PRACTICE 2

Reason Abstractly and Quantitatively

Practice 2: Reason abstractly and quantitatively

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (CCSSI 2010, p. 6)

Unpacking the Practice

Practice 2 highlights a general disposition and set of skills that we hope to see across all of mathematics. However, it has particularly strong ties to children's understanding of number and operations and algebra. While it is beyond the scope of this book to provide an in-depth analysis of connections to the National Council of Teachers of Mathematics (NCTM) Content Standards, after addressing the Process Standards we briefly highlight a few key connections to NCTM's Number and Operations Standard and Algebra Standard.

Problem Solving Standard

Practice 2 of the *Common Core State Standards for Mathematics* (CCSSM) begins with a focus on problem solving by highlighting how students should "make sense of quantities and their relationships *in problem situations*" (CCSSI 2010, p. 6; italics added), with a particular emphasis on the role of quantitative and abstract reasoning when solving problems. Although the practice is framed as a set of skills proficient students bring to bear on a problem, it is important to understand that abstract and quantitative reasoning develop *through* problem-solving opportunities. The first vignette in the examples section shows how problem-solving situations can be used to introduce students to or deepen their understanding of this practice. This connects to NCTM's Problem Solving Standard, which states that students should have opportunities to "build new mathematical knowledge through problem solving" (NCTM 2000, p. 52).

Another important aspect of NCTM's Problem Solving Standard is that students should "solve problems that arise in mathematics and *in other contexts*" (NCTM 2000, p. 52; italics added). Practice 2 emphasizes the relationship between contextualized problems and mathematical symbols, thus providing ample opportunity to connect mathematics to other contexts. This practice also highlights the importance of *Adding It Up*'s procedural fluency strand in problem solving. "*Procedural fluency* refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (NRC 2001, p. 121). While much of practice 2 focuses on contextualizing and decontextualizing (moving back and forth between the problem situation and its mathematical representation), this practice also states that students should "manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents" (CCSSI 2010, p. 6). As students develop a greater expertise and fluency with mathematical procedures, they can use the procedures as a tool without the need to constantly refer back to the problem situation. However, as this practice emphasizes, students need to check with reality frequently to be sure that their mathematical work remains connected to the problem context.

Representation Standard

Representation plays a central role in practice 2 and in the decontextualizing and contextualizing that the practice emphasizes in particular. This practice explicitly states that students should "abstract a given situation and *represent* it symbolically" (CCSSI 2010, p. 6; italics added) and that "quantitative reasoning entails habits of creating a coherent *representation* of the problem at hand…[and] attending to the meaning of quantities" (p. 6; italics added). Central to this practice is the use of the language of mathematics to represent a problem in a useful way, as well as the ability to make sense of symbolic representations of problems.

NCTM's (2000) *Principles and Standards for School Mathematics* states that "the term *representation* refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67). This dual nature of representation can be seen most clearly in the first vignette in the examples section. On the one hand, the teacher emphasizes the products by creating various symbolic representations of the students' strategies and using these representations of the problem to move forward. On the other hand, the entire vignette is an example of the process "of capturing a mathematical concept or relation in some form" (NCTM 2000, p. 67) because the teacher continually asks the students to explain the relationship between the symbolic representations and the problem context.

The importance of mathematical representation is also captured by the strategic competence strand from *Adding It Up*, which states that "with a formulated problem in hand, the student's first step in solving it is to represent it mathematically in some fashion, whether numerically, symbolically, verbally, or graphically" (NRC 2001, p. 124). By emphasizing the importance of contextualizing and "attending to the meaning of quantities" (CCSSI 2010, p. 6), this practice draws a clear distinction between reasoning about relationships among quantities and what *Adding It Up* refers to as "number grabbing":

> Becoming strategically competent involves an avoidance of "number grabbing" methods (in which the student selects numbers and prepares to perform arithmetic operations on them) in favor of methods that generate problem models (in which the student constructs a mental model of the variables and relations described in the problem). (NRC 2001, p. 124)

Notice that the emphasis in this practice is on "making sense of *quantities* and their relationships in problem situations" (CCSSI 2010, p. 6; italics added), not blindly computing with numbers or algebraic symbols. It is important to understand that quantity should be seen as a measureable attribute of an object and thus different from numbers. Thompson (1993) describes quantitative reasoning in the following way:

> A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter in to the primary analysis of a situation. What is important is relationships among quantities. In that regard, quantitative reasoning bears a strong resemblance to the kind of reasoning customarily emphasized in algebra instruction. (p. 165)

More specifically, Thompson argues that one can reason quantitatively without assigning specific measurements (numbers) and the use of numbers does not necessarily imply that one is reasoning quantitatively. He gives the example of being able to determine whether you or someone else is taller without actually measuring either person's height.

