course in statistics.

Table 4.7 illustrates the number of students taking each of the three different AP exams over the past decade. Also depicted in the table is the mean test score for each year based on reported AP test scores that range from 1 to 5. As seen in the table, the mean scores are consistent over the past decade, fluctuating very little for all three tests. The means for the Calculus AB test is around 2.9, the means for the BC test are about a point higher at approximately 3.7, and the means for the Statistics exam is around 2.8. Overall, the number of students taking the tests has increased. The numbers in 2019 reflect the vast popularity of AP testing in the United States, with more than 250,000 students taking the Calculus AB test, approximately 139,000 taking the Calculus BC test, and approximately 219,000 taking the AP Statistics test.

Table 4.7

Participation and Score Data for AP Tests from 2009–2019

Year	Number of Students Taking AP Calculus AB Test	Mean Score of AP Calculus AB Test	Number of Students Taking AP Calculus BC Test	Mean Score of AP Calculus BC Test	Number of Students Taking AP Statistics Test	Mean Score of AP Statistics Test
2009	230,588	2.99	72,965	3.72	116,876	2.83
2010	245,867	2.81	78,998	3.96	129,899	2.84
2011	255,357	2.82	85,194	3.77	142,910	2.82
2012	266,994	2.97	94,403	3.87	153,859	2.83
2013	282,814	2.96	104,483	3.73	169,508	2.80
2014	294,072	2.94	112,113	3.81	184,173	2.86
2015	302,532	2.86	118,707	3.72	195,526	2.80
2016	308,215	2.96	124,931	3.80	206,563	2.88
2017	316,099	2.93	132,514	3.78	215,840	2.72
2018	308,538	2.94	139,376	3.97	222,501	2.88
2019	300,659	2.97	139,195	3.80	219,392	2.87

The tests are taken by students across all four years of high school as well as some very accelerated special students. Students taking the examinations are sometimes offered advanced placement in the calculus program at a university; in other cases, students are given credit on the basis of their performances on the AP tests. Depending on the institution, these outcomes can range from no credit given but advancement to the second-semester course to credit given for the first two semesters of calculus.

International Tests and Comparisons

In 2015, the Trends in International Mathematics and Science study (TIMSS) assessed fourth-grade students in 43 countries and eighth-grade students in 34 countries across the world. TIMSS advanced data for twelfth-grade students were collected in 9 countries. TIMSS scores are reported on a scale of 0 to 1000. The mean scores for students in the U.S. were 539 for fourth-grade students and 518 for eighth-grade students. Both these mean scores were higher than the international mean. In fourth-grade mathematics, the U.S. average score was higher than 30 of the 42 countries. In eighth grade, the U.S. mean score was higher than 21 of the 33 countries tested. In the Advanced TIMSS mathematics assessment, the mean U.S. score was 485, 15 points lower than the center point.

The Program for International Student Assessment (PISA) is coordinated by the Organization for Economic Cooperation and Development (OECD). It measures the performance of 15-year-old students in mathematics within a real-world context. Thirty-five countries were assessed in 2015. U.S. students had a mean score of 470 on the 2015 assessment. This was lower than the international average of 490. U.S. students had a mean score lower than 27 of the 34 countries tested (Snyder, Brey, and Dillow 2018).

Chapter 5: State Standards, Policy Documents, Curricular Trends, and New Directions

It has been nearly a decade since the release of the *Common Core State Standards for Mathematics* (CCSSM; NGA and CCSSO 2010a, 2010b). Although the majority of states and the District of Columbia continue to implement the CCSSM or a revised version developed by the state, the number of states who have or are developing alternatives continues to increase. There continues to be no national mandate or even expectation that there will be a common set of mathematics standards across the states, nor does there seem to be any support for one. Professional organizations such as NCTM, ASA, AMATYC, SIAM, and COMAP have released documents that focus on recommendations for equitable structures and instructional practices as well as specific content of statistics and mathematical modeling that can inform state-level recommendations in mathematics education no matter what set of standards are adopted. This chapter discusses the status of state standards post-CCSSM, provides an overview of recent recommendations, and highlights the emergence of mathematics pathways at the high school and college level as a means of promoting equity and student success.

Status of State Standards

As mentioned in chapter 2, as of 2019, 39 states continue to implement CCSSM or a revised version, one state continues to implement while reviewing, one state is developing new standards to replace the Common Core, and six states that had formerly adopted the Common Core have implemented alternative statewide standards (ccrslegislation.info). The Thomas B. Fordham Institute has been reviewing state academic standards since 1997. The Fordham Institute's report *State of State Standards—and the Common Core—in 2010* (Fordham Institute 2010) stated that the Common Core State Standards for Mathematics were stronger than the mathematics standards in 39 states; the report encouraged these states to adopt them rather than developing or revising their own. The Institute was hopeful that CCSSM would be adopted by each state and that there would not be a need for further analysis of state standards. However, by 2018, it was clear to the Fordham Institute that it was time to evaluate state standards in mathematics and English language arts. It's new report, *The State of State Standards Post-Common Core* (2018), focuses on ten states that the panel of experts felt had made the most substantive changes or never adopted the Common Core State Standards for Mathematics. Based on a review of these states and a reexamination of CCSSM, the report highlights recommendations the panel felt to be worthy of broader adoption as well as mistakes that states should avoid. The Common Core standards and standards from the 10 states were evaluated for Content and Rigor on a scale of 1 to 7 and Clarity and Specificity on a scale of 1 to 3. Only one of the 10 states received a total score of 9 out of 10 (same as CCSSM). Standards from three of the states were rated as "good," five as "weak," and one as "inadequate."

The report provides detailed advice to the 10 states and identifies some positive national trends attributed in part to the influence of CCSSM:

- A stronger focus on arithmetic in K–grade 5, where the priority should be ensuring students' mastery of foundational skills, such as counting and flexibly computing with whole numbers, decimals, and fractions as well as their understanding of the place-value principle

- More coherent treatment of proportionality and linearity in middle school, including rates and ratios, slope, and linear relationships and functions (e.g., $y = mx + b$)
- An appropriate balance between conceptual understanding, procedural fluency, and application, each of which is an essential dimension of mathematical thinking
- Better organization and teacher supports, including focused introductions for individual grade levels and courses, mathematically coherent organizational approaches that highlight the connections between standards, and helpful ancillary materials. (p. 7)

Acknowledging that CCSSM is not perfect and adjustments might always be necessary, the report emphasizes that states can and should learn from the experience of other states that have continued to implement CCSSM. Broad recommendations that are worth noting include devoting resources to implementing "sustained, coherent, and subject-specific professional development" (p.8) and "articulating clear pathways in high school math that are specifically aligned with specific postsecondary and labor market outcomes" (p.9). The latter recommendation was based on data that suggest that, in most cases, although standards are listed for particular courses, it is not clear how the courses fit together or what the courses prepare students for. The report recommends that all paths include algebra, geometry, and statistics and probability and that students take four years of mathematics in high school (Fordham Institute 2018). These recommendations are consistent with those of NCTM's recent publication, *Catalyzing Change in High School Mathematics: Initiating Critical Conversations* (2018).

Catalyzing Change in High School Mathematics

NCTM's publication *Catalyzing Change in High School Mathematics* (2018) challenges high school mathematics instruction to meet the current and future needs of 21st-century students and teachers. *Catalyzing Change in High School Mathematics* was written with multiple audiences in mind, from teachers and teacher leaders to administrators and school boards, to high school counselors, and college and university admissions and faculty. It identifies existing challenges in high school mathematics and indicates directions for improvement. The Council decided to focus specifically on high school mathematics because at the high school level, mathematics achievement has been flat over the last thirty years, on the basis of the National Assessment of Education Progress (NAEP), as compared with the progress made at the elementary and middle levels (NCTM 2018, p. xii).

Catalyzing Change makes four key recommendations that it states must be addressed in order to achieve a vision of high school mathematics that works for each and every student. The recommendations are that—

- each and every student should learn the Essential Concepts in order to expand professional opportunities, understand and critique the world, and experience the joy, wonder, and beauty of mathematics;
- high school mathematics should discontinue the practice of tracking teachers as well as the practice of tracking students into qualitatively different or dead-end course pathways;
- classroom instruction should be consistent with research-informed and equitable teaching practices; and

- high schools should offer continuous four-year mathematics pathways with all students studying mathematics each year, including two to three years of mathematics in a common shared pathway focusing on Essential Concepts, to ensure the highest-quality mathematics education for all students. (NCTM 2018, p.7)

Among the structural obstacles addressed in *Catalyzing Change in High School Mathematics* are student and teacher tracking. *Catalyzing Change* recommends student tracking (in which students are placed into coursework or rigid pathways that do not prepare them for continued study of mathematics) be eliminated so that each and every student has access to high-quality content and instruction that will keep postsecondary study of mathematics an option. Teacher tracking is also a barrier to all students having access to high-quality instruction and support. *Catalyzing Change* recommends teachers within each school have balanced teaching assignments whenever possible. Balancing teacher assignments and collaborative development by teachers can lessen isolation and burn out, expand a teacher's repertoire of teaching strategies and activities, deepen overall knowledge of the curriculum, and build a collective sense of responsibility for all students. Balancing teacher assignments also strengthens the faculty and expands their capacity to support all students (Gutiérrez 2002; Strutchens, Quander, and Gutiérrez 2011). Classroom instruction for all students should be based on research-informed and equitable teaching practices such as those discussed in NCTM's *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014).

Catalyzing Change identifies a set of Essential Concepts in the content areas of number, algebra and functions, geometry and measurement, and statistics and probability. The Essential Concepts do not represent a set of standards but instead the "most critical content—the deep understandings that are important for students to remember long after they have forgotten how to carry out specific techniques or apply particular formulas" (p. 37). In addition to the Essential Concepts all students should engage in content and appropriate mathematical modeling activities and understand the importance of reasoning and proof in mathematics.

Catalyzing Change recommends four years of mathematics instruction for all students, starting with a common initial pathway through all of the Essential Concepts for the first two to three years of high school, recognizing that to address all the concepts in two years is unrealistic since at least two and a half years are needed to address all the Essential Concepts. A common pathway supports a single curriculum and universal rigorous and engaging instruction to all students in a school setting as a means to avoid the creation of separate and unequal tracks. Beyond the common pathway, students should continue to study mathematics that are necessary to graduate from high school prepared for a wide variety of choices among postsecondary study and career options as well as for active participation in a democratic society.

Catalyzing Change states that it is critical that the common shared pathway incorporating the Essential Concepts as well as any follow-up courses contain rigorous mathematics and multiple opportunities for reasoning and sense making. As defined by *Catalyzing Change,* such courses should have the following characteristics:

- Require clarity and precision in reasoning
- Have focused and significant mathematics learning standards
- Maintain the integrity of the mathematical standards

- Are part of a coherent mathematical learning progression (i.e., they are courses that prepare students to continue their study of mathematics; they are not dead-end courses)
- Approach the mathematics in an instructionally balanced way that includes attention to conceptual understanding, procedural fluency, problem solving, and mathematical reasoning and critical thinking practices (NCTM 2018, p. 84)

In April 2020, NCTM released two additional documents, *Catalyzing Change in Early Childhood and Elementary Mathematics* (NCTM 2020a) and *Catalyzing Change in Middle School Mathematics* (NCTM 2020b). These documents each present a set of key recommendations that parallel the high school recommendations (see table 5.1). The new publications provide detail on how the recommendations play out across PK–grade 12, highlighting which issues are common and which are unique to each level. They provide illustrative examples and conversation starters and, similar to the high school document, a set of potential actions for teachers, schools, parents and community members, policy makers, and postsecondary education as well as NCTM.

Table 5.1

Catalyzing Change in School Mathematics Key Recommendations (NCTM 2019. Used with permission.)

	Early Childhood and Elementary School	Middle School	High School
Broaden the Purposes of Learning Mathematics	Each and every child should develop deep mathematical understanding as confident and capable learners; understand and critique the world through mathematics; and experience the wonder, joy, and beauty of mathematics.	Each and every student should develop deep mathematical understanding; understand and critique the world through mathematics; and experience the wonder, joy, and beauty of mathematics, which all contribute to a positive mathematical identity.	Each and every student should learn the Essential Concepts in order to expand professional opportunities; understand and critique the world; and experience the wonder, joy, and beauty of mathematics.
Create Equitable Structures in Mathematics	Early childhood and elementary mathematics should dismantle inequitable structures, including ability grouping and tracking, and challenge spaces of marginality and privilege.	Middle school mathematics should dismantle inequitable structures, including tracking teachers as well as the practice of ability grouping and tracking students into qualitatively different courses.	High school mathematics should discontinue the practice of tracking teachers as well as the practice of tracking students into qualitatively different or dead-end course pathways.

(Continued)

	Early Childhood and Elementary School	Middle School	High School
Implement Equitable Mathematics Instruction	Mathematics instruction should be consistent with research-informed and equitable teaching practices that nurture children's positive mathematical identities and strong sense of agency.	Mathematics instruction should be consistent with research-informed and equitable teaching practices that foster students' positive mathematical identities and strong sense of agency.	Classroom instruction should be consistent with research-informed and equitable teaching practices.
Develop Deep Mathematical Understanding	Early childhood settings and elementary schools should build a strong foundation of deep mathematical understanding, emphasize reasoning and sense-making, and ensure the highest-quality mathematics education for each and every child.	Middle schools should offer a common shared pathway grounded in the use of mathematical practices and processes to coherently develop deep mathematical understanding, ensuring the highest-quality mathematics education for each and every student.	High schools should offer continuous four-year mathematics pathways with all students studying mathematics each year, including two to three years of mathematics in a common shared pathway focusing on the Essential Concepts, to ensure the highest-quality mathematics education for all students.

Statistics Education Efforts of the American Statistical Association

The American Statistical Association (ASA) is dedicated to and over the last decade, has become increasingly involved in enhancing statistics education at all levels. The ASA has several initiatives and committees dedicated to the advancement of education: namely, the ASA/NCTM Joint Committee on K–12 Statistics Education, the ASA/MAA Joint Committee on Undergraduate Statistics Education, and the ASA/AMATYC Joint Committee. The ASA also focuses on providing resources for K–12 teachers and teacher educators as well as developing national guidelines for statistics education and supporting projects related to education. In 2016, the ASA created a K–12 Statistical Ambassador position in order to emphasize its commitment to providing leadership in the creation and presentation of professional development materials for teacher educators and teachers. The ambassador presents at national conferences, conducts workshops, collaborates with ASA chapters to enhance their education initiatives, and assists in outreach to the STEM education community (https://www.amstat.org/ASA/Education/ASA-K-12-Statistical-Ambassador.aspx). Notable reports published by the ASA over the last decade include the following:

- *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A Pre-K–12 Curriculum Framework* (2007; PDF download) provides recommendations and a curriculum framework with examples for teaching statistics in PK–grade 12. A Spanish translation is available.

- *Statistical Education of Teachers (SET)* (2015; PDF download) outlines the content and conceptual understanding teachers need to know in assisting their students' development of statistical reasoning skills. SET is intended for everyone involved in the statistical education of teachers, both the initial preparation of prospective teachers and the professional development of practicing teachers.
- *Preparing K–12 Teachers of Statistics: A Joint Position Statement of the American Statistical Association and National Council of Teachers of Mathematics* (2019; PDF download) details the preparation and support teachers need to successfully support students' learning of statistics in the PK–grade 12 curriculum.
- Curriculum Guidelines for Undergraduate Programs in Statistical Science (2014; PDF download) provides recommendations for undergraduate programs in statistical science both for statistical science majors and students majoring in other subjects, seeking a minor or concentration in statistics.

In April 2020, the ASA released the report *Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education (GAISE) II* update. The GAISE II report enhances and updates the school-level GAISE of 2005 to adjust for the remarkable evolution in statistics over the past 15 years. Enhancements that are addressed include question posing in statistics, a wider variety of data types, multivariate thinking, using primary versus secondary data, probability in statistics, technology in statistics, and assessment in statistics. *Pre-K-12 GAISE II* is a must-read, not only for school-level teachers but also for all teachers of introductory statistics courses.

The ASA also continues to be involved in professional development for K–12 teachers. Workshops and webinars such as the Meeting within a Meeting (MWM) and Beyond AP Statistics support the teaching of statistics guided by the Common Core and Next Generation Science standards to foster conceptual understanding, active learning, real-world data applications, and appropriate use of technology. LOCUS (https://locus.statisticseducation.org) is a National Science Foundation–funded Discovery Research K12 (DRK12) Project (DRL 1118168) focused on developing assessments of statistical understanding. These assessments measure students' understanding across levels of development as identified in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A Pre-K–12 Curriculum Framework*. LOCUS assessments measure statistical understanding at two levels: Beginning/Intermediate and Intermediate/Advanced. The intent of these assessments is to provide teachers, educational leaders, assessment specialists, and researchers with a valid and reliable assessment of conceptual understanding in statistics consistent with the Common Core State Standards (CCSS).