Communication Standard

As written, practice 2 does not explicitly address mathematical communication. However, this practice is deeply connected to mathematical communication in two important ways. First, as illustrated in the elementary grades vignette in the examples section, engaging students in this practice provides ample opportunity for students to communicate about their mathematical thinking. When students must explain the relationship between the real-world context of a problem and their symbolic representation of that context, they must articulate their thinking to others and listen to and make sense of others' thinking and explanations. These ideas are reflected in the first three goals of NCTM's Communication Standard: Instructional programs… should enable all students to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;

- analyze and evaluate the mathematical thinking and strategies of others. (NCTM 2000, p. 60)

Therefore, we can see how this practice is supportive of NCTM's Communication Standard. Having students participate in abstract and quantitative reasoning provides rich opportunities for them to engage in mathematical communication. In addition, having students engage in mathematical communication is a key means of providing students with opportunities to reason abstractly and quantitatively.

Second, we can use mathematics as a means of communication, or as the fourth goal in NCTM's Communication Standard states, instructional programs should enable students to "use the language of mathematics to express mathematical ideas precisely" (2000, p. 60). Reasoning abstractly and qualitatively provides students with the opportunity to do exactly this. By translating real-world problem contexts into mathematical symbols, students have an opportunity to engage in a particularly precise form of communication that is available through the language of mathematics.

Connections Standard

Practice 2 provides key opportunities for students to make mathematical connections. NCTM's Connections Standard emphasizes that students should be able to "recognize and use connections among mathematical ideas" (NCTM 2000, p. 64) as well as "recognize and apply mathematics in contexts outside of mathematics" (p. 64). This practice provides opportunities for both. The first vignette in the examples section highlights the possibility of making connections between mathematical ideas by examining multiple symbolic representations of the same mathematical pattern. This practice also emphasizes students using mathematical symbols to represent problem situations and the need for them to relate their mathematical work back to those situations, providing the opportunity to connect mathematics to real-world contexts. Both types of connections support the development of *Adding It Up*'s productive disposition strand: Making connections between mathematical ideas helps students learn that "mathematics is understandable, not arbitrary" (NRC 2001, p. 131) and connections to real-world situations can help students see that mathematics is "both useful and worthwhile" (p. 131).

Number and Operations Standard

Reasoning abstractly and quantitatively requires that students understand multiple "ways of representing numbers [and the] relationships among numbers" (NCTM 2000, p. 32) as well as "understand meanings of operations and how they relate to one another" (p. 32). This might include renaming numbers in productive or useful ways, such as renaming 37 from "3 tens and 7 ones" to "2 tens and 17 ones" when using the standard U.S. algorithm for subtraction or relying on properties such as the commutative and associative properties of addition to simplify computations when solving a problem. Such work would reflect the idea that students should be able to "manipulate the representing symbols as if they have a life of their own" (CCSSI 2010, p. 6) and would highlight the importance of students "knowing and flexibly using different properties of operations and objects" (p. 6).

Algebra Standard

Practice 2 is arguably most closely related to NCTM's Algebra Standard. The practice points to the importance of students' ability to "*decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own" (CCSSI 2010, p. 6) and to "*contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved" (p. 6). A prototypical example of this practice might involve translating a problem situation into algebraic notation (decontextualizing), manipulating those symbols to arrive at a solution, and then reinterpreting that answer in terms of the original problem situation (contextualizing). The NCTM Algebra Standard emphasizes similar forms of reasoning, stating that students must be able to "represent and analyze mathematical situations and structures using algebraic symbols" (NCTM 2000, p. 37) and "use mathematical models to represent and understand quantitative relationships" (p. 37).

Classroom Examples

The examples below can be adapted to work across a broad range of grades and highlight two different aspects of reasoning abstractly and quantitatively. The elementary grades vignette focuses heavily on helping students learn to move back and forth between problem context (the number of seats at a table) and mathematical representations of this situation (such as symbolic expressions and equations). The middle and high school vignette offers a series of sample tasks that can be used to help students develop their understanding of *rate of change* (how one quantity changes in relationship to another quantity) and provides a clear emphasis on reasoning about relationships among quantities as opposed to considering specific numerical values.