The GAIMME Report

In 2016, following the lead taken by ASA, the Consortium for Applied Mathematics (COMAP) and the Society for Industrial and Applied Mathematics (SIAM) released their report *Guidelines for Assessment and Instruction in Mathematics Modeling Education* (GAIMME; https://comap.com/Free/GAIMME/index.html). The guidelines were developed as a resource for PK–12 teachers. A second edition was released in 2019 that contains revisions primarily for the chapter focused on the elementary and middle school levels. The report presents an overview of the modeling process and a rationale for why mathematical modeling is important. It contains a set of rich examples and best practices at each grade level as well as common themes that cut across the PK–12 curriculum. The examples include ways to transform more conventional problems into problems that provide opportunities for students to use the modeling process and the characteristics of assessment. The authors emphasize that the guidelines do not represent a curriculum or even a full set of modeling problems for each grade level.

As a follow-up to the publication of the GAIMME report, the Math Modeling Hub (MMHub) was created in 2018 to serve as an online community of practice. Organized and funded as a joint venture of COMAP, NCTM, and SIAM, the MMHub supports instruction in mathematics modeling by serving as a network of and discussion forum for K–16 educators interested in mathematical modeling. It is also a repository of open educational resources in mathematical modeling.

Currently there are four opportunities for high school students to engage in mathematical modeling challenges/competitions, far more than the number of traditional mathematical competitions focused on classical mathematical problem solving. These include, the High School Mathematical Contest in Modeling (HiMCM), the Mathematical Contest in Modeling, and the Interdisciplinary Contest in Modeling (MCM/ICM), the MathWorks Math Modeling Challenge (M3C), and the Modeling the Future Challenge. These contests and others are described in more detail in chapter 8.

Emergence of Alternative High School Course Pathways

As previously stated, one of the key recommendations of *Catalyzing Change in High School Mathematics* (NCTM 2018) is that all students complete four years of high school mathematics with the first two or three years in a common pathway that addresses all of the Essential Concepts in the key areas of number, algebra and functions, geometry and measurement, and statistics and probability. The mathematics that is studied should be based on the student's future goals and not on any perceived difference in mathematical ability that is determined by anyone else. The traditional high school course sequence has been algebra 1, geometry, and algebra 2, leading to calculus and a major in a STEM field. Although some students have been successful in this sequence, many more have found themselves repeating courses or stuck in dead-end pathways with no clear connection to postsecondary study or modern careers. *Catalyzing Change* outlines two sample pathways, a Geometry First pathway and an Integrated Approach pathway, intended not as a prescription of how a high school mathematics program should be structured but to initiate conversation about alternatives that result in more students having opportunities and being successful in mathematics.

There is evidence that this notion of creating alternative high school pathways that meet the college and professional goals for more students is gaining traction. Several school districts, including the Escondido Unified School District, Escondido California; San Francisco Unified School District, San Francisco; and Marlborough High School, Marlborough Massachusetts, have instituted course pathways at the high school level that provide students choices at certain points along each path. State-level efforts exist in Oregon and Alabama (Berry 2019; Daro and Asturias 2019; Jeffrey and Jimenez 2019). According to Phil Daro and Harold Asturias (2019) effective systems will have the following characteristics:

- Pathways as options that lead to postsecondary opportunities, with some flexibility to switch pathways
- Relevance of pathway content, expertise, and goals

- Recruitment of students to pathways
- Support for students within pathways (p. 16)

These elements are consistent with the *Catalyzing Change* recommendations, and can be identified in the aforementioned school and state-level efforts.

As previously stated, in April 2020, NCTM published two additional books in the Catalyzing Change Series. As NCTM's Robert Berry states in his December 2019 President's Message: "This series will broaden the pathways conversations to include the impact on high school mathematics that policies and structures of early childhood, elementary, and middle schools may have. I am excited about this series because it does not isolate policy and structural issues within a single grade band. This series pushes us to take a systemic approach to addressing teaching and learning issues for early childhood through postsecondary mathematics."

Another group of critical stakeholders that need to be involved in the high school pathways discussion are postsecondary institutions where admission requirements often determine what students take in high school. Articulation between high school and two- and four-year colleges has always been important, but it is key to the success of any high school pathways program. The Conference Board of the Mathematical Sciences (CBMS) in collaboration with the Charles A. Dana Center and Achieve has begun High School to College Math Pathways: Preparing Students for the Future, a project directed at bridging the gaps between school and college mathematics. It is a multistate effort; states applied to participate and 23 states were selected. State-level teams have representatives from across the spectrum: educators and administrators from high school, community colleges, and universities as well as nonprofit educational organizations, policy organizations, state boards, school district boards, and industry partners. The state-level teams attended a forum in May 2019 to set state-level goals, learn about best practices, and develop an implementation plan. The Dana Center is continuing to work with cohorts of states; a follow-up forum, where states will share progress, is scheduled for October 2020. Lessons learned from this project have the potential to inform the broader community.

Emergence of New Lower Division Pathways

The college mathematics requirement has long been a roadblock for student success and persistence in two-year and four-year colleges. A more recent trend, however, is to rethink how to support students, especially in developmental mathematics courses. Developmental mathematics courses are noncredit bearing courses and hence do not count toward a student's degree. These courses are for students who do not meet the requirements to enroll in a credit-bearing course, and they cover secondary school mathematics topics, such as fractions, proportions, and algebra. A 2016 report conducted by the U.S. Department of Education found that 59 percent of students at two-year colleges and 33 percent of students at four-year institutions are taking one or more developmental mathematics courses. Completion rate for these courses stands at 50 percent at two-year colleges and 58 percent at four-year colleges (Chen 2016).

Given the poor success rate of these courses, many educators are experimenting with alternative structures or formats that provide the necessary content remediation but in a manner that does not discourage the student aspiring to a higher education. One current and popular approach is what is referred to as the *corequisite* model. The corequisite model allows students who do not meet the requirements to enroll in a credit-bearing course to do so, but they must also take an additional support course with content aligned with the credit-bearing course. Many states have actually done away with noncredit-bearing courses and replaced them with the corequisite structure. For example, the Education Commission for the States (ECS 2018) found that 15 of the 19 states that had developmental course reforms either mandated or recommended the corequisite approach.

Another popular approach for helping non-STEM-intending students satisfy their mathematical college requirement is to forgo the college algebra requirement and replace it with one or more courses that develop quantitative thinking with content more pertinent to the student's major and professional goals. These alternative courses are often referred to as "multiple math pathways" and 41 percent of two-year colleges offer such courses for non-STEM career paths. For example, humanities majors might take a quantitative literacy course while social and health science majors would focus on statistics. Two popular models are the Carnegie Math Pathways, which includes Statway and Quantway, and the Dana Center Mathematics Pathways. Both of these models include pedagogical approaches that utilize evidence-based teaching strategies and an accelerated developmental math course in which students who place in developmental math can complete a credit-bearing mathematics course in one year (Zachry Rutschow 2019; Zachry Rutschow and Mayer 2018).

Chapter 6: Postsecondary Mathematics Education Professional Development

Professional development in mathematics education occurs across all levels of teaching, grades PK–16, for both in-service and preservice teachers. In this chapter, typical teacher preparation routes (preservice) and continuing professional development (in-service) offerings are outlined. In addition, recommended standards and professional development programs for university faculty in mathematics and mathematics education are described.

Teacher Preparation Programs at Postsecondary Institutions

Mathematics teacher preparation for grades PK–12 in the U.S. is typically a four-year program offered at colleges and universities, although many universities also offer teacher certification pathways that may require only one to two years of mathematics teacher preparation, provided the preservice teacher already holds an undergraduate degree in a similar content area.

For teacher preparation in secondary mathematics, university mathematics departments typically offer the mathematics and statistics courses taken by preservice and in-service teachers as part of their training and certification requirements. Mathematics and statistics courses required of preservice elementary school teachers may be offered in mathematics departments or education departments of postsecondary institutions. In these institutions, the method courses for the teaching of mathematics also may be taught in either the department of mathematics (if the mathematics educators are housed there) or the school or department of education.

Over the past two decades, the Conference Board of the Mathematical Sciences (CBMS) survey of undergraduate mathematics programs asked a special set of questions focused on coursework offered for prospective teachers of kindergarten through grade 12 (Blair, Kirkman, and Maxwell 2018). For universities at which teacher preparation programs exist, the questions focused on where such programs were housed and what courses and experience were required of the students in these programs.

The results of the 2015 survey indicated that 63 percent of the institutions had a K–8 teacher certification program. This was a decline from 72 percent in 2010 and 87 percent in 2005. An examination of some subareas within the data groups indicate that the major sources of the decline were at universities with PhD programs in mathematics and four-year colleges offering the BA as their highest degree in mathematics (Blair et al. 2018). In an era calling for more mathematics specialists in the K–8 years, the reasons for the decline in the percentage of institutions offering such a program are not clear. It is particularly perplexing to note this decline in program offerings at the K–8 level in university mathematics departments as 22 states now provide Elementary Mathematics Specialist certification options for teachers in K–grade 6. The year 2015 was the first that the CBMS survey had separate questions for K–grade 5 and grades 6–8 certification programs. Results showed about an equal number of programs across departments in the two grade bands (53 percent of departments offering K–5 programs and 51 percent offering 6–8 programs). The majority of these programs (63 percent and 64 percent, respectively) are offered in mathematics departments whose highest degree obtainable is at the master's level.

Examination of the data for four-year colleges that provide programs of preparation for either K–5 or 6–8 certification reflect that, on average, 30 percent of K–5 programs require no more than two mathematics courses, 35 percent require three mathematics courses, 34 percent require four or more mathematics courses. In contrast, only 8 percent of grades 6-8 programs require two or fewer mathematics courses, 8 percent require three mathematics courses, and 83 percent require four or more mathematics courses for certification. In general, the number of required courses or semester hours in mathematics saw a slight increase over data reported in the 2010 CBMS survey. Unlike previous CBMS surveys, the 2015 report did not include information on the range of mathematics content that courses and mathematical topics required for preservice teachers seeking certification at either the K–5 or 6–8 level (Blair et al. 2018).

For secondary mathematics teacher certification programs, common mathematics and statistics content requirements for certification programs included geometry (85 percent of reporting institutions), statistics (80 percent) and modern algebra (80 percent). In addition, many secondary certification programs housed in departments offering masters or doctoral degrees also required a course in advanced calculus/analysis (60 percent). Doctoral-level departments were also more likely to offer specialized mathematics courses for preservice secondary teachers. Other mathematics and statistics content courses generally taken by preservice secondary mathematics teachers include number theory, discrete mathematics, probability, and the history of mathematics. These results echo data reported in the 2010 CBMS survey (Blair, Kirkwood, and Maxwell 2012; Blair et al. 2018).

A more careful look at required courses for preparation and certification to teach statistics was part of the CBMS survey for the first time in 2015. Across all levels of statistics departments and mathematics departments that also teach statistics courses (bachelors-only, masters, and doctoral granting programs), 41 percent required an introductory statistics course for preservice teachers, and 42 percent required a calculus-based statistics and/or probability course. In addition, another 20 percent of reporting institutions noted they offered specialized statistics courses for preservice teachers. In contrast, 73 percent reported not requiring any statistics course for preservice teachers seeking K–5 certification, and 42 percent reported not requiring statistics for grades 6–8 certification although another 42 percent did require at least one statistics course for grades 6–8 certification programs (Blair et al. 2018). Because these survey questions regarding statistics requirements were new to the 2015 survey, no comparisons with prior years' surveys can be made.

Increasingly the licensing or certifying of teachers of mathematics is tied to examinations of mathematical content knowledge, particularly for teachers of grades 6–12. For example, most U.S. states require minimum scores on a Praxis Subject assessment of mathematics (which tests academic skills and knowledge of mathematics needed for teaching) offered by the Education Testing Service, and several states require future teachers to pass a state-specific or state-created mathematics content and pedagogy exam, or a practicum exam that includes submission of video recordings from actual instances of teaching. To maintain their ability to offer teacher preparation programs, colleges and universities may seek accreditation of their programs through the National Council for the Accreditation of Educator Preparation (CAEP) or may need to demonstrate that the program meets state-determined criteria. CAEP partners with organizations like NCTM to develop accreditation standards for teacher preparation programs and to review such programs for accreditation.

Guides and Recommendations for Teacher Preparation

The Conference Board of Mathematical Sciences published *The Mathematical Education of Teachers* in 2001, a framework of recommendations for the mathematics knowledge that teachers should acquire or know in order to teach mathematics in PK–grade 12. In 2012, a revised set of recommendations was published in the CBMS's *The Mathematical Education of Teachers II* (MET II), with updates that corresponded to the greater involvement of mathematicians and statisticians at the postsecondary level and also acknowledged the *Common Core State Standards for Mathematics*. The American Statistical Association's (ASA) 2015 document *The Statistical Education of Teachers* (SET) presented a similar framework, providing detailed outlines for statistical preparation of teachers to clarify the *MET II* recommendations and identify distinct features of statistical understanding and preparation for teaching (Franklin et al. 2015).

These two documents represent comprehensive frameworks for developing or modifying teacher preparation programs and also include suggestions for continuing professional development. These publications offer significant guidance regarding the nature of courses that would be valuable to teachers to build a strong understanding of mathematics and statistics for teaching. Both documents stress that such coursework should not focus on traditional mathematics content but instead the coursework should encompass content that teachers need to know and the ways by which they should come to know that content to be effective in the classroom. The principal intended audiences for these documents are postsecondary mathematics educators, mathematicians, and statisticians. Additionally, the framework for mathematical knowledge and professional development recommendations are valuable for state- and school-level policymakers as they seek to develop continuing education requirements that support the teaching and learning of mathematics and statistics (Franklin et al. 2015; CBMS 2001, 2012).

The ASA continues to take a very active role in teacher development and professional development in statistics education at the PK–12 level as well as in the postsecondary arena. In 2020, several new documents and initiatives will be released, such as an update of the *PK–12 Guidelines for Assessment and Instruction in Statistics Education* (GAISE II). Other initiatives include naming a Statistical Ambassador of the ASA to provide leadership in the area of professional development for statistics education at the secondary level as well as for in-service PK–12 teachers.

Another recent set of standards intended to guide teacher educators in the preparation of well-qualified mathematics teachers in PK–grade 12 was developed by the Association of Mathematics Teachers Educators (AMTE) and released in April of 2017. The *Standards for Preparing Teachers of Mathematics* (SPTM) articulate a national vision for the preparation of individuals who will teach mathematics at all levels. The standards focus on the development of successful teacher education programs as well as equitable practices in mathematics teaching. The publication includes a set of standards for well-prepared novice teachers as well as standards for the effective preparation of new teachers. The standards for well-prepared teachers elaborate on mathematical content consistent with MET II and SET, pedagogy and dispositions specific to teaching mathematics, knowledge of students, and awareness of the social contexts in which mathematics teaching and learning occurs. The standards, which are elaborated on for early childhood, elementary grades, middle school and high school, include—

- mathematical concepts, practices, and curriculum;
- pedagogical knowledge and practices for teaching mathematics;

- students as learners of mathematics; and
- social contexts of mathematics teaching and learning.

The standards for effective preparation of beginning teachers of mathematics illuminate the need for partnerships and input from a range of stakeholders as well as critical components of teacher preparation programs such as experience in clinical settings and multiple opportunities to learn mathematics and learn to teach mathematics. More specifically these standards include—

- partnerships;
- opportunities to learn mathematics;
- opportunities to learn to teach mathematics;
- opportunities to learn in clinical settings; and
- recruitment and retention of teacher candidates (AMTE 2017).

Assessment is seen as an important component of these AMTE standards, which elaborate on features of effective assessments, such as developing and administering relevant, transparent, and equitable assessments aligned with an institution's teacher preparation goals. The AMTE also advocates for assessments of teacher quality and assessments of teacher-preparation-program quality to ensure continued improvements and attainment of goals toward meeting the standards (AMTE 2017). More generally, the members and officers of AMTE hope to positively influence teacher preparation in mathematics so that all teachers have the knowledge, skills, and dispositions to help every student be successful in learning mathematics.

Teacher Education Requirements in Mathematics and Other Subjects

Teacher education requirements in the United States have not changed much over the past several decades. A bachelor's degree and a teaching certificate are all that's needed to teach in most public schools at any level, prekindergarten through secondary education. The teaching certificate is generally obtained through satisfactory completion of courses taken at the college level in combination with in-school experiences, such as observations and supervised practice teaching, in or around the grade levels at which the teaching certificate is requested. Most states also require that the teacher candidate pass one or more tests, which usually assess specific subject-matter knowledge and general knowledge about teaching and the education system. Although this paints a fairly uniform picture of teacher preparation and certification, it should be noted that state requirements vary widely.

All states, with the exception of Alaska, also provide some alternative route to teacher certification on the basis of an individual's prior experiences, education, and, potentially, a bundled set of courses and internship experiences (Snyder, Brey, and Dillow 2019) or through a set of examinations. NCES estimates, on the basis of data submitted by the states, indicate that approximately 18 percent of public school–teachers were issued certificates to teach through alternative routes in 2015–16. This is an increase from the 15 percent of alternatively certified teachers reported in the 2011–12 academic year and 13 percent in 2007–8. Also, in cases of teacher shortage or the movement of a teacher from one state to another, provisional certification is possible through state education officials until the relocated teacher has met all the requirements for full certification. Reports show that four percent of public school–teachers held provisional certificates while only one percent held emergency certifications or a certification waiver in 2015–16 in the case of teacher shortages (McFarland et al. 2019). It is worthy to note that in the 2019–2020 academic year, 48 states and the District of Columbia list mathematics as a critical teacher shortage subject (U.S. Department of Education 2020).

Teacher Preparation Trends and Programs in the U.S.

One response to the critical need for teachers in science and mathematics was the National Science Foundation's (NSF) creation of the Noyce Scholarship Program in 2002, which was reauthorized in 2007 and 2010. The program provides scholarships, stipends, and programmatic funding to postsecondary institutions and faculty to recruit and prepare preservice teachers in science, technology, engineering, and mathematics (STEM). The Discovery Research PK–12 (DRK–12) Program of the NSF is a program that brings together education researchers and PK–12 teachers and students for improvement of STEM education.

Another recent trend in STEM teacher preparation is the UTeach Program, which was developed at the University of Texas at Austin in 1997 with the goal of increasing the number of qualified STEM teachers in U.S. secondary schools. The program has since been replicated at 46 universities and four-year colleges across 22 states. One main feature of the UTeach programs is the requirement that students begin teacher preparation coursework and field experiences during their first year in the program. When teacher preparation coursework and experiences occur throughout the four years of postsecondary education, students have time for a concentrated focus on a STEM major while completing secondary teaching certification requirements, all within a four-year time frame. In addition to the secondary teacher preparation programs, UTeach has continued to expand its teacher preparation initiatives such as professional development for practicing teachers and outreach programs for secondary students (UTeach 2019).

One more effort of note is the Mathematics Teacher Education Partnership (MTE-Partnership) established by the Association of Public and Land Grant Universities (APLU) to provide "a coordinated research, development, and implementation effort to transform secondary mathematics teacher preparation programs" (APLU 2020). The partnership was established in 2012 and included representation from more than 100 postsecondary institutions, K–12 schools and districts, and a variety of others in the educational enterprise. With the goal of revamping mathematics teacher preparation programs according to the *Guiding Principles for Secondary Mathematics Teacher Preparation Programs* (created by the APLU and rereleased in 2014), the MTE-Partnership supports projects that move toward this goal. The work of this group that includes research action clusters addressing active learning, clinical experiences, and teacher recruitment and retention is documented in the book *The Mathematics Teacher Education Partnership: The Power of a Networked Improvement Community to Transform Secondary Mathematics Teacher Preparation* (Martin et al. 2020).

Professional Development for College Faculty of Mathematics, Mathematics Education, and Statistics

In the past decade the postsecondary mathematics community has become increasingly aware of the value of research-based, student-centered instructional approaches for student learning (Kober 2015). For example, in 2016, the Conference Board of the Mathematical Sciences (CBMS), which is the umbrella organization for the professional societies in mathematics and statistics, put out the following position statement in favor of active learning:

> [W]e call on postsecondary institutions, mathematics departments and the mathematics faculty, public policy-makers, and funding agencies to invest time and

> resources to ensure that effective active learning is incorporated into postsecondary mathematics classrooms.

Similarly, the *Common Vision for Undergraduate Mathematical Sciences Programs in 2025* report (Saxe and Brady 2015, p. 1) state that the "status quo is unacceptable" and call on postsecondary institutions to scale up the use of evidenced-based pedagogical methods. Complementing this call for increased use of student-centered instructional approaches is the call for the postsecondary mathematics community to focus on issues of equity and inclusive practices. For example, in 2018, the Special Interest Group of the Mathematical Association of America put out a comprehensive position statement on advancing equity in undergraduate mathematics education with respect to 1) participation within the community; 2) teaching practices; and 3) research.

Despite this growing awareness and calls from professional societies to increase the use of research-based instructional practices and to focus on issues of equity and inclusive practices, the uptake of such approaches has been slow (Apkarian et al. 2020; Eagan 2016; Stains et al. 2018). Hence the need for professional development for current, new, and future faculty in the mathematical sciences. Before providing an overview of these opportunities, we first highlight findings that point to the benefit for learners when instructors use research-based instructional strategies.

In the U.S., the terms *active learning* or *inquiry-based mathematics education* are often used as umbrella terms to capture the myriad forms that research-based, student-centered instructional approaches may take (Laursen and Rasmussen 2019). One of the most influential reports of the benefit for learners is that of Freeman and colleagues (2014), published in the proceedings of the National Academies of Science. To test the hypothesis that lecturing maximizes learning and course performance, these researchers conducted a meta-analysis of 225 studies that reported data on examination scores or failure rates when comparing student performance in postsecondary STEM courses under traditional lecturing versus active learning. They found that average examination scores improved by about 6 percent in active learning sections and that students in classes with traditional lecturing were 1.5 percent more likely to fail than were students in classes with active learning. They summarize these findings with the following provocative statement:

> If the experiments analyzed here had been conducted as randomized controlled trials of medical interventions, they may have been stopped for benefit—meaning that enrolling patients in the control condition might be discontinued because the treatment being tested was clearly more beneficial. (p. 8413)

In another study that focused solely on mathematics courses, Laursen and colleagues (Hassi and Laursen 2015; Kogan and Laursen 2014) examined student outcomes across a range of courses at multiple research-intensive institutions using measures such as grades and self-reported outcomes from surveys and interviews. Similar to the Freeman report, these studies broadly show greater benefits, both in terms of cognitive and affective gains, to students in inquiry-based classes compared to students in lecture-based courses. Moreover, Laursen and co-authors (2014) found that inquiry-based experiences result in considerable gains for women, thus raising serious equity concerns about lecture-oriented instruction. Finally, one concern from faculty is that less material is covered in classes that use active learning and hence students may be disadvantaged in subsequent courses. To address this concern, Laursen and colleagues examined student performance in downstream courses and found that students in inquiry-based classes did as well as or better than their counterparts in lecture classes and that high achieving students who took an inquiry-based course early on in the studies took more mathematics courses than their peers.

Professional Development Opportunities for Postsecondary Instructors

Unlike K–12 teachers, most postsecondary mathematics instructors in the U.S. did not receive pedagogical training as part of their graduate education. But what are the professional development opportunities for postsecondary mathematics instructors and which form do they take? We next review these opportunities for U.S. educators according to three groups of instructors: current faculty members, new faculty members, and future faculty members.

Current Faculty Members

There are several occasions in the U.S. for current instructors to learn about research-based, student-centered classrooms, although these opportunities are not abundant. Workshops put on by the Academy of Inquiry Based Learning (AIBL) have been offered since 2006. Since then, AIBL has conducted 16 intensive four-day inquiry-based learning workshops for more than 440 mathematics instructors. Rather than focusing on implementing particular curricula, workshops embrace a wide range of instructional practices and aim to help instructors decide for themselves when and why to implement various practices. The workshop design includes four main components: video lesson study of inquiry classrooms, educational research, facilitation skills, and personal work time to plan a course (Laursen et al. 2019). AIBL is also facilitating regional groups of instructors who are interested in using and disseminating inquiry-based approaches. As stated on their website, a supportive local community can assist members in their first-time teaching of a course using IBL and it can help them overcome documented barriers to transitioning their courses toward IBL techniques. It also effectively supports emerging and experienced practitioners of IBL by providing continuing professional development opportunities, offering support for trying a different type of IBL or beginning to teach a different course with IBL, and giving a sense of higher purpose by allowing for "giving back" in various ways.

The professional development approach taken by the AIBL is an example of a community-based strategy, which emphasizes the need for stable communities of teachers jointly engaging in long-term change processes. Another professional development approach is a materials-based strategy, which provides instructional resources that can then affect practice. In the U.S. there are several sources of instructional materials that are intended to be a resource for educators who want to incorporate various inquiry-based approaches into their instruction. For example, the Journal of Inquiry-Based Learning (JIBL) is an open-source repository of curricula ranging from calculus to graduate courses in topology. JIBL offers three types of freely downloadable professionally refereed materials: Course Notes, which are classroom-tested notes for full courses, User Reviews of Course Notes, and Modules, which are classroom-tested brief notes or activities addressing individual topics. Another repository of open-source inquiry-based materials is available through the Teaching Inquiry-oriented Mathematics: Establishing Supports (TIMES) project, which offers research-based curricula in abstract algebra, differential equations, and linear algebra.

For professional development in the domain of statistics, the Mathematical Association of American (MAA) in collaboration with the American Statistical Association (ASA) and the American Mathematical Association of Two-Year Colleges (AMATYC) offers the StatPREP program. As described on their website, StatPREP aims to foster the widespread use of data-centered methods and pedagogies in introductory statistics courses, especially for faculty at two-year institutions. StatPREP works directly with college-level instructors, both online and in community-based workshops, to develop the understanding and skills needed to work and teach with modern data. The project also seeks to initiate community transformation focused on modernizing undergraduate statistics education. The project, which combines both a community-based strategy and a materials-based strategy, is working to establish regional communities of practice to support instructors and to connect them in a national online support network with statistics education experts. The StatPrep program began in fall 2016, with the first summer workshops occurring in summer 2017. To date, 128 unique faculty have participated.

Other ASA professional development opportunities can be broadly described as curating web resources for educators, publishing publications statistics education journals, guidelines and reports, and sponsoring workshops and webinars. The professional development program is purposefully designed so that activities with the major components are complementary. For example, the release of a new guideline or report is supported by a virtual information session. Likewise, the work begun during the Meeting Within a Meeting (MWM) or Beyond AP Statistics (BAPS) workshops held in conjunction with the Joint Statistical Meetings is continued through virtual workshops throughout the subsequent school year. In 2018 prior to the Joint Statistical Meeting, the ASA piloted the Preparing to Teach Workshop. The effort continues, and it is hoped it will be an annual event. ASA also has developed recommendations for qualifications for teaching a modern introductory statistics course along with information and resources for assisting departments and faculty.

Finally, MAA also offers faculty members professional development through Preparation for Industrial Careers in Mathematical Sciences (PIC Math). This program offers a three-day summer workshop for faculty in which they learn about various nonacademic career options; receive guidance on making industry connections so they can provide real research experiences for their students that come directly from business, industry, or government; and develop their pedagogical expertise with the purpose of improving students' problem solving, critical thinking, independent thinking, teamwork, and communication skills. The program also provides follow-up support during the academic year by providing faculty with resources for a semester-long, credit-bearing course focused on solving industrial problems as well as opportunities for students to share the results of their solutions to real mathematical or statistical problems grounded in their business, industry, or government partners. PIC Math began in 2014; since its inception, a total of 227 faculty members have participated, some more than once.

New Faculty Members

Since 1994 MAA has a professional development opportunity called Project NExT for new or recent PhDs in the mathematical sciences. MAA Project NExT addresses all aspects of an academic career: improving the teaching and learning of mathematics, engaging in research and scholarship, finding exciting and interesting service opportunities, and participating in professional activities. Currently the program has a strong focus on helping its fellows adopt interactive pedagogies that actively engage their students (D. Kung, personal communication, January 8, 2020). MAA Project NExT also provides participants with a network of peers and mentors as they assume these responsibilities. Since its inception, there have more than 2,000 participants.

Similar to MAA Project NExT is the Service, Teaching, and Research (STaR) Fellows program for early-career faculty in mathematics education. The STaR Fellows program is provided by the Association of Mathematics Teacher Educators (AMTE) and is designed to address common challenges related to developing leadership and service skills, teaching mathematics content and methods courses to prospective and practicing K–12 teachers, instructing graduate courses in mathematics education, and establishing a research agenda. One important way the program negotiates these challenges is by providing opportunities to network with other new mathematics education faculty beyond the participant's home institution. To date, 345 early-career mathematics educators working at more than 100 postsecondary institutions have completed the program.

At the community college level, AMATYC offers new faculty a professional development opportunity through Project ACCCESS (Advancing Community College Careers: Education, Scholarship, and Service). Since its inception in 2004, Project ACCCESS has affected nearly 400 fellows. The goals of Project ACCCESS are to facilitate professional growth and to encourage leadership among new two-year college faculty while providing experiences that will help new faculty become more effective teachers and active members of the broader mathematical community (Watkins 2016). Project ACCESS fellows attend two consecutive AMATYC national meetings where they participate in a program developed specifically for new faculty. Fellows are also linked in an electronic network with each other and with a group of distinguished mathematics educators.

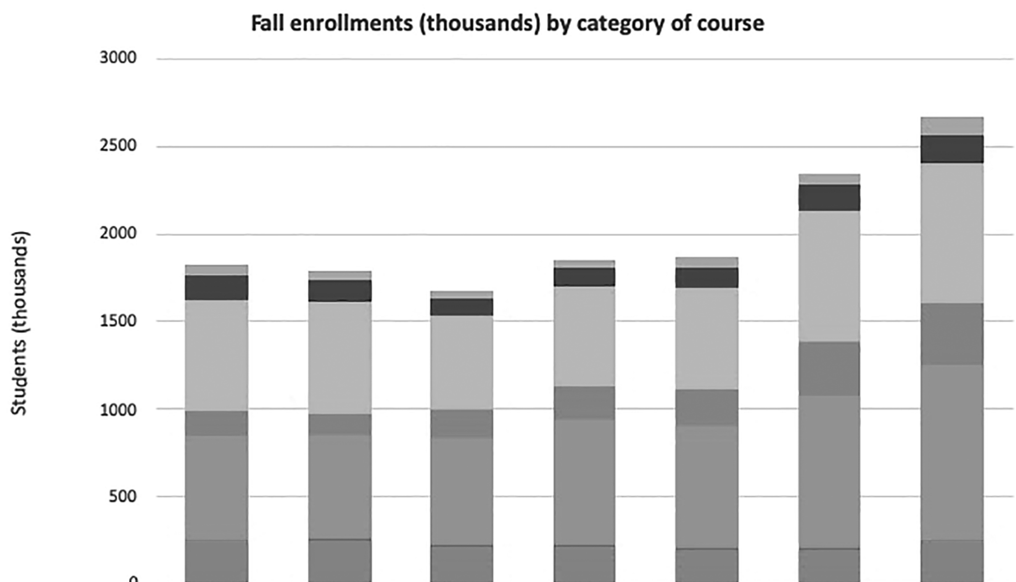
Future Faculty Members

Mathematics departments across the U.S. are increasingly turning their attention to providing professional development for the next generation of faculty, namely graduate teaching assistants (GTAs). In 2015 the MAA *Progress though Calculus* project conducted a census survey of all mathematics departments that offer a graduate degree in mathematics. They found that 83 percent of departments that offer a doctoral degree in mathematics have a required department-managed professional development program for GTAs who teach in the precalculus through calculus sequence. Moreover, 65 percent of these departments indicated that such programs are very important for student success in these courses (Rasmussen et al. 2019). These GTA professional development programs range from fairly low levels of preparation to quite extensive training. More specifically, 38 percent of doctoral-degree-granting departments require a half or one-day pre-semester workshop, 35 percent require a multiday pre-semester workshop, and 62 percent of departments require a term or year-long professional development course or seminar in addition to a pre-semester workshop (Apkarian and Kirin 2017).

While the numerous GTA professional development programs are largely being carried out by individual mathematics and statistics departments, MAA is providing support for these efforts through the College Mathematics Instructor Development Source(CoMInDS) project. This suite of resources is intended to enhance the mathematics community's ability to provide high-quality teaching-related professional development to graduate students. Project components are designed to address the needs of three core groups whose efforts have significant influence on the quality of undergraduate mathematics instruction: 1) faculty who provide professional development to graduate teaching assistants (GTAs), 2) faculty and graduate students whose research and other scholarly activities center on the teaching of undergraduate mathematics, and 3) graduate students whose responsibilities include teaching mathematics courses. This work includes establishing a professional community of practice in which experienced GTA professional development providers serve as mentors to other faculty wishing to start or improve a GTA professional development program, organize workshops for providers of GTA professional development, and establish an online resource suite of instructional materials and products of scholarly activity along with resources for assessment and evaluation of GTA professional development programs.

Chapter 7: Postsecondary Calculus

As shown in figure 7.1, mathematics enrollments in precollege, introductory level (college algebra and precalculus), introductory statistics, calculus level (calculus, linear algebra, and differential equations), advanced level, and advanced statistics at four-year institutions have all increased over the past decade. For example, from 2005 to 2010 enrollment in calculus level courses increased by 162,000 and from 2010 to 2015 by 59,000. Also noteworthy is the considerable growth in enrollments in advanced statistics, which nearly doubled from 60,000 to 110,000 in the period from 2010 to 2015.