Elementary Grades Vignette: How Many Seats?

The example below can be adapted to a broad range of elementary and middle grades. Work with younger children could focus more heavily on the earlier part of the vignette in which children share and explain their strategies with concrete numbers; see, for instance, the Dot Square problem (NCTM Problem Solving Standard; NCTM 2000, p. 185), which gives an example of a similar problem for grades 3 to 5. Work with older children could focus more heavily on introducing algebraic notation and developing a variety of algebraic expressions to match this situation. The vignette draws heavily from work with middle school students found in Bishop, Otto, and Lubinski (2001), Lannin (2003), and Boaler and Humphreys (2005), all three of which provide excellent examples of similar activities, how they can be used in the classroom, and the strategies children develop for solving these types of problems.

Teacher: Look at this picture I have on the board [*fig. 2.1*]. We are going to figure out how many people can sit at a table made out of squares. In the picture, there are four people sitting at just one square table. Figure out how many people can sit in the next three pictures.

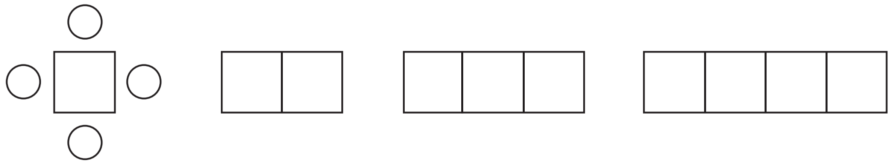

Fig. 2.1. Square table pattern

[*The students quickly complete this task, and the teacher has students explain their strategies for the fourth picture.*]

Teacher: Can someone come up and explain how they counted the number of seats for four squares?

Ashanti: Well this [*runs her finger along the top edge*] would be four chairs, and then the bottom would be another four, making eight. Then you would have these two [*points to the two ends*] makes ten.

Teacher: OK, did everyone see how she figured that out? I'm going to write that like this [*writes* $4 + 4 + 2$]. Ashanti, do you think that shows how you counted up the seats, or should I write it a different way?

Ashanti: That's how I did it.

Teacher: Did anyone count the number of people in a different way?

Khalil: I counted three on each end, so that's six. Then I counted two for each table in the middle, so that's four more. Six and four is ten.

Teacher: I'm not sure I followed all that. Can you come up and show us what you mean by three on each end?

Khalil: I mean on this table [*points to the first square*] there are three seats, one, two, three [*touches the top, left, and bottom edges*]. And then the same thing for this table [*points to the last square*]. So that's six.

Teacher: OK, and then you said "two for each table in the middle"?

Khalil: Yeah, these two in the middle have two seats, one on the top and one on the bottom. So two twos is four more seats. So that's ten total.

Teacher: I see now. Can I write your strategy like this? [*Writes* $3 + 3 + 2 + 2$.] Does that match your thinking?

Khalil: Yeah.

Teacher: OK, now I want you to imagine that you have ten square tables in a row. I want you to figure out how many people could sit there, but I want you to do it in three different ways. First, I want you to figure it out using

Ashanti's strategy. Then I want you to try Khalil's strategy. Then if you want you can come up with your own way.

[*The students work independently on this for a few minutes.*]

Teacher: OK, can someone besides Ashanti tell me how she would solve this problem?

Carlos: I can. First she would say there are ten seats all along the top. Then she would say there are ten more seats along the bottom. Then she would add two more seats for the ends. So that would be ten plus ten is twenty, plus two more is twenty-two.

Teacher: What do you think, Ashanti, does that fit your strategy from before?

Ashanti: Yeah, that's how I did it.

Teacher: OK, so if I write it the same way as before, so it kind of matches this [*points to 4 + 4 + 2 written earlier*], but for this new problem, what would I write?

Juanita: It would be ten plus ten plus two.

Teacher: [*Writes 10 + 10 + 2 next to 4 + 4 + 2.*] OK, and what does this first ten mean?

Juanita: Those are all the seats along the top.

Teacher: OK, and how do you know to write a ten there? Did you count each seat by ones?

Juanita: Well, you could count them by ones, but you can also just know.

Teacher: How would you just know? Anyone can answer, how would you just know that there are ten seats along the top without counting by ones?

Jaylen: Every table has one seat on top, and there are ten tables, so you don't have to count, you just can figure out that it would be ten.