	1985	1990	1995	2000	2005	2010	2015
Advanced Stats	63	52	44	45	57	60	110
Advanced Level	138	119	96	102	112	150	154
Calculus Level	637	647	538	570	587	748	807
Intro Stats	144	117	164	190	202	312	347
Intro Level	593	592	613	723	706	863	1000
Precollege	251	261	222	219	201	209	253

Fig. 7.1 Undergraduate enrollments by course category in mathematics and statistics departments at four-year institutions. (Source: Bressoud 2018)

Consistent with the enrollment growth in calculus, linear algebra, and differential equations is the increase in engineering majors. As shown in figure 7.2, the number of prospective engineering majors grew from 108,000 in 2005 to 156,000 in 2010, peaking at 194,000 in 2015. Similar increases in enrollment have occurred in the biological sciences.

In the remainder of this chapter, we highlight current knowledge about mainstream calculus in postsecondary institutions, where *mainstream* refers to calculus courses required of most STEM majors and which serve as a prerequisite for subsequent mathematics courses, such as differential equations and linear algebra. Our focus on introductory courses is due to their importance for student access to higher level mathematics and STEM careers. While calculus remains an essential required course for STEM majors, statistics is increasingly making its way into being a required

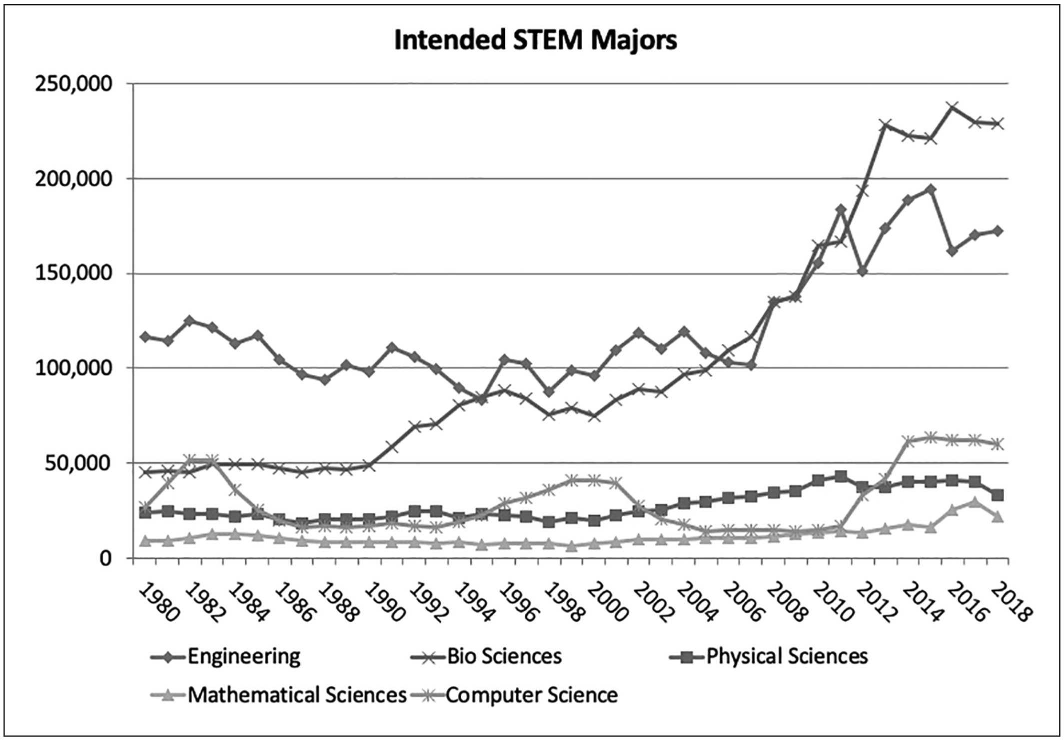


Fig. 7.2 Number of entering full-time first-year students at four-year institutions intending to major in one of the five core STEM disciplines. (Source: Stolzenberg et al. 2018; Bressoud 2015)

curriculum for STEM majors. However, while several national studies exist related to calculus, no such reports currently exist for statistics, thus our focus in the subsequent sections remain on calculus.

The MAA Characteristics of Successful Programs in College Calculus project

In 2010 the MAA *Characteristics of Successful Programs in College Calculus* (CSPCC; www.maa.org/cspcc) project carried out stratified random sample national surveys of mainstream calculus 1 courses at two-year colleges, four-year colleges, master's degree–granting mathematics departments, and PhD-granting mathematics departments. Student surveys, which were administered shortly after the start of the term and then again near the end of the term, were followed up two years later with case studies at 16 mathematics departments (four of each type of institution) that were identified through survey analysis as having a relatively more successful calculus 1 program compared to their counterpart institutions. In this study, success was determined by a combination of factors, including final course grades, student intention to continue in the calculus sequence, and affective measures of confidence, interest, and enjoyment of mathematics. *Insights and Recommendations from the MAA National Study of College Calculus,* edited by Bressoud, Mesa, and Rasmussen (2015), presents findings from both the survey and cases studies (see www.maa.org/cspcc for a free downloadable pdf).

Calculus 1 Student Career Goals by Demographics

The CSPCC survey provided detailed information on the intended career goals of calculus 1 students. As shown in table 7.3, there is considerable variation in career goals by gender, race, ethnicity, and institution type. For example, 25 percent of students enrolled

Table 7.1

Distribution of Career Goals by Gender, Race, Ethnicity, and Institution Type

		Gender		Ethnicity				Type of Institution			
	All N=9,445	**Male N=5,231**	**Female N=4,202**	**White N=7,166**	**Black N=500**	**Asian N= 1,255**	**Hispanic N=876**	**Univ (PhD) N=6,984**	**4Y Coll (BA) N= 11,965**	**Univ (MA) N=528**	**2Y Coll (AS) N=737**
Math	2%	2%	1%	2%	2%	2%	1%	1%	1%	3%	2%
Physical Science	4%	5%	4%	5%	5%	3%	3%	3%	4%	6%	5%
Engineer	31%	38%	14%	29%	26%	19%	26%	35%	20%	22%	29%
Comp Sci/IT	5%	10%	2%	6%	9%	6%	7%	4%	7%	7%	10%
Geo/Environ Science	2%	2%	3%	3%	2%	1%	2%	2%	3%	5%	2%
Bio Science Med.	30%	19%	43%	27%	36%	34%	33%	31%	30%	27%	28%
Total STEM	74%	76%	67%	72%	80%	65%	72%	76%	65%	70%	76%
Teacher	5%	4%	10%	8%	3%	3%	6%	3%	9%	13%	7%
Soc Science	1%	1%	2%	1%	2%	2%	2%	1%	2%	1%	1%
Business	7%	9%	7%	6%	7%	16%	8%	7%	8%	7%	7%
Other	4%	3%	5%	5%	4%	4%	4%	4%	5%	5%	3%
Undecided	8%	7%	9%	8%	5%	9%	7%	8%	10%	4%	6%

(Source: Bressoud 2015, p. 11).

in mainstream calculus 1 have intended careers that fall outside of the typical STEM career, where the CSPCC authors defined a typical STEM career as any of the choices above the Total STEM row. Considering students with typical STEM career goals, we see that 65 percent of Asian American calculus 1 students intend a STEM career, 80 percent of African American students intend a STEM career, and 72 percent of Hispanic students intend a STEM career. Approximately two-thirds of women taking calculus 1 intend a STEM career.

High School Preparation for College Calculus

The CSPCC survey also gathered data on calculus 1 students' high school courses. Table 7.2 presents a comparison of the percentage of students who have followed the progression that would normally lead to calculus in grade 12. It also shows the average grade and standard deviation in these courses. We see that the percentage of students enrolled in a PhD-granting institution who completed calculus by the end of high school was 67 percent. This decreased to 50 percent for students at a master-granting institution, to 40 percent for students at a four-year college, and to 22 percent for students enrolled in a two-year college.

Table 7.2

Percentage of Students' Enrolled in Various High School Mathematics Courses; Average Grade in Course with Standard Deviation (SD) by Type of Institution

	Univ (PhD) N = 7,174		4Y Coll (BA) N = 1,782		Univ (MA) N = 527		2Y Coll (AS) N = 740	
	%	**Grade (SD)**	**%**	**Grade (SD)**	**%**	**Grade (SD)**	**%**	**Grade (SD)**
Algebra 2 by end of 10th grade	78%	3.8 (0.5)	71%	3.7 (0.6)	59%	3.6 (0.7)	56%	3.4 (0.8)
Precalculus by end of 11th grade	67%	3.7 (0.6)	58%	3.5 (0.6)	46%	3.6 (0.6)	37%	3.3 (0.9)
Statistics by end of 12th grade	10%	3.8 (0.5)	11%	3.7 (0.6)	9%	3.6 (0.7)	8%	3.5 (0.8)
Calculus by end of 12th grade	67%	3.6 (0.6)	50%	3.5 (0.7)	40%	3.5 (0.7)	22%	3.4 (0.8)

(Source: Bressoud et al. 2015, p. 6).

The CSPCC survey also asked students Likert-scale questions to self-assess their preparedness. For example, on survey at the start of the term, students were asked to report how well their high school mathematics courses prepared them to factor expressions, solve inequalities, and solve word problems. As shown in table 7.3, students at all types of institutions reported being fairly confident in these skills. Students were also asked to self-assess how well they understood the mathematics they studied in high school and how ready for college calculus they were. Table 7.3 shows that across the board, students enter postsecondary calculus 1 believing that their high school experience prepared them well.

On the end-of-the-term survey, which was only completed by students who were successful in the course (roughly 40 percent getting an A, 40 percent a B, and 20 percent a C), students were asked again whether they were ready for college calculus. In comparison to the start-of-the-term survey, Table 7.4 shows a dramatic 25 percent decrease in the yes responses.

Table 7.3

Start-of-Term Percentage of Students' Self-Assessment of High School Preparation

		Univ (PhD) N = 7,440	4Y Coll (BA) N = 1,833	Univ (MA) N = 574	2Y Coll (AS) N = 781
Can factor expressions	Somewhat[c]	13%	14%	19%	17%
	Yes[d]	85%	83%	79%	77%
Can solve inequalities	Somewhat[c]	17%	18%	20%	21%
	Yes[d]	80%	80%	78%	74%
Can solve word problems	Somewhat[c]	27%	28%	28%	25%
	Yes[d]	69%	68%	66%	66%
Understand what I have studied[a]	Somewhat[c]	23%	28%	25%	24%
	Yes[d]	75%	69%	72%	73%
Ready for calculus[b]	Somewhat[c]	16%	19%	18%	17%
	Yes[d]	81%	79%	77%	81%

(Source: Bressoud et al. 2015, p. 14).

Notes: For the first three entries the prompts were the following:
[1]"My mathematics courses in high school have prepared me to factor expressions."
[2]"My mathematics courses in high school have prepared me to solve inequalities."
[3]"My mathematics courses in high school have prepared me to solve word problems."
[4]There were four letter choices from which the students could choose:
[a]"I understand the mathematics that I have studied."
[b]"I believe I have the knowledge and abilities to succeed in this course."
[c]Combines *Slightly Disagree* and *Slightly Agree*.
[d]Combines *Agree* and *Strongly Agree*.

Table 7.4

End-of-Term Percentage of Students' Self-Assessment of High School Preparation (Source: Bressoud. 2015, p. 14.)

		Univ (PhD) N = 3,664	4Y Coll (BA) N = 1,524	Univ (MA) N = 333	2Y Coll (AS) N = 441
Was ready for calculus[a]	Somewhat[b]	31%	33%	35%	31%
	Yes[c]	56%	54%	51%	57%

(Source: Bressoud et al. 2015, p. 14).

Notes:
[a]Students were asked for level of agreement with the statement, "My previous math courses prepared me to succeed in this course."
[b]Combines *Slightly Disagree* and *Slightly Agree*.
[c]Combines *Agree* and *Strongly Agree*.

Instructional Experience

The CSPCC end-of-term survey gathered extensive information on the nature of instruction in calculus 1. Sonnert and Sadler (2015) conducted a factor analysis of these data, and referred to one of the clusters as "ambitious teaching." Ambitious teaching includes the use of group projects, the inclusion of unfamiliar problems both in homework and on exams, requirements for students to explain how they arrived at their answers, and a decreased reliance on lecture as the primary mode of instruction. Such instructional practices are compatible with active learning strategies, which have been shown to increase student success (e.g., Freeman et al. 2012) and widely endorsed by professional societies (e.g., Common Vision [Saxe and Brady 2015]). Despite this one cluster, lecture was the dominant form of instruction. This was true at all institutions as well as those that were selected as being relatively more successful for follow-up case studies. However, as shown in figure 7.3, the extent to which lecture was the dominant mode of instruction at the case study institutions was somewhat less frequent.

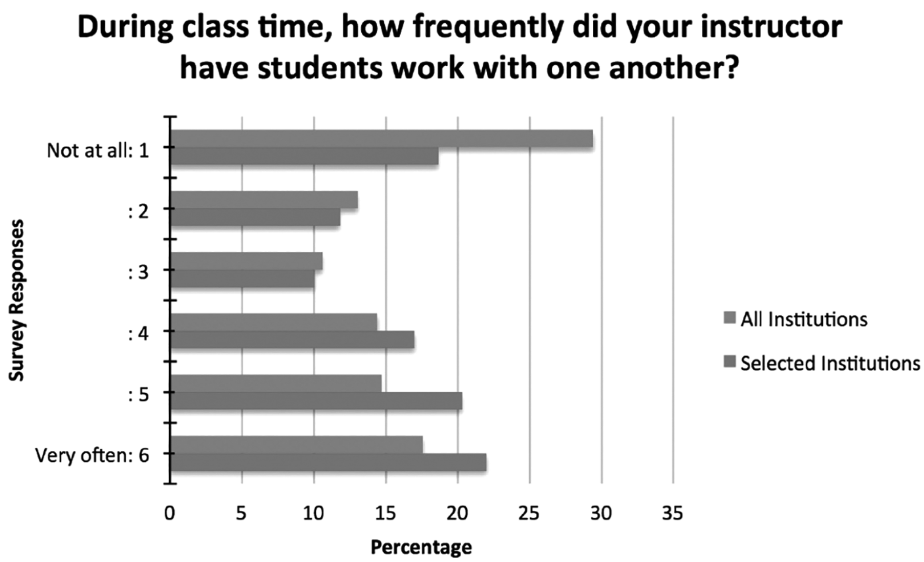


Fig. 7.3 Percentage of students reporting levels of lecture frequency at all institutions (N = 5,565) and at selected institutions (N = 1,221) (Source: Larsen, Glover, and Melhuish 2015, p. 95)

Questions about activities associated with ambitious teaching revealed a different pattern of responses between institutions selected for case study analysis and all institutions combined. Figure 7.4 reports students' responses on the frequency of working with other students during class time. *Not at all* (line 1) was the most common response

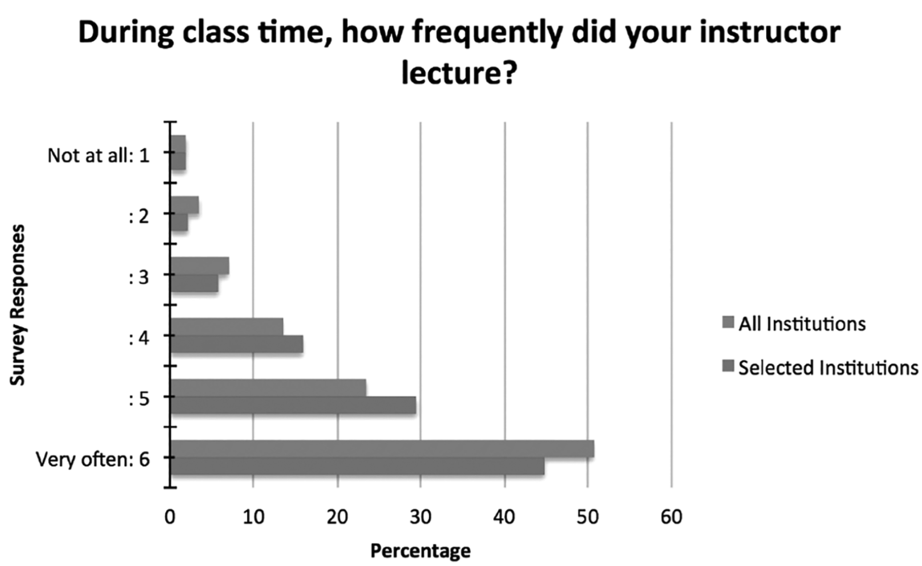


Fig. 7.4 Percentage of students reporting frequency levels of collaborative work at all institutions (N = 5,558) and at selected institutions (N = 1,220) (Source: Larsen, Glover, and Melhuish 2015, p. 95)

nationally (29 percent); however, frequent interaction (lines 5 and 6) were the next most reported options. At institutions selected for case study, only 19 percent of students selected *not at all.* This proportion was significantly ($z = 9.36$, $p < .01$) smaller than the proportion (33 percent) of students selecting *not at all* from non-selected institutions. Similar patterns were found for three other ambitious teaching activities: class time spent on working individually on problems, asking students to explain their thinking, and whole-class discussion (Larsen, Glover, and Melhuish 2015).

Persistence in the Calculus Sequence

The CSPCC survey also asked students Likert-scale questions about their confidence, interest, and enjoyment of mathematics, both at the start of the term and at the end of the term. Consistent with the self-reports on how well they felt they were ready for college calculus, students across institution types reported large drops in these affective measures. We next highlight an analysis that examines the relationship between confidence and persistence in the calculus sequence. In the following discussion, the term *switcher* refers to students who start the term intending to take more calculus but at the end of the term decide not to do so, effectively ending their ability to continue in a STEM major. In contrast, students who persist are those students that start and end the term intending to continue with the calculus sequence.

Controlling for student preparedness, intended career goals, institutional environment, and student perceptions of instructor quality and use of ambitious practices, Ellis, Fosdick, and Rasmussen (2016) found that women were 50 percent more likely to be switchers than men. Moreover, 35 percent of the STEM-intending women chose “I do not believe I understand the ideas of calculus I well enough to take Calculus II” as one of their reasons for switching. Only 14 percent of the men chose this reason. Conjecturing that capability might be a factor, Ellis et al. (2016) compared the change in student reported mathematical confidence among those students with standardized math scores at or above the national 85th percentile. Figure 7.5 shows that all such mathematically capable students lose mathematical confidence over the course of calculus 1, with

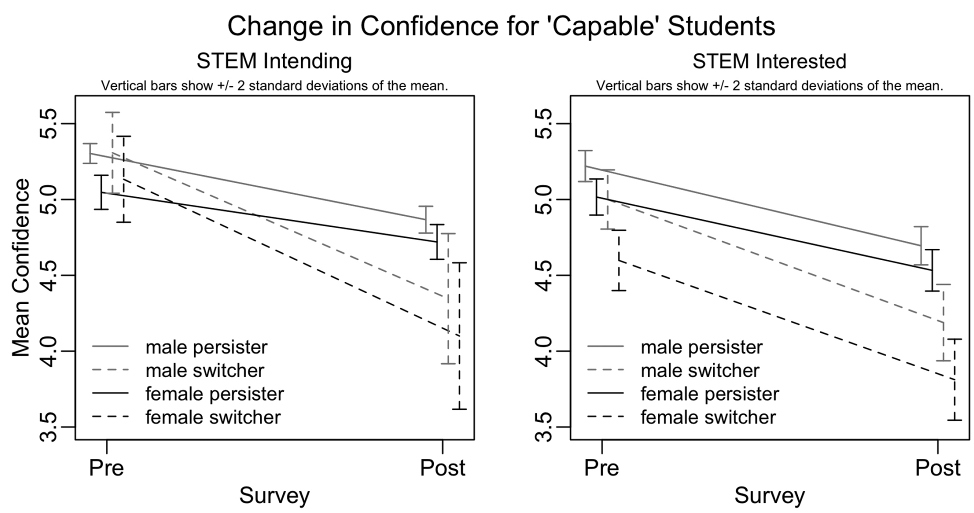


Fig. 7.5 Change in mathematical confidence at the beginning of the calculus 1 semester and at the end of the semester sorted by career intentions, gender, and persistence status, [*N* = 1524](Source: Ellis, Fosdick, and Rasmussen 2016, p. 9)

switchers reporting a greater decrease in confidence than students who persist. Moreover, women, while experiencing a similar decrease in mathematical confidence as men, start with a lower initial confidence level, and subsequently finish at an even lower level. This work points to female students' mathematical confidence or lack thereof when entering college as a major contributing factor to the persistence of women intending to pursue a STEM major.

Case Studies of Relatively More Successful Calculus 1 Programs

As stated at the start of the section on the CSPCC project, the national survey was followed by case studies of 16 mathematics departments whose survey results demonstrated some combination of better passing rates, greater persistence, and a smaller decline in student confidence, interest, and enjoyment in mathematics. Analysis of the PhD-granting departments by Rasmussen, Ellis, and Zazkis (2014) led to the identification of the following seven common features of their calculus 1 programs:

1. Attention to the effectiveness of placement procedures
2. Proactive student support services, including the fostering of student academic and social integration
3. Construction of challenging and engaging courses
4. Use of student-centered pedagogies and active-learning strategies
5. Coordination of instruction, including the building of communities of practice
6. Effective training of graduate teaching assistants
7. Regular use of local data to guide curricular and structural modifications

Each of these features is taken up in different chapters in *Insights and Recommendations from the MAA National Study of College Calculus* (Bressoud, Mesa, and Rasmussen 2015). We will return to these features in the overview of the MAA Progress through Calculus project.

The Factors Influencing College Success in Mathematics Project

One of the findings from the CSPCC study was that depending on the type of postsecondary institution, anywhere between a quarter and two-thirds of students enrolled in college calculus 1 took calculus in high school. A pertinent question is whether students who enroll in calculus 1 in college are well served by studying it first in high school. The national survey carried out as part of the *Factors Influencing College Success in Mathematics* (FICSMath) project provides an answer to this question.

To answer this question, the FICSMath project conducted a large-scale study of 10,437 students in mainstream calculus 1 in the fall of 2009 using a stratified random sample of 134 U.S. colleges and universities (Sadler and Sonnert 2018). As shown in figure 7.6, the authors found that taking calculus in high school led to an improvement in the college calculus grade for all levels of preparation. For example, for students in their first year of college with an average level of preparation, the benefit is five points, or half a grade. The authors also found that the benefit is greatest for students with the weakest preparation. Delaying calculus until the second year of college, however, lessens the benefit.

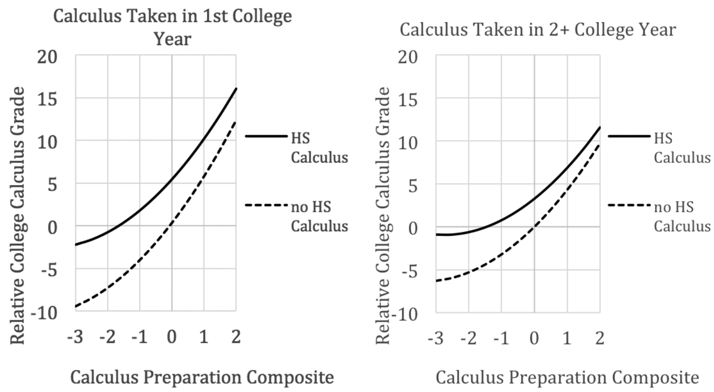
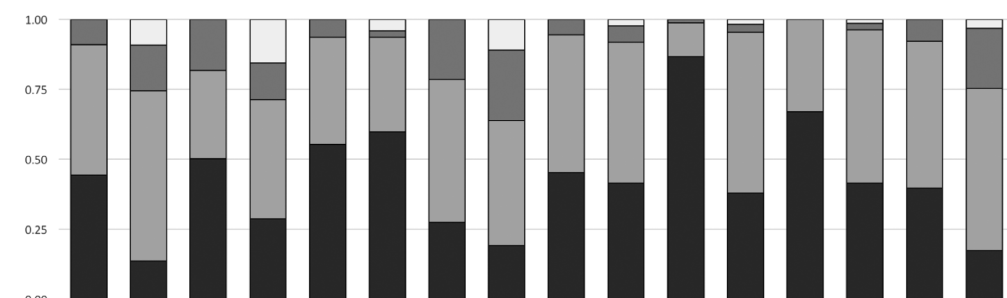


Fig. 7.6 Relationship between college calculus performance, high school preparation, taking high school calculus, and year taking calculus in college. (Sadler and Sonnert 2018, p. 319)

The MAA Progress through Calculus Project

In spring 2015 the MAA's *Progress through Calculus* project conducted a census survey of all U.S. mathematics departments that offer a graduate degree in mathematics. While the CSPCC study examined calculus 1 across four different types of institutions, this study narrowed the focus to PhD- and master-degree-granting departments but widened the investigation to include the entire precalculus through calculus 2 sequence. The broad aim of the survey was to learn about departmental practices, priorities, and concerns with respect to their mainstream courses in precalculus through single variable calculus. The response rate was an impressive 75 percent and 59 percent from PhD-degree-granting and master-degree-granting departments, respectively.

For each of the common features identified in the CSPCC study for the relatively more successful calculus 1 programs, the census survey asked department chairs to indicate (1) their perception of its importance for a strong precalculus through calculus 2 sequence and (2) a self-assessment of their program's success at implementation. Results are presented in figure 7.7. The red-arrow line segments highlight the difference in self-assessment for each feature being very important and their implication of that feature being very successful.



	1. Active learning		2. GTA professional development		3a. Uniform course components		3b. Regular instructor mtgs		4. Challenging courses		5. Student placement		6. Student support resources		7. Use of local data	
	Imp.	Suc.	Imp.	Suc.	Imp.	Suc.	Imp.	Suc.	Imp.	Suc.	Imp.	Suc.	Imp.	Suc.	Imp.	Suc.
N.A.		0.09		0.16		0.04		0.11		0.02		0.02		0.01		0.03
Not	0.09	0.16	0.18	0.13	0.06	0.02	0.22	0.25	0.06	0.06	0.01	0.03	0.00	0.02	0.08	0.21
Some	0.47	0.61	0.32	0.42	0.38	0.34	0.51	0.45	0.49	0.50	0.12	0.58	0.33	0.55	0.53	0.58
Very	0.44	0.14	0.50	0.29	0.55	0.60	0.27	0.19	0.45	0.42	0.87	0.38	0.67	0.42	0.40	0.17

Fig. 7.7 Perceived importance of and self-assessments of successful implementation of each feature of successful calculus 1 programs (Source: Rasmussen et al. 2019, p. 102)

One notable difference between the assessment of very important and very successful implementation is active learning. In particular, 44 percent of respondents said that active learning was very important for student success in precalculus through calculus 2, but only 15 percent reported that they were very successful at implementing active learning. Other similar and noteworthy differences exist when it comes to graduate teaching assistant (GTA) professional development, placement, and use of local data. In contrast, mathematics departments across the U.S. indicate relative parity between important and success in implementation when it comes to challenging and engaging courses and uniform course components. One way that the field can make use of these results is in the design of professional development opportunities (see chapter 6).

Chapter 8: Programs for Special Populations of Students

High school students who complete the standard college-bound curriculum that enables them to take precalculus before entering twelfth grade have three potential paths if they want to continue the study of mathematics in twelfth grade. First, if a student's school is very small and no two- or four-year college or university is nearby, then the student may be able to take an individualized course under a teacher's guidance or over the internet. Second, if the student's school is near a college or university, the student may be able to take a college course and apply the credit toward high school graduation. Third, if enough students in a school have completed the standard college-bound curriculum that enables them to take precalculus before entering twelfth grade, then the school may wish to offer Advanced Placement (AP) courses.

Advanced Placement Programs

In 1955, under the auspices of the College Board, the Educational Testing Service (ETS) created the Advanced Placement (AP) Program to enable students to take college-level work before graduating from high school (Handwerk et al. 2008). High schools participating in this program offer courses with syllabi designed to align with introductory college courses. In the 2018–19 school year, 38 AP courses were offered in seven different disciplinary areas, with more than 22,678 high schools worldwide participating and more than 2,825,710 individual students taking at least one examination. Most AP courses are a year in length. (https://secure-media.collegeboard.org/digitalServices/pdf/research/2019/Program-Summary-Report-2019.pdf)

When AP courses are taken in the eleventh grade or earlier, they can be considered with a student's application to a college and may factor into admissions decisions. Although scores on AP tests taken in the twelfth grade are not available to colleges before admission decisions are made, enrollment itself in AP courses tends to signify that an applicant is a serious student, and if the high school is known to be scholastically oriented, enrollment in an AP course can increase the student's chances of admission to some colleges.

The frameworks for Advanced Placement courses are developed, and modified periodically, by national committees of the College Board consisting of both secondary and university teachers from each particular content area. In 2019, the Course and Exam Descriptions for every Advanced Placement course were modified to bring them into alignment with all Advanced Placement offerings. Based on the Understanding by Design (Wiggins and McTighe 2005) model, the frameworks specify what students must know, be able to do, and understand in each course, with a focus on big ideas that encompass core principles, theories, and processes of the discipline.

The *Course and Exam Descriptions* (CEDs) for each Advanced Placement course organizes the content and skills that are the focus of the corresponding college course and that appear on the AP Exam into a series of units that represent a sequence found in widely adopted college textbooks. The CEDs provide suggestions for teaching core concepts that include activities, instructional approaches, and a library of AP exam questions. The new resources also include AP Classroom, a dedicated online platform featuring resources and tools to provide yearlong support to teachers and to enable students to receive meaningful feedback on their progress.

In May of each year, students in each Advanced Placement course take a nationwide exam, which is scored in June by teams of high school teachers and college instructors, with the scores sent to postsecondary institutions designated by the student. Colleges have the option of offering college credit, placing students in more advanced classes (with or without credit), or ignoring the scores that students receive. Many colleges take scores on AP tests into account when placing students into courses. Scores on AP tests range from a top score of 5 down to 1. The American Council on Education and the College Board have developed a matrix for universities to use in developing a policy to provide advanced placement or award credit for AP coursework (College Board CED 2019). For example, the College Board has suggested that scores of 5 down to 3 are related to levels of preparation and achievement in related university courses in the following manner: 5 is equivalent to extremely well qualified; 4 is equivalent to well qualified; and 3 is equivalent to qualified (College Board CED 2019). Lower scores are not suggested as constituting a basis for earned credit for advanced placement.

Advanced Placement Programs in Calculus

Two AP courses are offered in calculus: Calculus AB (since 1955/56) and Calculus BC (since 1968/69) (https://apcentral.collegeboard.org/courses). Calculus AB is designed to be the equivalent of a first-semester college calculus course devoted to topics in differential and integral calculus. AP Calculus BC is designed to be the equivalent to both first- and second-semester college calculus courses. AP Calculus BC applies the content and skills learned in AP Calculus AB to parametrically defined curves, polar curves, and vector-valued functions; develops additional integration techniques and applications; and introduces the topics of sequences and series (College Board CED 2019).

In May of each year, ETS administers nationwide exams for each of the AP calculus courses. Each of the exams is scheduled for three hours and fifteen minutes. In the 2018–19 school year, 300,659 students took the Calculus AB examination, and 139,195 took the Calculus BC examination. The percentages of 3 or better for the individual examinations were 63.3 percent for Calculus AB and 81.5 percent for Calculus BC (College Board 2019b Exam Distribution Scores: https://www.totalregistration.net/AP-Exam-Registration-Service/AP-Exam-Score-Distributions.php)

Students enrolled in Calculus BC receive a Calculus AB subscore for their performance on the items on the Calculus BC examination that examine Calculus AB content. This portion comprises about 60 percent of the BC examination. This subscore gives colleges and universities, as well as teachers and counselors at the students' secondary school, additional information about their performance on Calculus AB topics. Each examination is divided into two sections, each worth 50 percent of the total score. The multiple-choice portion consists of 45 questions to be completed in 105 minutes. These questions are divided into two parts, part A and part B. Part A consists of 30 items to be completed in 60 minutes and does not allow the use of a calculator. Part B, the second major part of the multiple-choice items, consists of 15 questions to be completed in 45 minutes and includes some questions for which a graphing calculator is required. The second section of the examination is a constructed-response section, which consists of six problems to be completed in 90 minutes and is also divided into parts A and B. Part A consists of two problems to be completed in 30 minutes and requires the use of a graphing calculator. Part B consists of four problems to be completed in 60 minutes and does not allow the use of a calculator (College Board 2014).

In 2014, the AP Calculus Curriculum Framework was redesigned to bring a sharper focus to the course and provide a conceptual framework for students to connect and apply the central knowledge. As part of the larger College Board initiative described above, the course was modified slightly in 2019 to encompass the Big Ideas, threads that run throughout the course, serve as the foundation of the course, and allow students to create meaningful connections among concepts. The Big Ideas in Calculus are Change, Limits, and Analysis of Functions.

The *Course and Exam Description* (CED) for Calculus AB and BC includes connections to mathematical practices associated with students' development of the depth of understanding desired for these big ideas. The mathematical practices for AP Calculus (MPACs) define ways of thinking that allow a successful learner to develop understanding of the big ideas while relating them to other broad sets of cognitive skills that allow such students to apply their knowledge across the course. The four MPACs for AP Calculus are—

1. Implementing Mathematical Processes: Determine expressions and values using mathematical procedures and rules;
2. Connecting Representations: Translate mathematical information from a single representation or across multiple representations;
3. Justification: Justify reasoning and solutions; and
4. Communication and Notation: Use correct notation, language, and mathematical conventions to communicate results or solutions.

These MPACs link the big ideas to enduring understandings related to each big idea, learning objectives specific to the course, and essential knowledge related to the facts, concepts, and principles central to knowing and being able to creatively and productively use the knowledge of calculus that is key to each AP course.

Advanced Placement Program in Statistics

The AP Statistics course was first offered in 1997 with 7,600 students taking the exam that year. The course introduces students to the major concepts and the tools for collecting, analyzing, and drawing conclusions from data. Four themes are highlighted throughout the content, skills, and assessment in the AP Statistics course: 1) exploring data, 2) sampling and experimentation, 3) probability and simulation, and 4) statistical inference. Students use technology, investigations, problem solving, and writing to build conceptual understanding. The only prerequisite is the successful completion of a second-year course in algebra. The course is designed to prepare students for advanced coursework in statistics or other fields using statistical reasoning and for active, informed engagement with the data they will encounter in the world, that is, to be able to interpret such data and use it to make informed decisions. The AP Statistics course is equivalent to a one-semester, introductory, non-calculus-based college course in statistics.