[*The conversation continues, and the teacher then facilitates a similar conversation about Khalil's strategy. The teacher then asks the class to repeat this activity for a row of one hundred tables, conducts another class discussion of the strategies, and then finally asks the whole class to consider if there were "n" tables pushed together.*]

Teacher: OK, so we have figured this out for ten tables and one hundred tables. What if we knew there were a lot of tables in a row, but we did not know exactly how many? When mathematicians have a problem like that, they use a letter to stand for the number of tables. So instead of saying there are ten tables or one hundred tables, they would say, "we don't know how many tables, so we will say there are *n* tables." And this *n* can be any number of tables. So think about how Ashanti would figure out how many people could sit down. Talk to your partners about how Ashanti could do this problem.

[*After a brief discussion in partners, the teacher calls the class back together.*]

Teacher: OK, any ideas? How could Ashanti use her strategy now? I know this is harder because we don't actually know how many tables there are. David, would you be willing to share what you and Rosa were talking about?

David: Well, we said it might be *n* plus *n* plus two.

Teacher: Where did you get that idea from?

David: Well, when it was four, it was four plus four plus two, and then with ten it was ten plus ten plus two, and the same with one hundred. So we just said "*n*" instead of the number.

Teacher: OK, so, I can write this $n + n + 2$, is that what you're saying?

David: Yeah, that's what we wrote down.

Teacher: OK, so, and anyone can answer this question, can someone explain what this means? What does this first *n* mean?

Rosa: Well, we were saying that the *n* is like how many seats there are on the top. Just like in the other problems.

Teacher: But how do you know how many there will be? How do you know it will be *n*?

Rosa: Because it's always the same as the number of tables.

[*The lesson continues with a transition to problems where the teacher tells the class the number of people and asks how many tables there must have been.*]

Practice 2 essentially describes a three-step process: (1) decontextualizing problems by representing a problem context using mathematical symbols; (2) manipulating symbols, such as performing calculations or solving an algebraic equation; and (3) contextualizing problems by periodically connecting the mathematical symbols back to the problem context. The vignette above focuses primarily on the first and third steps. The teacher solicits strategies and then demonstrates how they can be represented symbolically. However, she continually draws the students' attention to the meaning of the symbols in terms of the problem context (people sitting around a table). Thus the teacher is demonstrating how we can move back and forth between a problem context and a symbolic representation of that context.

Traditional mathematics instruction frequently involves teaching algebraic and symbolic rules first, without a meaningful problem context, and then later having students apply them to problem situations. This vignette shows how taking the opposite approach allows students to build meaning for mathematical symbols out of the sense making they have engaged in with the problem context (people sitting at a table), thus grounding the mathematical symbols in the students' reasoning instead of emphasizing an abstract set of rules and procedures to memorize.

Middle and High School Vignette: Rate of Change

Understanding rate of change is important for understanding functions and graphs and laying a foundation for calculus. An important aspect of understanding functional relationships involves coming to think of how one quantity (treated as the dependent variable) varies based on a relationship with another quantity (treated as the independent variable). Rate of change involves understanding how the dependent variable changes in response to changes in the independent variable. Is it increasing, constant, or decreasing, and is the rate of change steady, speeding up, or slowing down? A focus on rate of change can begin even before algebra, as can be seen in van Dyke and Tomback's (2005) article discussing how they collaborated to introduce algebra first through "qualitative graphs (graphs without scale), then quantitative graphs (graphs with scale)" (p. 237). The examples that follow are re-created from van Dyke and Tomback's work.

An early task can involve asking students to match graphs to a situation that is described in words, such as in figure 2.2. Initial problems such as these allow students to begin focusing on the relationship between quantities and when the dependent variable is increasing or decreasing as the independent variable increases. Later problems can provide a greater focus on the *intensity* of the rate of change, for instance considering when something is speeding up or slowing down as seen in figure 2.3.

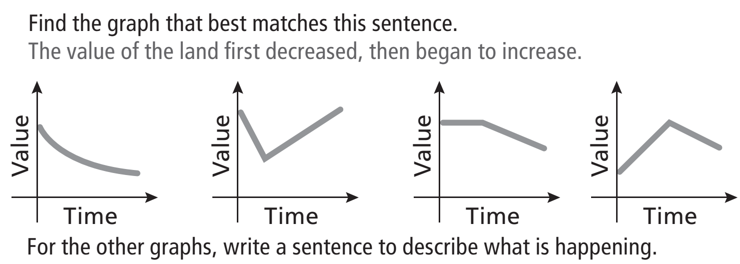

Fig. 2.2. Match a graph to a story
(adapted from van Dyke and Tomback [2005, pp. 237–38])

Janet, Gail, and Susan all walked away from the railroad station.
Janet walked at a steady pace, Gail speeded up as she walked,
and Susan slowed down.
Decide which graph pictures each girl's walk. Explain your reasoning.