The enrollment in AP Statistics grew rapidly; two years after its introduction in 1997, more than 25,000 students took the exam in 1999. By 2019, statistics was the tenth largest Advanced Placement examination with 219,392 students from 9,412 schools taking the exam (combined calculus AB and BC is the second most taken examination after English language and composition). Of those students, 59.2 percent earned a score of 3 or better.

The Advanced Placement Statistics exam is three hours long and consists of Section I and Section II. Section I is composed of 40 multiple-choice questions to be done in

90 minutes and counts for 50 percent of the total score. Section II consists of a 90-minute block composed of Part A, five free-response questions recommended to be done in 65 minutes for 37.5 percent of the total score, and Part B, one investigative task recommended to be done in 25 minutes for 12.5 percent of the total score. Students are given formulas and tables but are expected to bring a graphing calculator with statistical capabilities to the exam.

Similar to AP Calculus, in 2019, the *Course and Exam Description* (CED) for AP Statistics was modified to bring it into alignment with the other Advanced Placement offerings and to make clear the integration of the big ideas throughout the course. The Big Ideas in Statistics are variation and distribution; patterns and uncertainty; and data-based predictions, decisions, and conclusions. The *Course and Exam Description* (CED) for Statistics describes skills aligned to specific learning objectives that a student should be able to do while exploring course concepts. The skill categories for AP Statistics are—

1. Selecting statistical methods: Select methods for collecting and/or analyzing data for statistical inference;
2. Data analysis: Describe patterns, trends, associations, and relationships in data;
3. Using probability and simulation: Explore random phenomena; and
4. Statistical argumentation: Develop an explanation or justify a conclusion using evidence from data, definitions, or statistical inference (College Board 2019a).

Advanced Placement Program in Computer Science

The College Board offers two Advanced Placement (AP) examinations focused on computer science, AP Computer Science A and AP Computer Science Principles. The course AP Computer Science A focuses on object-oriented programming methodology and imperative problem solving, concentrating on problem solving and algorithm development. The course is meant to be the equivalent of a first-semester college-level course in computer science. It also includes the study of the organization of data structures and algorithm design. The course employs a subset of the Java programming language that is described in the course description (College Board 2014b).

The course AP Computer Science Principles focuses on introducing students to computer science concepts and ways of exploring data and other forms of information. Students explore applications of computer science techniques in a variety of disciplines that enhance their skill and analysis tools for applications in later coursework. Techniques include working with computer graphics to illustrate a process and manipulating and computing with large data sets in studying trends. The course is structured around big ideas in a manner similar to AP Calculus.

Special Schools and Programs for Students in Mathematics

The National Consortium of Secondary STEM Schools (NCSSS) includes more than 100 institutional members with 40,000 students and 1,600 educators. The goal of the consortium, as its name indicates, is to foster, support, and advance the efforts of specialized schools to attract students and prepare them academically for leadership in the subject areas of mathematics, science, and technology. Some members are boarding schools requiring state residence and highly competitive examinations for entrance, a few are specialized local high schools, and others are regional centers that students may attend for a half- or full-day for a single year. Twenty-six states have one or more schools that are institutional members of NCSSS (NCSSS 2020).

Programs for K–12 students

In addition to the specialized STEM schools described above, students can take advantage of advanced mathematics programs through public and private schools, universities, or other organizations. The first type of program is a diploma-based program that follows an international curriculum managed by the International Baccalaureate Organization (IBO), headquartered in The Hague, The Netherlands. More than 1.3 million students were enrolled in some type of IB program worldwide in 2019. In the United States, 1,853 schools are authorized to offer the IBO programs in some form; more than 5,000 schools are authorized worldwide in more than 150 countries. Of the 1,853 schools in the United States, 949 offer the Diploma Program, a demanding two-year pre-college program that leads to examinations and is designed for students who are 16 to 19 years of age. The remaining schools offer the Middle Years Program (669) or the Primary Years Program (579), both of which are designed for younger students. In 2012, the IBO initiated the Career-Related Program, aimed at meeting the needs of students engaged in career-related programs. This program is currently offered in 120 schools in the United States (IBO 2018).

A second type of special program is a university-centered program offered in two formats as a summer program in mathematics for very capable secondary students. The first format follows a model initiated by the late Julian Stanley at Johns Hopkins University in the 1970s, identifying talent in the upper elementary or middle school grades and offering accelerated courses (usually in the summer but sometimes during the school year) and online courses to enable those students to study more advanced mathematics as well as other disciplines at a younger age. Today, the Johns Hopkins Center for Talented Youth (CTY) program serves gifted students in grades 2–12 through a wide range of programs across the U.S. and in Hong Kong (Johns Hopkins University 2020).

The second format for summer mathematics enhancement follows a model initiated around the same time as the CTY program at Johns Hopkins University. The Ross Mathematics Program was developed by Arnold Ross at Notre Dame University in 1957. In the Ross Program, precollege students are taught mathematics in a different way from the approach that they would normally be exposed to in school. They are expected to solve problems and deduce propositions in somewhat the same manner as a professional mathematician—by working through the problems on their own or in collaborative groups with some outside hints from mentors. Now operating out of Ohio Dominican University, the Ross Program recruits both regionally and nationally, and provides opportunities for students across the entire nation. Ross Math Asia also offers a summer mathematics program following the same format but in Nanjing, China (Ross Mathematics Program 2020).

A third approach to mathematics enrichment and advancement for secondary school students comes through mathematics clubs. The largest organization of mathematics clubs in the United States is Mu Alpha Theta, founded in 1957. Mu Alpha Theta has more than 2,600 high school and community college chapters and more than 124,000 student members across the United States. Its purpose is to stimulate interest in mathematics by providing recognition of superior mathematical scholarship in students. In addition to holding regional meetings and an annual national meeting, Mu Alpha Theta also publishes a newsletter and provides several other resources for its student members (Mu Alpha Theta 2020).

Programs for Undergraduate and Graduate Students

The National Science Foundation (NSF) funds a large number of research opportunities for undergraduate students through its Research Experiences for Undergraduates (REU) programs (NSF 2020). A REU site consists of a group of 10 to 15 undergraduates who work on aspects of the active research programs of the sponsoring college or university. Each student is associated with a specific research project and works closely with the faculty and other researchers involved in that program. Students are granted stipends and, in many instances, assistance with housing and travel. Undergraduate students supported with NSF funds must be citizens or permanent residents of the United States or its territories. In 2019–20, 58 REU sites with research opportunities were available in mathematics. A list of the REU sites for 2019–20 can be found at NSF's REU website (NSF 2020).

The National Heart, Blood and Lung Institute of the U.S. Department of Health and Human Services offers another federally funded summer institute for undergraduates: the Summer Institute in Biostatistics (SIBS). This program provides a six- to seven-week training course for undergraduates and beginning graduate students interested in learning about biostatistics. Additional information on the program can be found at the SIBS website: https://www.nhlbi.nih.gov/grants-and-training/summer-institute-biostatistics.

Mathematics Competitions for K–12 Students

Mathematics competitions in the United States are voluntary for both individuals and schools. Some middle schools and high schools have mathematics teams, often competing in events operated by local professional organizations. Descriptions follow of the larger competitions of national scope.

MATHCOUNTS

The National Society of Professional Engineers, the CNA Foundation, and NCTM founded MATHCOUNTS in 1983 to increase interest and involvement in mathematics and to assist in developing a technologically literate population. The MATHCOUNTS Foundation now operates the competition. Sponsors include the National Society of Professional Engineers, the National Council of Teachers of Mathematics, CNA, Raytheon Company, Northrup Grumman Foundation, the U.S. Department of Defense, Phillips 66, Texas Instruments Incorporated, 3M Foundation, Art of Problem Solving (AoPS), and Next Thought. Participation is restricted to students in grades 6, 7, and 8. MATHCOUNTS activities involve school-based club and competition activities. Operating on a broader range are competitions at the chapter level of the Society of Professional Engineers, the state level, and the national level.

American Mathematics Competitions (AMC)

This program, administered by the MAA, is a series of examinations and curriculum materials that build problem-solving skills and mathematical knowledge in middle and high school students. The series includes AMC 8, an examination in middle school mathematics that provides the opportunity for middle school students to develop positive attitudes toward analytical thinking and mathematics, AMC10 for students in grade 10 or below, and AMC12 for students in grade 12 or below. High scoring AMC10 and AMC12 participants are invited to participate in the American Invitational Mathematics Examination (AIME); high-scoring AIME participants are invited to participate in AMC's top invitational competition, the United States of America Mathematical Olympiad (USAMO), a six-question, six-hour exam that is used to determine the U.S. team members for the International Mathematical Olympiad (IMO).

The American Statistical Association and National Council of Teachers of Mathematics Poster Competition and Project Competition

The ASA/NCTM Joint Committee on the Curriculum in Statistics and Probability and the ASA's education department sponsor an annual Data Visualization Competition for students in grades K–12 and a Statistics Project Competition for middle and high school students. Posters are to be developed and constructed on flat poster board by a student or group of students. For K–grade 3, there is no limit on the number of students in the group. Above that level, the maximum number of students who may work on a poster is four. The subject matter is of the student's or students' own choosing, but the submitted posters are assessed on "demonstration of important relationships and patterns, obvious conclusions, and ability to stand alone, even without the explanatory paragraph on the back" of the poster. The posters are classified into grade intervals for judging: K–3, 4–6, 7–9, and 10–12. Original research studies and results are accepted along with the data, statement of purpose, and method of collection of the data.

Projects, like the posters, have subject matter that is selected by the participants themselves. A group of students competing together may not have more than four students working on a project. The students may have some adult guidance, but the amount of guidance must be detailed in the project write-up. The statistical methods in the projects are limited to what might be found in an introductory statistics class. Statisticians and teachers use the stages of a statistical experiment in evaluating the projects.

The entry rules and regulations for both posters and projects can be found at http://www.amstat.org/education/posterprojects/posterrules.cfm. Additional information and education available for teachers and students at the ASA website is detailed at http://amstat.org/education.

The Math League

Founded in 1977, the Math League specializes in mathematics contests, books, and computer software designed to stimulate interest and confidence in mathematics for students from fourth grade through high school. These contests involve students in individual and team-based competitions. Contest problems are designed to cover a range of mathematical knowledge for each grade level and require no additional knowledge of mathematics beyond the grade level that they test. More information can be found at https://www.mathleague.com.

The American Regions Mathematics League (ARML)

ARML, begun in 1976 as the Atlantic Region Mathematics League, organizes a competition of teams of high school students who represent their school, local area, state, or country (outside the United States). This contest takes place during April of the school year. Teams of students from different schools compete in a contest to solve a set of honors-level problems in a 45-minute period of time. The papers are then mailed in and evaluated by a team of judges. This national competition, which takes place toward the end of the school year, occurs at three sites. (http://www.arml2.com/arml_2019/page/index.php?page_type=public&page=home)

High School Mathematical Contest in Modeling (HiMCM)

Sponsored by the Consortium for Mathematics and its Applications (COMAP), HiMCM is an open competition for teams of up to four students. The competition was designed to provide students with an opportunity to work as members of a team on a mathematical modeling problem testing their capabilities to merge their mathematical knowledge with knowledge of a particular context, their ability to use technology where necessary, and their capability to write a complete and coherent explanation of their model and demonstrate its value in determining one or more solutions to the problem posed. The problem statements are available to students in mid-November and the teams have about two weeks to work on the solution. The contest currently costs schools $100 per team to compensate the applied mathematicians who review and assess the students' work. Certificates and plaques are awarded to the participants. Sample problems can be found at the HiMCM website: http://www.comap.com/highschool/contests/himcm/index.html.

The MathWorks Math Modeling Challenge (M3C)

Sponsored by MathWorks (creators of MATLAB software), this is a 14-hour competition for teams of up to five high school students. There is no cost, and each high school can sponsor up to two teams. The competition is held on a weekend in March. The top six teams present their work to a panel of mathematicians for final judging. Scholarship funds are awarded to the top teams.

Modeling the Future Challenge

This competition is hosted by The Actuarial Foundation in partnership with The Institute of Competition Sciences. The challenge is an extended modeling opportunity that has several rounds with different problems focusing on a specific theme. The 2019–20 theme was Agriculture, Water, and Climate Change. The challenge begins in the fall and ends in the spring of each school year. The competition allows teams of up to five students to compete and is free for all participants. A panel of actuarial scientists judge the student submissions with scholarship funds going to the winners.

Mathematics Competitions for Undergraduate Students

A number of competitions are open to undergraduates in the United States. The following contests are among the most prominent.

Student Mathematics League (SML)

The SML competition is for students enrolled in two-year colleges. Originally founded in 1970 by Nassau Community College in Garden City, New York, the competition came under the sponsorship of the American Mathematical Association of Two-Year Colleges (AMATYC) in 1981. SML involves more than 8,000 two-year college students from 165 colleges in 35 states and Bermuda in its annual cycle of two examinations, one in November and the other in March. In each academic year, a team and an individual champion are determined, and a scholarship is awarded as well. In addition to the national results, regional individual and team standings are determined for the eight regional sectors of the United States in the AMATYC governing structure. Each set of these regional results, individual and team, ranks the top five entrants, individual or team. The examinations are based on the standard syllabus in college algebra and trigonometry and may involve precalculus-level algebra, trigonometry, synthetic and analytic geometry, and probability. All questions are short-answer or multiple choice. (https://amatyc.site-ym.com/page/StudentMathLeague)

Mathematical Contest in Modeling (MCM) and the Interdisciplinary Contest in Modeling (ICM)

This competition sponsored by COMAP is a four-day competition similar to HiMCM for teams of three students. The contest is designed for collegiate modeling teams, but high schools can participate. Like HiMCM, the contest currently costs schools $100 per team with certificates and plaques for the participants. (https://www.comap.com/undergraduate/contests/index.html)

Undergraduate Statistics Project Competition (USPROC; Consortium for the Advancement of Undergraduate Statistics [CAUSE])

This competition is open to undergraduate students studying statistics at the introductory or intermediate level. Teams of one or two students select a topic requiring the application of statistics to solve, then carry out the appropriate statistical analysis on the collected data, and report the conclusions. The contest is held every other year, and cash prizes are awarded to the top three teams. Additional information about CAUSE can be found at https://www.causeweb.org.

William Lowell Putnam Mathematical Competition

Begun in 1938, The Putnam Competition is for undergraduate mathematics students and is administered annually by MAA on the first Saturday in December. The competition is perhaps the most rigorous and prestigious mathematics examination held each year. The competition consists of two 3-hour sessions, one in the morning and one in the afternoon. Students work individually on six challenging mathematical problems. Institutions are ranked according to the sum of the scores of their three highest scoring participants. (https://www.maa.org/math-competitions/putnam-competition)

Chapter 9: Professional Organizations Supporting the Mathematical Sciences

Teachers, administrators, parents, and the public often want access to information on professional mathematics organizations in the United States that are involved in supporting the mathematical sciences. The following list of U.S. professional organizations is separated according to whether membership is by appointment (closed) or is open to anyone on the basis of their own desires, and gives their contact points, a brief overview, and any regular publications that they may produce. (References to the organizations in this chapter are given in full and will not be repeated in the Bibliography.)

Closed-Membership Organizations

Members of the following two organizations are selected or appointed by the organizations themselves or the National Research Council.

Conference Board of the Mathematical Sciences

Founded in 1960, the Conference Board of the Mathematical Sciences (CBMS) is an umbrella organization consisting of the major professional societies in the mathematical sciences in the United States and composed of the CBMS Executive Committee and the presidents and executive directors of the member societies. Its purpose is to promote understanding and cooperation among the national professional organizations in mathematics so that they can work together, support one another in research, advocate for the improvement of education, and amplify the expansion of the mathematical sciences. These organizations currently belong to CBMS: American Mathematical Association of Two-Year Colleges (AMATYC), American Mathematical Society (AMS), Association of Mathematics Teacher Educators (AMTE), American Statistical Association (ASA), Association for Symbolic Logic (ASL), Association for Women in Mathematics (AWM), Association of State Supervisors of Mathematics (ASSM), Benjamin Banneker Association (BBA), Institute of Mathematical Statistics (IMS), Institute for Operations Research and the Management Sciences (INFORMS), Mathematical Association of America (MAA), National Association of Mathematicians (NAM), National Council of Supervisors of Mathematics (NCSM), National Council of Teachers of Mathematics (NCTM), Society for Industrial and Applied Mathematics (SIAM), Society of Actuaries (SOA), TODOS: Mathematics for All (TODOS), and Women and Mathematics Education (WME). (Email: bressoud@macalester.edu)

United States National Commission on Mathematics Instruction (USNC/MI)

The U.S. National Academy of Sciences (NAS) is the national adhering body to the International Commission on Mathematical Instruction (ICMI). Acting through the National Research Council (NRC), the United States National Commission on Mathematics Instruction (USNC/MI), which was founded in 1978, conducts the work of the ICMI in the United States and fosters other international collaborations in mathematics education (ICMI 2015). The NRC Board of Mathematical Sciences, CBMS, and NCTM suggest nominees for the USNC/MI.

Open-Membership Organizations

Numerous mathematics organizations in the United States have open memberships—that is, the members self-select and join on their own, often through the payment of an annual membership fee. The listings are arranged by the primary focus of the organizations.

Kindergarten–Grade 12

- **National Council of Supervisors of Mathematics (NCSM;** founded 1969)
  Email: office@mathedleadership.org; Website: www.mathedleadership.org/
  Journal: *Journal of Mathematics Education Leadership*
- **National Council of Teachers of Mathematics (NCTM;** founded 1920)
  Email: nctm@nctm.org; Website: www.nctm.org
  Journals: *Mathematics Teacher Educator* (joint publication with AMTE); *Journal for Research in Mathematics Education; Mathematics Teacher: Learning and Teaching PK–12* (first issue January 2020, replacing three journals: *Teaching Children Mathematics* [TCM], *Mathematics Teaching in the Middle School* [MTMS], and *Mathematics Teacher* [MT])
- **School Science and Mathematics Association (SSMA;** founded 1901)
  Email: office@ssma.org; Website: www.ssma.org
  Journal: *School Science and Mathematics*
- **Women and Mathematics Education (WME;** founded 1978)
  Website: www.wme-usa.org

Postsecondary

- **American Mathematical Association of Two-Year Colleges (AMATYC;** founded 1974)
  Email: amatyc@amatyc.org; Website: www.amatyc.org
  Journal: *MathAMATYC Educator*
- **American Mathematical Society (AMS;** founded 1888)
  Email: ams@ams.org; Website: www.ams.org
  Journals: *Notices of the American Mathematical Society*, *Bulletin of the American Mathematical Society*
- **American Statistical Association (ASA;** founded 1839)
  Email: asainfo@amstat.org; Website: www.amstat.org
  Journal: *The American Statistician*, *Chance*, *Significance*, and others devoted to research in statistics
- **Mathematical Association of America (MAA;** founded 1915)
  Email: maahq@maa.org; Website: www.maa.org
  Journals: *The American Mathematical Monthly*, *College Mathematics Journal*, *Mathematics Magazine*, *Math Horizons*
- **National Association of Mathematicians (NAM;** founded 1969)
  Website: www.nam-math.org
  Journal: *NAM Newsletter*

Special Groups in Mathematics Education

- **Association of Mathematics Teacher Educators (AMTE;** founded 1993)
  Website: www.amte.net
  Journals: *Mathematics Teacher Educator*, *AMTE Connections*

- **Benjamin Banneker Association (BBA;** founded 1986)
  Website: www.bbamath.org
- **National Association of Community College Teacher Education Programs (NACCTEP;** founded 2003)
  Website: www.nacctep.org
- **Research Council on Mathematics Learning (RCML;** founded 1974)
  Website: www.rcml-math.org
  Journal: *Investigations in Mathematics Learning*
- **Special Interest Group for Research in Mathematics Education (SIGRME)**
  Website: www.sigrme.org
- **TODOS: Mathematics for ALL** (founded 2003)
  Email: requests@todos-math.org; Website: www.todos-math.org
  Journal: *Noticias de TODOS*

References

Achieve. 2014. *Closing the Expectations Gap: 2013 Annual Report on the Alignment of State K–12 Policies and Practice with the Demands of College and Careers*. Washington, DC: Achieve. https://www.achieve.org/achieving-common-core.

———. 2020. Graduation Requirements Data Explorer. Washington, DC: Achieve. https://highschool.achieve.org/graduation-requirements-data-explorer.

American Mathematical Association of Two-Year Colleges (AMATYC). 2006. *Beyond Crossroads: Implementing Mathematics Standards in the First Two Years of College.* Memphis, TN: AMATYC, Southwest Tennessee Community College. http://beyondcrossroads.matyc.org/doc/PDFs/BCAll.pdf.

———. 2018. *IMPACT: Improving Mathematical Prowess and College Teaching.* Memphis, TN: AMATYC.

American Statistical Association (ASA). 2007. *Guidelines for Assessment and Instruction in Statistics Education (GAISE Report): A PreK-12 Curriculum Framework.* Alexandria, VA: ASA. https://www.amstat.org/asa/files/pdfs/GAISE/GAISEPreK-12_Full.pdf.

———. 2016. *Guidelines for Assessment and Instruction in Statistics Education College Report*. Alexandria, VA: ASA. https://www.amstat.org/asa/files/pdfs/GAISE/GaiseCollege_Full.pdf.

———. 2020. *Preparing K–12 Teachers of Statistics: A Joint Position Statement of the American Statistical Association and National Council of Teachers of Mathematics*. Alexandria, VA: ASA. https://www.amstat.org/asa/files/pdfs/POL-JointASA-NCTM-Statement.pdf.

———. 2020. *Guidelines for Assessment and Instruction in Statistics Education (GAISE II Report): A PreK–12 Curriculum Framework.* Alexandria, VA: ASA.

Apkarian, Naneh, Estrella Johnson, Jeffrey Raker, Marilyne Stains, Charles Henderson, and Melissa Dancy. 2020. “Assessing the Uptake of Research Based Instructional Strategies by Postsecondary Mathematics Instructors.” *Proceedings of the 23rd Conference on Research in Undergraduate Mathematics Education*, Boston, MA, February 27–29.

Apkarian, Naneh, and Dana Kirin. 2017. *Progress through Calculus: Census Survey Technical Report*. Washington, DC: MAA. https://bit.ly/PtCCensusReport.

Association of Mathematics Teacher Educators (AMTE). 2017. *Standards for Preparing Teachers of Mathematics.* Raleigh, NC: AMTE. amte.net/standards.

Association of Public and Land-Grant Universities (APLU). 2020. *Mathematics Teacher Education Partnership*. Washington, DC: APLU. https://www.aplu.org/projects-and-initiatives/stem-education/mathematics-teacher-education-partnership.

Bargagliotti, Anna and Christine Franklin. Forthcoming 2021. *Statistics and Data Analysis for Teachers*. To be published jointly by the American Statistical Association (ASA) and the National Council of Teachers of Mathematics (NCTM).

Berry, Robert Q, III. 2019. “Let’s Talk about Mathematics Pathways.” *NCTM President’s Message.* December 2019. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Robert-Q_-Berry-III/Lets-Talk-About-Mathematics-Pathways/.

Blair, Richelle, Ellen E. Kirkman, and James W. Maxwell. 2018. *Statistical Abstracts of Undergraduate Programs in the Mathematical Sciences in the United States: Fall 2015 CBMS Survey.* Providence, RI: American Mathematical Society.

———. 2012. *Statistical Abstracts of Undergraduate Programs in the Mathematical Sciences in the United States: Fall 2010 CBMS Survey.* Providence, RI: American Mathematical Society.

Bressoud, David. 2015."The Calculus Students." In *Insights and Recommendations from the MAA National Study of College Calculus*, edited by David Bressoud, Vilma Mesa, and Chris Rasmussen, pp. 1–16. Washington, DC: Mathematical Association of America Press.

Bressoud, David. 2018. "Explosive Growth of Advanced Undergraduate Statistics." *MAA Launchings*, blog, June 1, 2018. http://launchings.blogspot.com/2018/06/explosive-growth-of-advanced.html.

Bressoud, David, Vilma Mesa, and Chris Rasmussen. 2015. *Insights and Recommendations from the MAA National Study of College Calculus.* Washington, DC: The Mathematical Association of America.

Carmichael, Sheila Byrd, W. Stephen Wilson, Kathleen Porter-Magee, and Gabrielle Martino. 2010. *State of the State Standards—and the Common Core—2010.* Washington, DC: The Thomas B. Fordham Institute. https://fordhaminstitute.org/national/research/state-state-standards-and-common-core-2010.

Chen, Xianglei Duprey. 2016. *Remedial Course Taking at U.S. Public 2- and 4-Year Institutions: Scope, Experiences, and Outcomes* (NCES 2016-405). Washington, DC: U.S. Department of Education National Center for Education Statistics. http://nces.ed.gov/pubsearch.

College Board. 2006. *College Board Standards for College Success: Mathematics and Statistics.* New York: College Board. Accessed January 13, 2020. http://www.asainstitute.org/conference2008/featuredsessions/collegeboard-Math-Standards_92206.pdf

———. 2019a. *AP Statistics Course and Examination Description.* New York: College Board.

———. 2019b. *AP Calculus AB and BC Course and Examination Description.* New York: College Board.

———. 2019c. *Trends in College Pricing 2019*. New York: College Board. https://research.collegeboard.org/trends/college-pricing.

———. 2019d. *Trends in Student Aid 2019*. New York: College Board. https://research.collegeboard.org/pdf/trends-student-aid-2019-full-report.pdf.

Conference Board of the Mathematical Sciences (CBMS). 2001. *The Mathematical Education of Teachers I*. Providence, RI: American Mathematical Society; Washington, DC: Mathematical Association of America.

———. 2012. *The Mathematical Education of Teachers II*. Providence, RI: American Mathematical Society; Washington, DC: Mathematical Association of America. https://www.cbmsweb.org/the-mathematical-education-of-teachers.

———. 2016. *Active Learning in Post-Secondary Mathematics Education*. Washington, DC: CBMS. https://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education.

Cunningham, Josh. 2018. *Comprehensive School Choice Policy: A Guide for Legislators.* Washington, DC: National Conference of State Legislatures. http://www.ncsl.org/research/education/comprehensive-school-choice-policy.aspx.

Daro, Phil and Harold Asturias. 2019. "Branching Out: Designing High School Math Pathways for Equity." *Just Equations.*

Eagan, Kevin. 2016. *Becoming More Student-Centered? An Examination of Faculty Teaching Practices across STEM and Non-STEM Disciplines between 2004 and 2014*. Los Angeles: Higher Education Research Institute–University of California, Los Angeles.

Education Commission of the States. 2018. *Age Requirements for Free and Compulsory Education*. Denver: Education Commission of the States. https://www.ecs.org/age-requirements-for-free-and-compulsory-education/.

———. 2018. *50 State Comparison of Developmental Education Policies*. Denver: Education Commission of the States. https://www.ecs.org/50-state-comparison-developmental-education-policies/.

EdReports. 2020. *Compare Materials: Math K–8 and High School*. https://www.edreports.org/compare/results/math-k8.

Ellis, Jessica, Bailey K. Fosdick, and Chris Rasmussen. 2016. "Women 1.5 Times More Likely to Leave STEM Pipeline after Calculus Compared to Men: Lack of Mathematical Confidence a Potential Culprit." *PLoS ONE* 11, no. 7 (July). https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0157447.

Fairtest. 2019. *More Than 1060 Accredited Colleges and Universities That Do Not Use ACT/SAT Scores to Admit Substantial Numbers of Students into Bachelor-Degree Programs*. Cambridge, MA: The National Center for Fair and Open Testing. https://www.fairtest.org/university/optional.

Franklin, Christine, Anna Bargagliotti, Catherine Case, Gary Kader, Richard Scheaffer, and Denise Spangler. 2015. *The Statistical Education of Teachers*. Alexandria, VA: American Statistical Association. https://www.amstat.org/asa/files/pdfs/EDU-SET.pdf.

Freeman, Scott, Sarah L. Eddy, Miles McDonough, Michelle K. Smith, Nnadozie Okoroafor, Hannah Jordt, and Mary Pat Wenderoth. 2014. "Active Learning Increases Student Performance in Science, Engineering, and Mathematics." *Proceedings of the National Academy of Sciences* 111, no. 23.

Friedberg, Solomon, Diane Baron, Juliana Belding, Andrew Chen, Linda Dixon, Francis (Skip) Fennell, Douglas Fisher, Nancy Frey, Roger Howe, and Tim Shanahan. 2018. *The State of State Standards Post Common Core*. Washington, DC: Thomas B. Fordham Institute. https://fordhaminstitute.org/national/research/state-state-standards-post-common-core.

Gewertz, Catherine. 2017. "What States Are Using PARCC and Smarter Balanced?: An Interactive Breakdown of States' 2016–2017 Testing Plans." *Education Week*. February 15, 2017. Updated April 9, 2019. https://www.edweek.org/ew/section/multimedia/states-using-parcc-or-smarter-balanced.html.

Glander, Mark. 2017. *Selected Statistics from the Public Elementary and Secondary Education Universe: School Year 2015–16* (NCES 2018-052). U.S. Department of Education. Washington, DC: National Center for Education Statistics. https://nces.ed.gov/pubs2018/2018052/index.asp.

Gutiérrez, Rochelle. 2002. "Enabling the Practice of Mathematics Teachers in Context: Toward a New Equity Research Agenda." *Mathematical Thinking and Learning* 4, no. 2–3: 145–87.

Handwerk, Philip, Namrata Tognatta, Richard J. Coley, and Drew H. Gitomer. 2008. *Access to Success: Patterns of Advanced Placement Participation in U.S. High Schools.* Princeton, NJ: Educational Testing Service.

Haertel, Edward. 2016. "Future of NAEP Long-Term Trend Assessments." White paper prepared for National Assessment Governing Board (NAGB). Washington, DC: NAGB, December 9, 2016. https://www.nagb.gov/content/nagb/assets/documents/newsroom/naep-releases/naep-long-term-trend-symposium/long-term-trends.pdf.

Hassi, Marja-Liisa, and Sandra L. Laursen. 2015. "Transformative Learning: Personal Empowerment in Learning Mathematics." *Journal of Transformative Education* 13, no. 4: 316–40. https://doi.org/10.1177/1541344615587111.

Higher Educational Research Institute. 2010, 2011, 2012, 2013, and 2014. *The American Freshman: Expanded National Norms Fall.* Los Angeles: Cooperative Institutional Research Program, Graduate School of Education and Information Studies, University of California. http://www.heri.ucla.edu/tfsPublications.php.

Home School Legal Defense Association (HSLDA). 2020. *Home School Laws in Your State*. Purcellville, VA: Home School Legal Defense Association. https://hslda.org/content/laws.

International Baccalaureate Organization (IBO). 2018. *About the IB's Programmes.* Bethesda, MD: International Baccalaureate Organization IB Global Centre. https://www.ibo.org/about-the-ib/.

Jimenez, Jeffrey, Ashley Jimenez, and Laura Jimenez. 2019. "Math Pathways: The Way Forward." https://www.americanprogress.org/issues/education-k-12/reports/2019/09/19/474645/math-pathways/.

Johns Hopkins University. 2020. *CTY Mission and History.* Baltimore: The Johns Hopkins Center for Talented Youth. https://cty.jhu.edu/about/mission.

Kober, Nancy. 2015. *Reaching Students: What Research Says About Effective Instruction in Undergraduate Science and Engineering*. Washington, DC: National Academies Press.

Koedel, Cory, and Morgan Polikoff. 2017. "Big Bang for Just a Few Bucks: The Impact of Math Textbooks in California." *Evidence Speaks Reports* 2, no. 5. Economic Studies at Brookings. https://www.brookings.edu/wp-content/uploads/2017/01/es_20170105_polikoff_evidence_speaks.pdf.

Kogan, Marina, and Sandra L. Laursen. 2015. "Assessing Long-Term Effects of Inquiry-Based Learning: A Case Study from College Mathematics." *Innovative Higher Education* 39: 183–99. https://doi.org/10.1007/s10755-013-9269-9.

Larsen, Sean, Erin Glover, and Kathleen Melhuish. 2015. "Beyond Good Teaching: The Benefits and Challenges of Implementing Ambitious Teaching." In *Insights and Recommendations from the MAA National Study of College Calculus*, edited by David Bressoud, Vilma Mesa, and Chris Rasmussen, pp. 93–105. Washington, DC: Mathematical Association of America Press.

Laursen, Sandra L., Marja-Liisa Hassi, Maria Kogan, and Timothy J. Weston. 2014. "Benefits for Women and Men of Inquiry-Based Learning in College Mathematics: A Multi-Institution Study." *Journal for Research in Mathematics Education* 45, no. 4 (July): 406–18.

Laursen, Sandra L., and Chris Rasmussen. 2019. "I on the Prize: Inquiry Approaches in Undergraduate Mathematics Education." *International Journal of Research in Undergraduate Mathematics Education* 5, no. 1: 129–149.

Laursen, Sandra L., Carol Schumacher, Matthew Jones, Charles Hayward, and Stan Yoshinobu. Under review 2019. A Broad Doorway to the Big Tent: A Four-Strand Model for Discipline Based Academic Development on Inquiry-Based Learning.

Maneki, Al. 2015. *Guidelines for Collegiate Faculty to Teach Mathematics to Blind or Visually Impaired Students.* Washington, DC: MAA Committee on the Undergraduate Program in Mathematics.