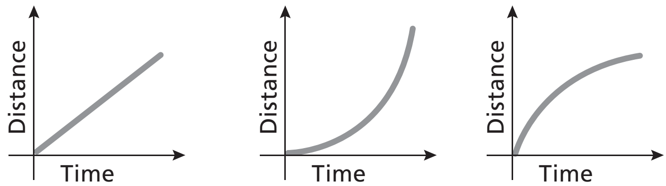

Fig. 2.3. Speeding up and slowing down
(re-created from van Dyke and Tomback [2005, p. 239])

Examples such as these are important for developing students' understanding of relationships between quantities and rate of change in particular. However, they also lay an important foundation for understanding calculus. Johnson (2012) provides an example of a student who developed such reasoning prior to taking calculus. Johnson found that rich understanding of rate of change involved the ability to systematically vary one variable, which acted as the independent variable (such as the side length of a square) and examine the change in the second variable (such as the area of the square). Importantly, as discussed in the examples above, the student could not only attend to the direction of change (was it increasing or decreasing) but to the intensity of this change (was it changing quickly or slowly, and was it a steady change or was it speeding up or slowing down).

One of the activities Johnson (2012) used was to create a square with the Geometer's Sketchpad software (Jackiw, 2001) and then allow the student to drag a corner of the square. The software would display the side length, the area, and the perimeter of the square. Johnson also later gave a set of two tables: one showing side length versus perimeter and another showing side length versus area. This activity allowed the student to explore how perimeter and area change in relationship to changes in side length. Two features of this problem that may help focus students' attention on the intensity of the rate of change are (1) students can examine when the area is growing faster than the perimeter and vice versa, and (2) students can investigate if the area changes at a steady rate or if it increases "faster and faster."

A common theme that cuts across these examples from van Dyke and Tomback (2005) and Johnson (2012) is the clear emphasis on the relationships among the various quantities *before* turning attention to the specific numerical values. This echoes the point raised above that quantity is not number (Thompson 1993); quantitative reasoning is fundamentally about considering the relationships among quantities within the problem context.

Resources

This resource provides a broad range of examples for incorporating algebraic reasoning into the elementary grades, including patterning activities and generalizing around simple story problems.

- Blanton, M. L., and J. J. Kaput. "Developing Elementary Teachers' Algebra Eyes and Ears." *Teaching Children Mathematics* 10, no. 2 (2003): 70–77.

These resources were drawn on heavily in the elementary-grades vignette above. They detail different forms of student thinking when making generalizations of patterns and the role of the teacher in accurately capturing students' thinking.

- Bishop, J. W., A. D. Otto, and C. A. Lubinski. "Promoting Algebraic Reasoning Using Students' Thinking." *Mathematics Teaching in the Middle School* 6, no. 9 (2001): 508–14.
- Lannin, J. K. "Developing Algebraic Reasoning through Generalization." *Mathematics Teaching in the Middle School* 8, no. 7 (2003): 342–48.

These resources connect to the secondary examples above in focusing on connecting qualitative graphs (graphs without scale) to real world contexts.

- Maus, J. "Every Story Tells a Picture." *Mathematics Teaching in the Middle School* 10, no. 8 (2005): 375–79.
- van Dyke, F., and J. Tomback. "Collaborating to Introduce Algebra." *Mathematics Teaching in the Middle School* 10, no. 5 (2005): 236–42.

This resource provides a clear emphasis on the relationship between algebraic symbols and problem contexts (the decontextualizing and contextualizing emphasized in practice 2). It includes problem contexts such as painting a room, identifying the number of dots in a star pattern, and summing consecutive numbers.

- Philipp, R. A., and B. P. Schappelle. "Algebra as Generalized Arithmetic: Starting with the Known for a Change." *The Mathematics Teacher* 92, no. 4 (1999): 310–16.

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Thompson, P. W. "Quantitative Reasoning, Complexity, and Additive Structures." *Educational Studies in Mathematics* 25, no. 3 (1993): 165–208.

van Dyke, F., and J. Tomback. "Collaborating to Introduce Algebra." *Mathematics Teaching in the Middle School* 10, no. 5 (2005): 236–42.