Martin, W. Gary, Brian R. Lawler, Alyson E. Lischka, and Wendy M. Smith, eds. 2019. *The Mathematics Teacher Education Partnership: The Power of a Networked Improvement Community to Transform Secondary Mathematics Teacher Preparation*. In Association of Mathematics Teacher Educators Professional Book Series vol. 4, edited by Babette M. Benka. Charlotte, NC: Information Age Publishing.

Mathematical Association of America (MAA). 2015. *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences.* Washington, DC: MAA, Committee on the Undergraduate Program in Mathematics (CUPM). https://www.maa.org/sites/default/files/CUPM%20Guide.pdf.

———. 2018. *Guidelines to Evidence-Based Instruction Practices in Undergraduate Mathematics.* Washington, DC: MAA. https://www.maa.org/programs-and-communities/curriculum%20resources/instructional-practices-guide.

McFarland, Joel, Jishan Cui, Amy Rathbun, and Juliet Holmes. 2018. *Trends in High School Dropout and Completion Rates in the United States: 2018* (NCES 2019-117). U.S. Department of Education. Washington, DC: National Center for Education Statistics. http://nces.ed.gov/pubsearch.

McFarland, Joel, Bill Hussar, Jijun Zhang, Xiaolei Wang, Ke Wang, Sarah Hein, Melissa Diliberti, Emily Forrest Cataldi, Farrah Bullock Mann, and Amy Barmer. 2019. *The Condition of Education 2019* (NCES 2019-144). U.S. Department of Education. Washington, DC: National Center for Education Statistics. https://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2019144.

McKnight, Curtis C., F. Joe Crosswhite, John A. Dossey, Edward Kifer, Jane O. Swafford, Kenneth J. Travers, and Thomas J. Cooney. 1987. *The Underachieving Curriculum: Assessing U.S. School Mathematics from an International Perspective.* Champaign, IL: Stipes Publishing.

McLeod, Douglas B., Robert E. Stake, Bonnie P. Schappelle, Melissa Mellissinos, and Mark J. Gierl. 1996. "Setting the Standards: NCTM's Role in the Reform of Mathematics Education." In *Bold Ventures: Case Studies of U.S. Innovations in Mathematics Education*, vol. 3, edited by Senta A. Raizen and Edward D. Britton, pp. 13–132. Dordrecht, The Netherlands: Kluwer Academic.

Meany, Karen, and Kathy Mickey. 2019. *K–12 Mathematics Market Survey Report 2019.* Rockville, MD: Simba Information, Market Research LLC.

Mu Alpha Theta. 2020. *Mu Alpha Theta: National High School and Two-Year College Mathematics Honor Society*. Norman, OK: Mu Alpha Theta, University of Oklahoma. https://www.mualphatheta.org.

Mullis, Ina V. S., Michael O. Martin, Albert E. Beaton, Eugenio J. Gonzalez, Dana L. Kelly, and Teresa A. Smith. 1998. *Mathematics and Science Achievement in the Final Year of Secondary School: IEA's Third International Mathematics and Science Study*. Chestnut Hill, MA: Boston College. http://timss.bc.edu/timss1995i/TIMSSPDF/C_full.pdf.

National Assessment Governing Board (NAGB). 2015. *Frequently Asked Questions*. Washington, DC: NAGB.

National Research Council (NRC). 2001. *Adding it Up: Helping Children Learn Mathematics.* Edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell. Washington, DC: National Academies Press.

———. 2019. *Mathematics Framework for the 2019 National Assessment of Educational Progress*. Washington, DC: NAGB. https://www.nagb.gov/content/nagb/assets/documents/publications/frameworks/mathematics/2019-math-framework.pdf.

NAEP Data Explorer (NDE). 2015a. *Student Questionnaires Results: Classroom Instruction for Mathematics, Reading and Science*. Washington, DC: U.S. Department of Education, National Center for Education Statistics. https://www.nationsreportcard.gov/sq_classroom/#mathematics?anchor=section-1.

———. 2015b. *Survey Questionnaires Results: Students' Computer Access and Use*. Washington, DC: U.S. Department of Education, National Center for Education Statistics. https://www.nationsreportcard.gov/sq_computer/.

National Center for Education Statistics (NCES). 2005. *Dual Enrollments of High School Students at Postsecondary Institutions: 2002–03*. Washington, DC: U.S. Department of Education, National Center for Education Statistics. https://nces.ed.gov/surveys/peqis/inc/displaytables_inc.asp.

———. 2013. *Dual Enrollment Programs and Courses for High School Students at Postsecondary Institutions: 2010–11.* Washington, DC: U.S. Department of Education, National Center for Education Statistics. https://nces.ed.gov/pubs2013/2013002/tables/table_01.asp.

———. 2013. *Trends in Academic Progress 2012*. Washington, DC: U.S. Department of Education, NCES.

———. 2015. *NAEP Data Explorer.* Washington, DC: U.S. Department of Education, National Center for Education Statistics. https://www.nationsreportcard.gov/ndecore/xplore/NDE.

———. 2016. *School and Student Participation Rates in NAEP Mathematics at Grade 8, by State/Jurisdiction: 2015*. Washington, DC: U.S. Department of Education, NCES.

National Conference of State Legislatures. 2019. *Common Core State Map*. Washington, DC: National Conference of State Legislatures. https://www.ccrslegislation.info/ccr-state-policy-resources/common-core-status-map/.

National Consortium for Secondary STEM Schools (NCSSS). 2020. *Mission and Vision*. Chevy Chase, MD: NCSSSMST. http://ncsss.org/about/mission-vision.

National Council of Teachers of Mathematics (NCTM). 1980. *An Agenda for Action: Recommendations for School Mathematics in the 1980s.* Reston, VA: NCTM.

———. 1989. *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

———. 1991. *Professional Standards for Teaching Mathematics.* Reston, VA: NCTM.

———. 1993. *Assessment Standards for School Mathematics.* Reston, VA: NCTM.

———. 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

———. 2006. *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.* Reston, VA: NCTM.

———. 2009. *Focus in High School Mathematics: Reasoning and Sense Making.* Reston, VA: NCTM.

———. 2014. *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: NCTM.

———. 2018. *Catalyzing Change in High School Mathematics: Initiating Critical Conversations.* Reston, VA: NCTM.

———. 2020a. *Catalyzing Change in Early Childhood and Elementary Mathematics: Initiating Critical Conversations.* Reston, VA: NCTM.

———. 2020b. *Catalyzing Change in Middle School Mathematics: Initiating Critical Conversations.* Reston, VA: NCTM.

National Council for the Social Studies (NCSS). 2010. *National Curriculum Standards for Social Studies: A Framework for Teaching, Learning, and Assessment*. Silver Spring, MD: NCSS.

National Governors' Association Center for Best Practices (NGA) and Council of Chief State School Officers (CCSSO). 2010a. *Common Core State Standards for Mathematics. Common Core State Standards (College- and Career-Readiness Standards and K–12 Standards in English Language Arts and Math).* Washington, DC: NGA and CCSSO. http://www.corestandards.org.

———. 2010b. *Appendix A: Common Core State Standards for Mathematics*. Washington, DC: NGA and CCSSO. http://www.corestandards.org/assets/CCSSI_Mathematics_Appendix_A.pdf.

Next Generation Science Standards (NGSS) Lead States. 2013. *Next Generation Science Standards: For States, By State*s. Washington, DC: National Academies Press.

National Research Council (NRC). 2011. *Successful K–12 STEM Education: Identifying Effective Approaches in Science, Technology, Engineering, and Mathematics.* Committee on Highly Successful Science Programs for K–12 Science Education; Board on Science Education and Board on Testing and Assessment, Washington, DC: National Academies Press. https://www.nap.edu/catalog/13158/successful-k-12-stem-education-identifying-effective-approaches-in-science.

———. 2012. *Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century.* Washington, DC: National Academies Press.

———. 2013a. *The Mathematical Sciences in 2025*. Committee on the Mathematical Sciences in 2025; Board on Mathematical Sciences and Their Applications; Division on Engineering and Physical Sciences. Washington, DC: National Academies Press. http://www.nap.edu/catalog/15269/the-mathematical-sciences-in-2025.

———. 2013b. *Monitoring Progress toward Successful K–12 STEM Education: A Nation Advancing?* Committee on the Evaluation Framework for Successful K–12 STEM Education; Board on Science Education and Board on Testing and Assessment; Division of Behavioral and Social Sciences and Education. Washington, DC: National Academies Press. https://www.nap.edu/catalog/13509/monitoring-progress-toward-successful-k-12-stem-education-a-nation.

National Science Board. 2014. *Science and Engineering Indicators 2014*. NSB 14-01. Arlington VA: National Science Foundation. http://www.nsf.gov/statistics/seind14 /content/etc/nsb1401.pdf.

National Science Foundation (NSF). 2020. *Research Experiences for Undergraduates (REU) Sites: Mathematical Sciences.* Arlington, VA: NSF. https://www.nsf.gov/crssprgm/reu/list_result.jsp?unitid=5044.

Partnership for Assessment of Readiness for College and Careers (PARCC). 2015. *States and Assessments*. Washington, DC: PARCC. http://www.parcconline.org.

Perie, Marianne, Rebecca Moran, and Anthony D. Lutkus. 2005. *NAEP 2004 Trends in Academic Progress: Three Decades of Student Performance in Reading and Mathematics*. Washington, DC: National Center for Education Statistics.

Polikoff, Morgan S. 2015. "How Well Aligned Are Textbooks to the Common Core Standards in Mathematics?" *American Educational Research Journal* 52, no. 6: 1185–1211.

Presidential Council of Advisors on Science and Technology (PCAST). 2012. *Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics.* Washington, DC: Office of the President.

Rampey, Bobby D., Gloria S. Dion, and Patricia L. Donahue. 2009. *NAEP 2008 Trends in Academic Progress*. Washington, DC: U.S. Department of Education, National Center for Education Statistics.

Rasmussen, Chris, Naneh Apkarian, Jessica Ellis Hagman, Estrella Johnson, Sean Larsen, and David Bressoud. 2019. "Characteristics of Precalculus through Calculus 2 Programs: Insights from a National Census Survey." *Journal for Research in Mathematics Education* 50, no. 1 (January): 24–37.

Rasmussen, Chris, Jessica Ellis, and Dov Zazkis. 2014. "Lessons Learned from Case Studies of Successful Calculus Programs at Five Doctoral Degree Granting Institutions." In *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education*, edited by Tim Fukawa-Connelly, Gulden Karakok, Karen Keene, and Michelle Zandieh, pp. 999–1004. Denver, Colorado, June 5–7.

Ray, Brian D. 2019. *Research Facts on Homeschooling*. Salem, OR: National Home Education Research Institute. https://www.nheri.org/research-facts-on-homeschooling/.

Resnick, Robert M., and Glenn Sanislo. 2015. *The Complete K–12 Report: Market Facts & Segment Analyses 2015*. Rockaway Park, NY: Education Market Research/Simba Information.

Reys, Barbara, ed. 2006. *The Intended Mathematics Curriculum as Represented in State-Level Curriculum Standards: Consensus or Confusion?* Charlotte, NC: Information Age.

Ross Mathematics Program. 2015. *The Ross Mathematics Program*. Columbus, OH: Ohio Dominican University. https://rossprogram.org.

Sadler, Phillip, and Gerhard Sonnert. 2018. "The Path to College Calculus: The Impact of High School Mathematics Coursework." *Journal for Research in Mathematics Education* 49, no. 3 (May): 292–329.

Saxe, Karen, and Linda Braddy. 2015. *A Common Vision for Undergraduate Mathematical Sciences Programs in 2025.* Washington, DC: Mathematics Association of America.

Schmidt, William. 2000. "Researchers Find Systematics Problem in U.S. Mathematics and Science Education." *New Education.* East Lansing, MI: College of Education, Michigan State University. http://www.educ.msu.edu/neweducator/fall00/timss.htm.

Smarter Balanced Assessment Consortium (SBAC). 2015. *Understand the Individual Student Report*. SBAC. http://smarterbalanced.org.

Smith, Margaret S., and Mary Kay Stein. 2018. *5 Practices for Orchestrating Productive Mathematics Discussion*. 2nd ed. Reston, VA: National Council of Teachers of Mathematics.

Snyder, Thomas D., Cristobal de Brey, and Sally A. Dillow. 2018. *Digest of Education Statistics: 2016*. Washington, DC: National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education.

Snyder, Thomas D., and Sally A. Dillow. 2015. *Digest of Education Statistics 2013*. Washington, DC: National Center for Education Statistics. http://nces.ed.gov.

———. 2019. *Digest of Education Statistics: 2017*. Washington, DC: National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education.

Society for Industrial and Applied Mathematics (SIAM) and the Consortium for Mathematics and Its Applications (COMAP). 2016. *Guidelines for Assessment and Instruction in Mathematics Modeling Education (GAIMME)*. Philadelphia, PA: SIAM; Bedford, MA: COMAP. http://www.siam.org/Portals/0/Publications/Reports/gaimme-full_color_for_online_viewing.pdf?ver=2018-03-19-115454-057.

Sonnert, Gerhard, and Phillip Sadler. "The Impact of Instructor and Institutional Factors on Students' Attitudes." 2015. In *Insights and Recommendations from the MAA National Study of College Calculus*, edited by David Bressoud, Vilma Mesa, and Chris Rasmussen, pp. 17–29. Washington, DC: Mathematical Association of America.

Stains, Marilyne, Jordan Harshma, Megan K. Barker, Stephanie V. Chasteen, Reneé Cole, S. E. DeChenne-Peters, M. K. Eagan Jr., J. M. Esson, J. K. Knight, F. A. Laski, M. Levis-Fitzgerald, C. J. Lee, S. M. Lo, L. M. McDonnell, T. A. McKay, N. Michelotti, A. Musgrove, M. S. Palmer, K. M. Plank, T. M. Rodela, E. R. Sanders, N. G. Schimpf, P. M. Schulte, M. K. Smith, M. Stetzer, B. Van Valkenburgh, E. Vinson, L. K. Weir, P. J. Wendel, L. B. Wheeler, and A. M. Young. 2018. "Anatomy of STEM Teaching in North American Universities." *Science* 359, no. 6383 (March 30): 1468–70. https://doi.org/10.1126/science.aap8892.

Stetser, Marie C., and Robert Stillwell. 2014. *Public High School Four-Year On-Time Graduation Rates and Event Dropout Rates: School Years 2010–11 and 2011–12—First Look*. NCES 2014-391. Washington, DC: National Center for Education Statistics. http://nces.ed.gov/pubs2014/2014391.pdf.

Stolzenberg, Ellen B., Kevin Eagan, Edgar Romo, Elaine J. Tamargo, Melissa C. Aragon, Madeline Luedke, and Nathaniel Kang. 2019. *The American Freshman: National Norms Fall 2018*. Los Angeles: Higher Education Research Institute.

Struchens, Marilyn E., Judith R. Quander, and Rochelle Gutiérrez. 2011. "Mathematics Learning Communities That Foster Reasoning and Sense Making for All High School Students." In *Focus on Reasoning and Sense Making for All Students,* edited by Marilyn E. Struchens and Judith R. Quander, pp. 101–13. Reston, VA: National Council of Teachers of Mathematics.

StudyPoint. 2020. *Dual Enrollment Programs: The Pros and Cons*. Boston: StudyPoint. https://www.studypoint.com/ed/dual-enrollment.

U.S. Department of Commerce. 2018. *Current Population Survey* (CPS). Washington, DC: U.S. Census Bureau, American Community Survey Office. https://www.census.gov/programs-surveys/cps.html.

U.S. Department of Education. 2008. *No Child Left Behind*. Washington, DC: U.S. Department of Education. http://www2.ed.gov/nclb/landing.jhtml.

———. 2019. *Every Student Succeeds Act*. Washington, DC: U.S. Department of Education. https://www.ed.gov/essa.

———. 2020. *Teacher Shortage Area Report.* Washington, DC: U.S. Department of Education. https://tsa.ed.gov/#/reports.

UTeach. 2019. UTeach website. Austin, TX: The University of Texas at Austin. Information retrieved 01/06/2020 from https://uteach.utexas.edu/about.

van den Ham, Ann-Katrin, and Aiso Heinze. 2018. "Does the Textbook Matter? Longitudinal Effects of Textbook Choice on Primary School Students' Achievement in Mathematics." *Studies in Educational Evaluation* 59:133–40.

Watkins, Laura. 2016. "Project ACCCESS." *AMATYC News* 31, no. 2 (April): 6.

Wiggins, Grant, and Jay McTighe. 2005. *Understanding by Design.* 2nd ed. Alexandria, VA: Association for Supervision and Curriculum Development.

Zachry Rutschow, Elizabeth. 2019. *Commissioned Paper for The National Academies of Sciences, Engineering, and Medicine Workshop on Understanding Success and Failure of Students in Developmental Mathematics: Developmental Mathematics Reforms.* https://sites.nationalacademies.org/cs/groups/dbassesite/documents/webpage/dbasse_191791.pdf.

Zachry Rutschow, Elizabeth Mayer, and Alexander Mayer. 2018. *Early Findings from a National Survey on Developmental Education Practices.* New York: Center for the Analysis of Post-secondary Readiness (CAPR).